ON A RATIONAL APPROXIMATION PROBLEM IN THE REAL HARDY SPACE $\mathbf{H}_2$

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1 Introduction.

The problem of finding a rational approximation to a holomorphic function in a domain Ω often arises in practice. Let us give an example from system theory which has first motivated our work [1].

System theory is concerned with the study of physical systems that can be described by a relation between inputs and outputs. When considering discrete time systems, it is a convention to denote by $u(z) = \sum_{k \geq k_0} u_k z^{-k}$, $u_k \in \mathbf{R}$, $k_0 \in \mathbf{Z}$, a sequence of inputs where u_k is applied at time $t = k$, and by $y(z) = \sum_{j \geq j_0} y_j z^{-j}$, $y_j \in \mathbf{R}$, $j_0 \in \mathbf{Z}$, a sequence of outputs where y_j occurrs at time $t = j$.
So, a discrete time system can be described by a map

$$\begin{array}{rccc} \sigma : & \mathbf{R}((1/z)) & \rightarrow & \mathbf{R}((1/z)) \\ & u & \rightarrow & y \end{array}$$

where $\mathbf{R}((1/z))$ is the set of truncated Laurent series.
Usually, further assumptions are made on the system (linearity, causality, stationarity) so that σ becomes a $\mathbf{R}((1/z))$-homomorphism, and y can be written as a product of formal series

$$y = fu,$$

where f is an element of $\mathbf{R}[[1/z]]$, the set of formal power series in $1/z$. The study of such systems has been completely achieved in the case where f is rational.
In practical situation, one would like to deduce some useful description of a system from experimental input-outputs data. Here, "useful description" means a description of the previous form characterized by a rational f of small size. Though such a model may fail to be exact, you can always compute some f from some data. However, the more data you have at hand, the more the f computed fits the situation; but at the same time, the bigger gets its size.
So the best thing one can hope is to find a rational function r which could replace f in a satisfactory way.
A mathematical meaning can be given to this problem through approximation theory. Indeed, to provide such an r, the mathematician makes the assumption that f lies in a

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normed space and looks for a rational function which minimizes the norm of the difference $f - r$, and whose order doesn't exceed a given integer n. Since $f \in \mathbf{R}[[1/z]]$, this space will be a set of holomorphic functions outside the unit disk.

The purpose of this paper is to present some results that we have obtained on this problem by using differential tools, when the space is the real Hardy space H_2 .
Since the commonly used optimization algorithms give no solution to this problem, because of its hill conditioning, our main goal was to propose a specific algorithm providing a local rational approximant. This was done thanks to some helpful recursive structure we put in evidence.
It must be noticed that there may be several local approximants and even several global ones. However, from a practical point of view, the existence of several global approximants doesn't make much sense and raises the question of the well-posedness of the problem. Moreover, up to this day, it hasn't been proved that our algorithm can find all the local approximants and thus the global ones. In view of these considerations, the question of the uniqueness of the approximant seems of the greatest importance. We shall give some results obtained in that direction.

2 Rational Approximation in H_2^-.

Following the introduction, we settle ourselves in the set of holomorphic functions in the complement of the closed unit disk on the Riemann sphere, and we choose the measure of closeness between f and a rational function r to be the Hardy norm

$$(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\ e^{it}) - r(\ e^{it})|^2 dt)^{1/2}.$$

Of course, $f(e^{it})$ is not a priori defined, and for the above formula to make sense, we will have to restrict ourselves to a special class of functions f that we now proceed to define.

Here and after, Ω stands for the complement of the closed unit disk on the Riemann sphere, T for the unit circle and U for the open unit disk.

The Hardy space $H_2(\Omega)$ is the space of functions f, holomorphic in Ω, and satisfying

$$\sup_{r>1}(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(r\ e^{it})|^2 dt)^{1/2} < \infty. \tag{1}$$

If we write down the power series expansion, converging for $|z| > 1$,

$$f(z) = \sum_{k\geq 0}\frac{a_k}{z^k}, \tag{2}$$

it follows at once from Parseval's equality that the integral in the left hand-side of (1) is equal to

$$\sum_k \frac{|a_k|^2}{r^{2k}},$$

so that f belongs to $H_2(\Omega)$ if and only if

$$\sum_k |a_k|^2 < \infty. \tag{3}$$

This establishes a one-to-one correspondance between $H_2(\Omega)$ and the subspace of $L^2(\mathbf{T})$ consisting of functions whose Fourier coefficients of positive rank are zero. This correspondance associates with f defined by (2) the function f^* in $L^2(\mathbf{T})$ defined by

$$f^*(e^{i\theta}) = \sum_{k\geq 0} a_k e^{-ik\theta}.$$

Thus, by definition, the coefficients of the power series expansion of f at infinity are the Fourier coefficients of f^*. A more subtle relation between these two functions is that f^* is equal almost everywhere to the radial limit of f [4]. As a consequence, f^* is the natural extension of f to the boundary of Ω, namely to the unit circle, and we shall no longer distinguish between f and f^* unless otherwise stated. This allows us to consider $H_2(\Omega)$ as a closed subspace of $L^2(\mathbf{T})$, and thus as a Hilbert space.

In the sequel, we shall only consider functions with real Fourier coefficients, or equivalently functions assuming real values for real arguments. They form a *real* subspace of $H_2(\Omega)$ denoted by $H_{2,\mathbf{R}}(\Omega)$, which inherits a structure of *real* Hilbert space.

Changing z into z^{-1} in the above construction defines another Hilbert space, known as the Hardy space of the unit disk, denoted by $H_2(\mathbf{U})$. This space consists of functions g holomorphic in $\mathbf{U}$, satisfying the growth condition (1) , where this time the *sup* should be taken over $r < 1$. If we write the power series expansion representing g in the unit disk:

$$g(z) = \sum_{n\geq 0} a_n z^n,$$

we see as before that (3) is a necessary and sufficient condition for g to belong to $H_2(\mathbf{U})$. In the same fashion, $H_2(\mathbf{U})$ can be identified with the closed subspace of $L^2(\mathbf{T})$ consisting of functions whose Fourier coefficients of negative rank do vanish. Again, this identification arises in fact by taking radial limits.

Restricting ourselves to functions with real Fourier coefficients, we introduce a *real* subspace of $H_2(\mathbf{U})$ that we should logically denote by $H_{2,\mathbf{R}}(\mathbf{U})$. We shall nevertheless prefer the notation H_2^+ for this real Hilbert space, and this discrepancy will soon disappear.

Now, define the *degree* of a rational function to be $max\{d^\circ p, d^\circ q\}$ where p/q is an irreducible representation.
The approximation problem that we want to study is the following:
given an integer n and $f \in H_{2,\mathbf{R}}(\Omega)$, find a minimum of the squared Hardy norm

$$\|f - \frac{p}{q}\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\,e^{it}) - \frac{p}{q}(\,e^{it})|^2 dt\,,$$

where p/q ranges over rational functions of degree at most n in $H_{2,\mathbf{R}}(\Omega)$.
Observe that a rational function belongs to $H_{2,\mathbf{R}}(\Omega)$ if and only it can be written p/q,

where p and q are real polynomials satisfying $d^\circ p \leq d^\circ q \leq n$, and q has all its roots in the open unit disk.

At this point it may be helpful for the reader to make some remarks. On the first hand, we restrict ourselves to real functions since we were first interested in this question through the identification problem described in the introduction. However, most of the results presented here are likely to have their complex counterparts though we shall not consider them. On the other hand, if we stated the problem in $H_{2,\mathbf{R}}(\Omega)$ and not in H_2^+ which looks more natural, it was not solely to follow a system theory convention. It turns out that $H_{2,\mathbf{R}}(\Omega)$ is technically easier to handle, because the poles of the functions remain bounded. Moreover, the two formulations are really equivalent. This can be readily checked for $z \to 1/z$ is an isometry between $H_{2,\mathbf{R}}(\Omega)$ and H_2^+ that preserves rational functions and their degree.

We shall first reduce the question to the case of functions f vanishing at infinity, or equivalently whose Fourier coefficient of rank 0 is zero. These functions form a closed subspace of $H_{2,\mathbf{R}}(\Omega)$ denoted by H_2^-, and a rational function of $H_{2,\mathbf{R}}(\Omega)$ belongs to H_2^- if and only if the degree of the numerator is *strictly* less than the degree of the denominator. Upon Euclidean division, any rational function $p/q \in H_{2,\mathbf{R}}(\Omega)$ can be written $c + r/q$, where $c \in \mathbf{R}$ and $r/q \in H_2^-$ has the same degree as p/q. Similarly, if $f \in H_{2,\mathbf{R}}(\Omega)$ is given by (2), we have that $f = a_0 + f_1$ where $f_1 \in H_2^-$. From Parseval's equality, it follows that

$$\|f - \frac{p}{q}\|^2 = |c - a_0|^2 + \|f_1 - \frac{r}{q}\|^2.$$

It is thus clear that the best we can do to minimize the above expression, is to choose $c = a_0$, and to find the best approximation to f_1 among rational functions of degree at most n in H_2^-.

The original question now turns into the following one: given an integer n and some $f \in H_2^-$, find a minimum of the squared norm

$$\|f - \frac{p}{q}\|^2, \tag{4}$$

where p/q ranges over rational functions in H_2^-, subject to the constraint $d^\circ q \leq n$.

If we introduce the real subspace $L^2_{\mathbf{R}}(\mathbf{T}) \subset L^2(\mathbf{T})$ consisting of functions having real Fourier coefficients, we have an orthogonal decomposition

$$L^2_{\mathbf{R}}(\mathbf{T}) = H_2^+ \oplus H_2^-.$$

The scalar product of $L^2_{\mathbf{R}}(\mathbf{T})$ is by definition

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\, e^{it}) \overline{g(\, e^{it})} \, dt = \frac{1}{2i\pi} \int_T f(z) \overline{g(z)} \, \frac{dz}{z}.$$

As $\bar{z} = z^{-1}$ on T and the coefficients in the power series expansion of g are real, we also have

$$< f, g >= \frac{1}{2i\pi} \int_T f(z) g(\frac{1}{z}) \frac{dz}{z}.$$

This scalar product satisfies the two following obvious properties which will be of use in the sequel:

1) for all $k \in \mathbf{Z}$, the multiplication by z^k is an isometry of $L^2_{\mathbf{R}}(\mathbf{T})$, i.e. for all $f,\ g \in L^2_{\mathbf{R}}(\mathbf{T})$,

$$< z^k f, z^k g >=< f, g >; \tag{5}$$

2) for all $f,\ g,\ h$ in $L^2_{\mathbf{R}}(\mathbf{T})$ such that fg and $f(1/z)h$ are in $L^2_{\mathbf{R}}(\mathbf{T})$,

$$< fg, h >=< g, f(1/z)h > . \tag{6}$$

The norm associated with this scalar product will still be denoted by $\| \ \|$, since it obviously induces the Hardy norm on H_2^+ and H_2^-.

The purpose of this paper is to present some results that have been obtained by the authors on this problem by using differential tools. Since we could not include all the proofs and still get a reasonably size paper, we had to refer the reader to the litterature on a couple of occasions.

2.1 A differential formulation.

Let us introduce the function

$$\phi : (p, q) \to \|f - \frac{p}{q}\|^2,$$

where p belongs to the set of polynomials of degree at most $n-1$, denoted by $\mathbf{R}[z]_{n-1}$, and q belongs to the set of monic polynomials of degree n whose roots are inside the unit disk, denoted by $\mathbf{R}[z]_n^-$. Since the correspondance $(p, q) \to p/q$ fails to be injective due to possible cancellation of common factors, we may find several pairs which minimize ϕ but correspond to the same argument of (4). In fact, this will only happen in a trivial case.

Proposition 1 *If $f \in H_2^-$ is not a rational function of degree strictly less than n, then the argument of any local minimum of (4) is an irreducible fraction whose degree is exactly n.*

Here, local is to be understood in the sense of the topology induced by H_2^-.

Proof: Assume that p_0/q_0, is a local minimum of (4), where p_0 and q_0 are coprime and $d^\circ q_0 < n$. Then there will be an open set U of $\mathbf{R}$ containing zero such that

$$\forall a \in U, \quad \forall b \in \mathbf{R}, \ |b| < 1, \quad \left\| f - (\frac{p_0}{q_0} + \frac{a}{z-b}) \right\| \geq \left\| f - \frac{p_0}{q_0} \right\| .$$

Expanding this expression yields

$$< \frac{a}{z-b}, \frac{a}{z-b} > -2 < f - \frac{p_0}{q_0}, \frac{a}{z-b} > \geq 0.$$

As a tends to zero, the principal part of the left-hand side becomes

$$-2a < f - \frac{p_0}{q_0}, \frac{1}{z-b} > .$$

For the inequality to be satisfied, we must have

$$\forall b, \ |b| < 1, \quad < f - \frac{p_0}{q_0}, \frac{1}{z-b} >= 0,$$

or equivalently

$$\int_T z^{-1}(f - \frac{p_0}{q_0})(z^{-1})\frac{dz}{z-b} = 0.$$

From Cauchy's formula applied to the function $z^{-1}(f - p_0/q_0)(z^{-1}) \in H_2^+[4]$, we get

$$(f - \frac{p_0}{q_0})(\frac{1}{b}) = 0,$$

where b is any real number such that $|b| < 1$. By analytic continuation,

$$f = \frac{p_0}{q_0}$$

so that f is indeed rational of degree less than n. Q.E.D.

In the sequel, we assume implicitly that f is not rational of order $\leq n$ when we perform rational approximation at order n.

Proposition 1 proves that local minima of ϕ coincide with local minima of (4), just because restricted to pairs (p,q) such that p/q is irreducible, the function $(p,q) \to p/q$ has a good behavior (it is an embedding into H_2^-). Now, the function ϕ is differentiable and we will study its critical points, namely zeros of its gradient vector fields. As we shall see later, the minimum we seek is indeed matched, and therefore lies among the critical points of ϕ.

The domain $\mathbf{R}[z]_{n-1} \times \mathbf{R}[z]_n^-$ of ϕ is isomorphic to an open set of $\mathbf{R}^{2n}$, by identifying the polynomials

$$p(z) = p_0 + p_1 \ z + p_2 \ z^2 + ... + p_{n-1} \ z^{n-1},$$
$$q(z) = q_0 + q_1 \ z + q_2 \ z^2 + ... + q_{n-1} \ z^{n-1} + z^n,$$

with the points $(p_{n-1}, ..., p_1, p_0)$ and $(q_{n-1}, ..., q_1, q_0)$ of $\mathbf{R}^n$ respectively.
Differentiating with respect to the p_i's, we see that every critical point (p,q) satisfies

$$\forall i \in \{1...n-1\}, \ < f - \frac{p}{q}, \frac{z^i}{q} >= 0. \tag{7}$$

Hence, if we denote by V_q the n-dimensional linear subspace of H_2^- generated by the z^i/q for $i = 0, ..., n-1$, we have that p/q is the projection of f onto V_q. This means that in our search for critical points, we can systematically restrict ourselves to pairs (p,q) where p/q is the projection of f onto V_q. In this way, p becomes a function of q denoted by $L(q)$.

Next, we shall use the following

Lemma 1 *Let h be a function of H_2^- orthogonal to the linear space V_q, where q is any polynomial of $\mathbf{R}[z]_n^-$. Then every root of order m of the polynomial q is a zero of order m of the function $h(1/z)/z$ of H_2^+.*

Proof: The function h is orthogonal to V_q if and only if for any complex polynomial u of degree at most $n-1$, we have

$$< h, \frac{u}{q} >= 0,$$

or equivalently, by definition of the scalar product

$$\int_T h(1/z) \frac{u}{q}(z) \frac{dz}{z} = 0.$$

If α is a root of order m of the polynomial q, this implies

$$\forall l \in \{1, ..., m\}, \int_T h(1/z)/z \frac{dz}{(z-\alpha)^l} = 0.$$

Then, by the residue formula in H_2^+, the following derivatives should vanish:

$$\forall l \in \{0, ..., m-1\}, [h(1/z)/z]^{(l)}(\alpha) = 0.$$

Therefore, α is a zero of order m of $h(1/z)/z$. Q.E.D.

Let us introduce the function g of H_2^+ defined by

$$g(z) = f(1/z)/z, \tag{8}$$

and the reciprocal polynomial of q (of degree n):

$$\tilde{q}(z) = z^n \; q(1/z),$$

which is of degree at most n and whose roots are the inverses of those of q. Finally, though the degree of $L(q)$ might be strictly less than $(n-1)$, we shall also use the notation

$$\widetilde{L(q)}(z) = z^{n-1} L(q)(1/z).$$

Now, $L(q)$ is a solution to the set of equations (7) in the variable p if and only if $f - L(q)/q$ is orthogonal to V_q. Then, by Lemma 1, every root of order m of the polynomial q is a zero of order m of $g - \widetilde{L(q)}/\tilde{q}$.
Thus, the polynomial $\widetilde{L(q)}$ interpolates $g\tilde{q}$ at the roots of q, and it is uniquely determined by this property.

Proposition 2 *The polynomial $\widetilde{L(q)}$ is the remainder of the division in H_2^+ of $g\tilde{q}$ by q,*

$$g\tilde{q} = v(q) \; q + \widetilde{L(q)}. \tag{9}$$

Moreover, the function

$$v(q) = \frac{g\tilde{q} - \widetilde{L(q)}}{q}$$

is the analytic part of the meromorphic function $g\tilde{q}/q$, and we have the integral representation formula (cf. [5])

$$\forall z \in \mathbf{U},\ v(q)(z) = \frac{1}{2i\pi}\int_{\mathbf{T}} \frac{g\tilde{q}}{q}(\xi)\frac{d\xi}{(\xi - z)},$$

from which we deduce an integral representation formula for $\widetilde{L(q)}$.

$$\forall z \in \mathbf{U},\ \widetilde{L(q)}(z) = \frac{1}{2i\pi}\int_{\mathbf{T}} \frac{g\tilde{q}}{q}(\xi)\frac{q(\xi) - q(z)}{(\xi - z)}d\xi. \tag{10}$$

If we introduce the map ψ_n defined on $\mathbf{R}[z]_n^-$ by

$$\psi_n(q) = \|f - \frac{L(q)}{q}\|^2,$$

then the expression (10) for $\widetilde{L(q)}$ ensures that:

Proposition 3 *The map ψ_n is a smooth function. Its critical points are the same as those of ϕ.*

It must be noticed that ψ_n can easily be expressed by means of the quotient $v(q)$ appearing in (10).

Proposition 4 *Let q be a point of $\mathbf{R}[z]_n^-$ and $v(q)$ the corresponding quotient, then*

$$\psi_n(q) = \|v(q)\|^2.$$

Proof: Using (5) and (6), the value of the criterion at q is:

$$\psi_n(q) =< f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >=< g - \frac{\widetilde{L(q)}}{\tilde{q}}, g - \frac{\widetilde{L(q)}}{\tilde{q}} > .$$

Therefore,

$$\psi_n(q) = \left\| g - \frac{\widetilde{L(q)}}{\tilde{q}} \right\|^2 = \left\| \frac{qv(q)}{\tilde{q}} \right\|^2 = \|v(q)\|^2,$$

the last equality coming out directly from the definition of the norm in $L^2_{\mathbf{R}}(\mathbf{T})$. Q.E.D.

2.2 Critical points of ψ_n.

We now turn to some divisibility properties at critical points.

Proposition 5 *For $q \in \mathbf{R}[z]_n^-$, the following assertions are equivalent:*
(i) q is a critical point of ψ_n.
(ii) q divides $v(q)L(q)$.
Thus, if q is an irreducible critical point (i.e. $\gcd(L(q), q) = 1$*), q divides $v(q)$.*

Proof: $q = z^n + q_{n-1}z^{n-1} + ... + q_0$ is a critical point iff:

$$\forall i \in \{0, ..., n-1\}, \frac{\partial \psi_n}{\partial q_i}(q) = 0.$$

These partial derivatives are computed as

$$\begin{aligned}\frac{\partial \psi_n}{\partial q_i}(q) &= -2 < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i}(\frac{L(q)}{q}) > \\ &= -2 < f - \frac{L(q)}{q}, \frac{\frac{\partial}{\partial q_i}(L(q))}{q} > +2 < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} > .\end{aligned}$$

Since $L(q)/q$ is the orthogonal projection of f on the space V_q, the first term of the right-hand side is zero, and

$$\frac{\partial \psi_n}{\partial q_i}(q) = 2 < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} > .$$

By Lemma 1, this set of derivatives vanishes iff every root of order m of the polynomial q is a zero of order m of $v(q)L(q)$. This proves the equivalence of the two assertions and proposition 5. Q.E.D.

Concerning reducible critical points, we have the following result:

Proposition 6 *Let $p \in \mathbf{R}[z]_n^-$ be a critical point such that the fraction $L(p)/p$ is irreducible. Let $r \in \mathbf{R}[z]_m^-$ and $q = pr$. Then*
(i) $L(q) = rL(p)$ iff r divides $v(p)$.
(ii) if $L(q) = rL(p)$, we have the following equivalence: q is a critical point of degree $m+n$ iff p is a critical point of degree n and r divides the quotient $v(p)/p$.

Proof: Apply the division (9) to $g\tilde{q}$ and $g\tilde{p}$:

$$g\tilde{q} = v(q)q + \widetilde{L(q)},$$

$$g\tilde{p} = v(p)p + \widetilde{L(p)}.$$

Multiply the second equation by $\tilde{r}$:

$$g\tilde{q} = v(p)\tilde{r}p + \tilde{r}\widetilde{L(p)}. \tag{11}$$

Now divide $v(p)\tilde{r}$ by r:

$$v(p)\tilde{r} = v'r + r'.$$

Plugging this expression in (11), we get

$$g\tilde{q} = v'q + (r'p + \tilde{r}\widetilde{L(p)})$$

where the second term on the right-hand side is of degree strictly less than $m+n$. Thus we have

$$\begin{cases} v(p)\tilde{r} &= \quad v(q)r + r' \\ \\ \widetilde{L(q)} &= \quad pr' + \tilde{r}\widetilde{L(p)}. \end{cases} \tag{12}$$

In order to prove (i), assume first that $L(q) = rL(p)$ holds. Substituting in the second equation of (12), we have

$$pr' = 0$$

and hence

$$r' = 0.$$

The previous pair of equations becomes then

$$\begin{cases} L(q) &= \quad rL(p), \\ \\ v(p)\tilde{r} &= \quad v(q)r. \end{cases} \tag{13}$$

Since r and $\tilde{r}$ are coprime, the second equation of (13) shows that r divides $v(p)$.
Suppose, conversely, that $r|v(p)$. The first equation of (12) implies that $r|r'$. Since the degree of r' is stricly less than that of r, it must be zero reducing equation (12) to equation (13).

Assume now that $L(q) = rL(p)$ and let us prove (ii).
If q is a critical point, proposition 5 implies that $q|v(q)L(q)$, that is by (13)

$$pr|v(p)\tilde{r}L(p)$$

and since $\tilde{r}$ and pr are coprime,

$$pr|v(p)L(p).$$

In particular, $p|v(p)L(p)$ i.e. p is critical. Note also that $r|(\frac{v(p)}{p})L(p)$. Moreover, by (i), $r|(\frac{v(p)}{p})p$. As $L(p)$ and p are relatively prime, we deduce that $r|\frac{v(p)}{p}$.
Conversely, the last relation implies that $q|v(p)$. By the second equation of (13), we get $q|v(q)\frac{r}{\tilde{r}}$. Since the roots of $\tilde{r}$ lie in the complement of the unit disk, q divides $v(q)r$ hence also $v(q)rL(p)$. Thus q divides $v(q)L(q)$ and q is critical by proposition 5. Q.E.D.

2.3 Qualitative properties of the minima.

We are primarily concerned with the question of the existence of a minimum. It has been proven for a more general case that a minimum is always matched (cf.[5] or [1] for

a shorter proof in a more general case).
Another important question is uniqueness. It has been proven that uniqueness of the best approximant is a strongly generic property, that is, true on an open dense subset of H_2^- (cf. [1]).

However, situations may arise where there are several absolute minima. For instance, consider a non-rational even function, for example $f(z) = e^{1/z^2} - 1$. Since the norm is invariant under the transformation $z \rightarrow -z$, if $r_0(z)$ is a best approximant, so is $r_0(-z)$. By Proposition 1 they are both of order n and it is easily shown that they cannot coincide if n is odd. In fact the problem is not well-posed in the neighborhood of such an f, since one can perturb f slightly so as to obtain a best approximant which is close to $r_0(z)$ or to $r_0(-z)$ alternatively.
Thus it would be interesting to find conditions on the function to be approximated that ensure the uniqueness of an absolute minimum.

As far as local minima are concerned, it is not even known whether their number is generically finite. It is clear that there can be no generic bound on this number.
Consider the function of H_2^+ defined by

$$\forall z \in \mathbf{U},\ g(z) = sin\frac{1}{1+\epsilon-z},$$

associated with some $f \in H_2^-$ by (8) .
This function vanishes at an arbitrary large number of points as $\epsilon \rightarrow 0$. But, if ξ is a zero of g, then $L(z-\xi) = 0$, and $\psi_1(z-\xi) = \|f\|^2$. Since $L(q)/q$ is the projection of f onto V_q, we have

$$\forall q \in \mathbf{R}[z]_n^-,\ \|f - \frac{L(q)}{q}\| \leq \|f\|,$$

and thus $z - \xi$ is a maximum of ψ_1.
Now, by Rolle's theorem, the function ψ_1 has an arbitrarily large number of local minima, and this property remains valid when the map is slightly deformed.

An even more pathological example is the following. Let g be a Blaschke product with an infinite sequence of real zeros $\{\alpha_n\}$ converging to 1. For instance let $\alpha_n = 1 - 1/n^2$, and put

$$\forall z \in \mathbf{U},\ g(z) = \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \alpha_n z},$$

where the product converges since $\sum_{n=1}^{\infty}(1 - |\alpha_n|) < \infty$ [4]. Since g has infinitely many zeros inside the unit disk, the previous argument shows that the corresponding function ψ_1 has infinitely many minima.

To avoid such problems we shall assume that the function g is analytic in a wider domain. Such a condition prevents g from having a sequence of zeros accumulating on the unit circle. More precisely,unless otherwise indicated, we shall assume that the function g is holomorphic in an open disk U_r, of radius r, with $r > 1$. Hereafter, we shall use the notation U_μ for the open disk of radius μ and T_μ for the circle of radius μ.

3 Extension of ψ_n to a compact set.

Let Δ_n be the closure in $\mathbf{R}^n$ of the set $\mathbf{R}[z]_n^-$; this compact set consists of monic polynomials of degree n whose roots are of modulus less or equal to 1.
Our assumption previously made on g allows us to proceed to such an extension. In fact, it allows us to extend ψ_n to a smooth function in a neighborhood of Δ_n and this will be useful in the sequel.
Denote by P_r the open neighborhood of Δ_n consisting of monic polynomials whose roots are of modulus strictly less than r.

Proposition 7 *If g is holomorphic in U_r then ψ_n extends to a smooth function*

$$\psi_n : P_r \to \mathbf{R}.$$

Proof: We first extend the function L by putting

$$\forall q \in P_r,\ \widetilde{L(q)}(z) = \frac{1}{2i\pi}\int_{T_\mu} \frac{g\tilde{q}}{q}(\xi)\frac{q(\xi)-q(z)}{\xi - z}d\xi \tag{14}$$

where $\mu < r$ is an upper bound for the moduli of the roots of q.
By Cauchy's formula this expression agrees with (10) when $q \in \mathbf{R}[z]_n^-$. This extension is clearly smooth. Note that $\widetilde{L(q)}$ is still the remainder of the division (9).

Now, it follows from (7) that for all $q \in \mathbf{R}[z]_n^-$,

$$< f - L(q)/q, L(q)/q >= 0,$$

so that ψ_n writes

$$\psi_n(q) = \|f\|^2 - < f, L(q)/q > .$$

Therefore, it is sufficient to extend smoothly the functions

$$q \to < f, \frac{\xi^k}{q} >= \frac{1}{2i\pi}\int_{\mathbf{T}} \frac{g(\xi)\xi^k}{q(\xi)}d\xi.$$

As before, we can define these functions on P_r by putting

$$q \to \frac{1}{2i\pi}\int_{T_\mu} \frac{g(\xi)\xi^k}{q(\xi)}d\xi. Q.E.D.$$

3.1 The structure of Δ_n.

Proposition 8 *The set Δ_n is homeomorphic to the closed unit ball B^n of $\mathbf{R}^n$.*

For the proof of this proposition we refer to [2].

By this result, the boundary $\partial\Delta_n$ of Δ_n is homeomorphic to the (n-1)-dimensional sphere S_{n-1}. However it is not smooth as one can see by looking at Δ_2, which is a triangle. The smooth part of $\partial\Delta_n$ consits of those polynomials having exactly one irreducible factor over $\mathbf{R}$ whose roots are of modulus 1. Indeed, define the maps

$$\begin{array}{cccc} \phi_1: & \dot{\Delta}_{n-1} & \to & \Delta_n \\ & p & \to & (z-1)\ p \end{array}$$

$$\begin{array}{cccc} \phi_{-1}: & \dot{\Delta}_{n-1} & \to & \Delta_n \\ & p & \to & (z+1)\ p \end{array}$$

$$\begin{array}{cccc} \phi_c: &]-1,1[\times\dot{\Delta}_{n-2} & \to & \Delta_n \\ & (\alpha,p) & \to & (z^2+2\alpha z+1)\ p \end{array},$$

where $\dot{\Delta}_n$ is the interior of Δ_n. One can check that they are embeddings. The smooth part of $\partial\Delta_n$ is thus the union of the images of ϕ_1, ϕ_{-1} and ϕ_c.

3.2 Properties of ψ_n on $\partial\Delta_n$.

Proposition 9 *Let $q \in \partial\Delta_n$ be such that $q = \chi\ p$, where all the roots of χ are of modulus 1, and $p \in \dot{\Delta}_k$ where $d^\circ p = k$. Then $L(q) = \chi L(p)$ and $\psi_n(q) = \psi_k(p)$.*

Proof: Write the division of $g\tilde{p}$ by p,

$$g\tilde{p} = v(p)p + \widetilde{L(p)}.$$

Multiplying by $\tilde{\chi}$ yields

$$g\tilde{q} = v(p)p\tilde{\chi} + \tilde{\chi}\widetilde{L(p)}.$$

Since the roots of χ are of modulus 1, $\tilde{\chi} = \pm\chi$, hence

$$\widetilde{L(q)} = \tilde{\chi}\widetilde{L(p)}.$$

The second assertion is then obvious. Q.E.D.

The next result concerns the behavior of the gradient $\nabla\psi_n$ of the function ψ_n on $\partial\Delta_n$.

Proposition 10 *Let $p \in \dot{\Delta}_{n-1}$, and $x = \phi_1(p)$, or $\phi_{-1}(p)$). The projection of $\nabla\psi_n(x)$ on $\partial\Delta_n$ coincides with $\nabla\psi_{n-1}(p)$. If $x = \phi_c(\alpha,p)$, $\alpha \in]-1,1[$, then the projection of $\nabla\psi_n(x)$ on the tangent space $T_x\partial\Delta_n$ lies in the subspace $\phi_c(\alpha,\dot{\Delta}_{n-2})$, where it coincides with $\nabla\psi_{n-2}(p)$.*

Proof: Observe first that $\Phi_1(\dot{\Delta}_{n-1})$ is an open subset of a $(n-1)$-dimensional linear subspace of $\mathbf{R}^n$ (consisting of polynomials whitch have the root 1). Thus it makes sense to speak of the projection on this part of the boundary.

Now, we show the result for $x = \phi_1(p)$.
From proposition 9, the following diagram commutes

$$\begin{tikzcd} \dot{\Delta}_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d, "\phi_1"'] & \mathbf{R} \\ P_r \arrow[ur, "\psi_n"'] & \end{tikzcd}$$

Differentiating using the chain rule, we obtain the equality between Jacobian matrices

$$J(\psi_n)_x \ J(\phi_1)_p = J(\psi_{n-1})_p,$$

where the notation $J(\psi_n)_x$ refers to the Jacobian matrix of ψ_n at the point x.
But $J(\psi_n)_x = {}^t\nabla\psi_n(x)$ and the columns of $J(\phi_1)_p$ generate the tangent space $T_x\partial\Delta_n$, and this proves our contention.
The proof is similar in the remaining cases. Q.E.D.

Proposition 11 *Let p be a minimum of ψ_{n-1}, and $x = \phi_1(p)$, or $\phi_{-1}(p)$ or else $\phi_c(\alpha, p)$ with $\alpha \in]-1, 1[$, then $\nabla\psi_n(x)$ is orthogonal to $T_x\partial\Delta_n$ and points outwards(if non zero).*

Proof: Write down the Taylor expansion of ψ_n in the neighborhood of x

$$\psi_n(x+h) = \psi_n(x) + \nabla\psi_n(x).h + \epsilon(h)\|h\|^2,$$

where "." stands for the scalar product in $\mathbf{R}^n$.
If h is sufficiently small, the sign of $\psi_n(x+h) - \psi_n(x)$ agrees with the sign of $\nabla\psi_n(x).h$, provided $\nabla\psi_n(x) \neq 0$.
Now denote by η_x the outward normal vector to the boundary $\partial\Delta_n$ [3]. We have

$$x + h \in \dot{\Delta}_n \iff h.\eta_x < 0.$$

Suppose that $\nabla\psi_n(x)$ points inwards. For all h such that $x + h \in \dot{\Delta}_n$, we have

$$h.\nabla\psi_n(x) > 0,$$

and thus

$$\psi_n(x+h) > \psi_n(x).$$

This implies $L(x)/x$ is a minimum of (4). Since $L(x)/x = L(p)/p$ with $d^\circ p = n-1$, this contradicts proposition 1. Q.E.D.

3.3 An algorithm.

We shall make two extra-assumptions in this section. For one thing, we shall assume that ∇_f^k does not vanish on $\partial\Delta_k$ if $1 \leq k \leq n$. For another thing, we shall ask all critical points of ψ_f^k in Δ_k to be nondegenerate for k as above, *i.e.* to have a second derivative which is a nondegenerate quadratic form. These two properties hold generically [1]. They ensure in particular that critical points in Δ_k are finite in number.

The algorithm proceeds as follow.

(0) Choose an initial point q_0.

(1) Integrate the vector field $-\nabla\psi_n$ from the initial conditions $(q_0, \psi_n(q_0))$.

- we reach a local minimum $\rightsquigarrow$ end.
- we reach the boundary $\partial\Delta_n \rightsquigarrow 2$.

(2) You are at the point q_b of $\partial\Delta_n$:

$$q_b = \chi q_i,\ \ \chi(\alpha) = 0 \Rightarrow |\alpha| = 1,\ \ q_i \in \dot{\Delta}_k$$

Integrate the vector field $-\nabla\psi_k$ from the initial conditions $(q_i, \psi_k(q_i))$.

- we reach a minimum at order $k < n \rightsquigarrow (3)$.
- we reach the boundary of $\Delta_k \rightsquigarrow (2)$ replacing n by k.

(3) You are at a minimum $q_m \in \dot{\Delta}_k$ of ψ_k:
Integrate the vector field $-\nabla\psi_{k+1}$ from the initial conditions $((z+1)q_m, \psi_k(q_m))$.

- we reach a minimum at order $k+1$
  - If $k+1 < n \rightsquigarrow (3)$ replacing k by $k+1$.
  - If $k+1 = n \rightsquigarrow$ end.
- we reach $\partial\Delta_{k+1} \rightsquigarrow (2)$ replacing n by $k+1$.

Let us make some remarks concerning this algorithm.
1)As saddles are unstable critical points we cannot stop on them, and thus we only meet minima.
2)This algorithm is based on the recursiveness of the function ψ_n described in propositions 9 and 11. Proposition 9 provides new initial conditions when we meet the boundary. Proposition 11 allows to penetrate inside Δ_{k+1} from a boundary point which corresponds to a minimum at order k.
3)For $q = \chi\ p$, where χ has all its roots of modulus 1, and $p \in \dot{\Delta}_k$, define a function ψ by $\psi(q) = \psi_k(p)$. At each step of the algorithm, the function ψ decreases. Thus, we cannot meet twice the same minimum. Since the number of minima is finite under the foregoing hypothesis, the algorithm necessarily comes to an end.

It must be noted that this algorithm provides us with a local minimum and not necessarily a global one. We do not even know whether there exists a finite set of initial points (for example all the boundary points corresponding to minima at order $n-1$) which allows us to exhaust the set of local minima in Δ_n. Since the number of such minima can be arbitrarily large (cf. 1.3) anyway, the combinatorial complexity might increase too much with the order.
On the contrary, a particularly favourable circumstance occurs when the function has only one local minimum. It would be of most interest to find some classes of functions to be approximated that exhibit this property. We shall report on this problem in the next section, but this research is yet still under investigation.

3.4 The index theorem and its applications.

In [2] we proved an analogous to the Poincaré-Hopf Index Theorem [3] for all vector fields $\nabla\psi_n$ on Δ_n associated with our approximation problem .
Recall the index of a vector field $\vec{v}$ at an isolated zero x_0 is the degree of the directional map

$$\begin{array}{ccc} S_\epsilon & \to & S^{n-1} \\ x & \to & \dfrac{\vec{v}(x-x_0)}{\|\vec{v}(x-x_0)\|} \end{array}$$

where S_ϵ is any small sphere around x_0.
Our assumption on g ensures that the index of $\nabla\psi_n$ at a critical point of ψ_n is generically well defined. When defined, the index is equal to $(-1)^e$, where e is the number of negative eigenvalues of the Hessian matrix of ψ_n, provided the latter is nondegenerate

Theorem 1 *If g is holomorphic in U_r, then the sum of the indices of $-\nabla\psi_n$ over all critical points is equal to 1.*

In particular, this result has the advantage of bringing down the global problem of the uniqueness of a minimum into a local one.
Indeed, if for some f, the index at any critical points of ψ_n is equal to 1, then the function ψ_n has necessarily a unique minimum from the index theorem.
For instance, this gives some results on Stieltjes functions,

$$f(z) = \int_0^1 \frac{d\mu(t)}{z-t},$$

where μ is a positive mesure.
We prove that there exists a value $0 < \lambda < 1$, such that Stieltjes functions for which the support of the mesure μ lies in $[0, \lambda]$ have a unique local minimum, hence a fortiori a unique approximant.
This result is to appear in a forthcoming paper.

4 Asymptotic properties.

Throughout this section, we assume that f is not a rational function.

4.1 Asymptotic behaviour of critical points

We denote by V_n the subset of $\mathbf{R}[z]_n^-$ containing the critical points at order n. By choosing points in the union of the sets of critical points at each order, we construct a sequence of quotients (v_n) of the form $v(q_n)$, for some sequence q_n of critical points.

In order to prove that the family of functions (v_n) is normal, we use the integral representation:

$$v(q)(z) = \frac{1}{2i\pi} \int_{T_\mu} \frac{g(\xi)\tilde{q}(\xi)}{q(\xi)} \frac{d\xi}{\xi - z}$$

where μ is any real number such that $1 < \mu < r$. Using this expression we show the

Lemma 2 *Let μ' be a real number such that $1 < \mu' < \mu$. There exists on the open set $U_{\mu'}$ an uniform bound for the set of functions (v_n) which depends only on the function g.*

Proof: On the unit circle T, the quotient $\tilde{q}/q$ is of modulus 1. Then by using the maximum principle over the complement of the unit disk U, we get

$$\forall \xi \in \mathbf{C} - U, \left| \frac{\tilde{q}(\xi)}{q(\xi)} \right| \leq 1.$$

This inequality is true on the circle T_μ so that:

$$\forall z \in U_{\mu'}, |v(q)(z)| \leq \frac{1}{2\pi} \left(\sup_{T_\mu} |g| \right) \int_{T_\mu} \frac{d\xi}{|\xi - z|}.$$

But for $z \in U_{\mu'}$, $|\xi - z|$ is greater than $\mu - \mu'$ and we get the bound in question. Q.E.D.

Every sequence of quotients (v_n) associated with critical points of ψ_n is thus normal on the open disk U_μ.

We are interested in those sequences in which the order of the points tends to infinity i.e. there is no subsequence with an infinite number of elements sharing the same order.

We shall first discard reducible points from the sequence (v_n) so that we get a new sequence denoted by (w_n).
From such a sequence, take a subsequence (w_p) which converges to a limit w_{lim} uniformly over all compact subsets of U_μ. Let μ' such that $1 < \mu' < \mu$ and suppose that the analytic function w_{lim} has no zeros on the circle $T_{\mu'}$. Then

$$\exists N, \forall n \geq N, \forall z \in T_{\mu'}, |w_n(z) - w_{lim}(z)| < |w_{lim}(z)|.$$

By Rouché's theorem, w_n and w_{lim} will have the same number of zeros in the open set $U_{\mu'}$, but using proposition 5, a quotient corresponding to an irreducible critical point of order n has at least n zeros in U. As the order of points in the subsequence (w_p) tends to infinity, w_{lim} must be equal to zero. This is a contradiction with the assumption made on the circle $T_{\mu'}$. By letting μ' vary continuously, we get a compact circular annulus containing infinitely many zeros for w_{lim} and thus this limit must vanish on the open disk U_μ. We just showed that every convergent subsequence of (w_n) converges to zero uniformly on every compact set of U_μ. But then, it is true for the sequence (w_n) itself. If not, there would exist an $\epsilon > 0$ and a subsequence (w_p) such that:

$$\forall p, \exists z, |w_p(z)| > \epsilon.$$

But from the normal sequence (w_p), we can extract a convergent subsequence which will converge to zero by the previous argument. This is a contradiction and we are done. By using proposition 4, we get the l^2-convergence to f of any sequence $L(q_n)/q_n$ where q_n is a sequence of irreducible critical points, as there order tends to infinity.

If critical points are irreducible, there exists an order over which the corresponding quotients $v(q)$ have more than any preassigned number of zeros. To get a more general result, we prove that such an order exists even in the case of reducible points. Following proposition 6, these points are generated by adjoining to irreducible critical points q of lower order, zeros from $v(q)/q$. We show that for a fixed order, the number of such zeros is bounded from above. Let W_n be the subset of V_n containing irreducible critical points of order n and let $q \in W_n$. We denote by $Z(v(q)/q)$, the number of zeros of the quotient $v(q)/q$ in the disk U. Then $Z(v(q)/q)$ is finite. Indeed, with the assumption made on f, the quotient $v(q)$ is analytic on the open set U_r which contains the compact disk $\bar{U}$. If $Z(v(q)/q)$ is not finite, $v(q)$ vanishes in U which means that the function f to approximate is already a rational fraction, but we discarded this case. Let us set one more notation:

$$R_n = \max\{Z(v(q)/q), q \in W_n\},$$

then R_n is finite. This is obvious when W_n itself is finite. Otherwise, let us suppose that R_n is not finite, then we can select a sequence of critical points (q_l) in W_n whose corresponding quotients (v_l) have a number of zeros growing to infinity. From this sequence, we can extract as before a subsequence which tends to zero. But this means that there is a sequence of critical points of order n which converges to the function f. We have then

$$g\widetilde{q_l} = v_l q_l + \widetilde{L(q_l)}. \tag{15}$$

The functions v_l converge uniformly to zero on $\bar{U}$ as before, and the polynomials q_l and $\widetilde{q_l}$ are also bounded on $\bar{U}$ as their degree and their coefficients are. Then, by (15), $\widetilde{L(q_l)}$ is bounded. We can successively extract two subsequences such that $\widetilde{L(q_l)}$ and $\widetilde{q_l}$ will converge respectively to some polynomials p and q, uniformly on $\bar{U}$. Taking the limit equation (15) becomes

$$gq = p,$$

and thus f is equal to $\tilde{p}/\tilde{q}$, contradicting again our assumptions.
As a conclusion, at order $n+R_n$, quotients $v(q)$ corresponding to irreducible critical points as well as reducible ones which come from irreducible points of order n have all at least n zeros. At order

$$\max_{p \leq n}\{p + R_p\} + 1,$$

no critical point comes from an irreducible one of order less than or equal to n. Thus all the corresponding quotients have more than n zeros. This is the result we needed and finally, we proved the

Theorem 2 *Let f, be a function in the Hardy space H_2^-, distinct from a rational fraction, analytic on a open domain containing the complement of the unit disk U. Let (v_n) be a sequence of quotients corresponding to critical points q_n whose orders tend to infinity. Then the sequence (v_n) converges uniformly to zero on every compact subset of an open set containing the closed unit disk $\bar{U}$. Consequently, the sequence of critical points $(L(q_n)/q_n)$ tends to the function f, accordingly to the l^2-norm.*

4.2 Finiteness of the number of orders where local maxima appear

We shall first investigate the case where the critical point q in $\mathbf{R}[z]_n^-$ is irreducible. The partial derivatives of the criterion $< f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >$ at q vanish i.e.:

$$\forall i \in \{0, ..., n-1\}, < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right) >= 0,$$

or

$$\forall i \in \{0, ..., n-1\}, < f - \frac{L(q)}{q}, \frac{\frac{\partial}{\partial q_i}(L(q))}{q} > - < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} >= 0.$$

As $\frac{L(q)}{q}$ is the orthogonal projection of f on the space V_q, we know that

$$\forall k \in \{0, ..., n-1\}, < f - \frac{L(q)}{q}, \frac{z^k}{q} >= 0, \tag{16}$$

and the last equality reduces to

$$\forall i \in \{0, ..., n-1\}, < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} >= 0. \tag{17}$$

Combining (16) and (17), we get

$$< f - \frac{L(q)}{q}, \frac{r_1 L(q) + r_2 q}{q^2} >= 0$$

where r_1 and r_2 are any polynomials in $\mathbf{R}[z]_{n-1}$. But q and $L(q)$ are relatively prime and the last equality is equivalent to

$$< f - \frac{L(q)}{q}, \frac{P}{q^2} >= 0, \tag{18}$$

P being any polynomial of $\mathbf{R}[z]_{2n-1}$.

Let us come back to (16). Taking partial derivatives yiels

$$\forall i \in \{0, ..., n-1\}, - < \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right), \frac{z^k}{q} > - < f - \frac{L(q)}{q}, \frac{z^{k+i}}{q^2} >= 0.$$

Using (18), the second term in the left-hand side is zero and we get the orthogonality relations:

$$\forall k \in \{0, ..., n-1\}, < \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right), \frac{z^k}{q} >= 0. \tag{19}$$

Now, it follows from Lemma 1 that $\tilde{q}$ divides $\frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right)$. Consequently, there exist polynomials ν_i of $\mathbf{R}[z]_{n-1}^-$ such that:

$$\forall i \in \{0, ..., n-1\}, \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right) = \frac{\tilde{q}}{q^2}\nu_i.$$

Moreover, the polynomials ν_i, for $i = 0,..,n-1$, are linearly independent. Indeed, let $(\lambda_i)_{i=0,..,n-1}$ be a family of real numbers such that

$$\sum_{i=0}^{n-1} \lambda_i \nu_i = 0.$$

Then

$$\sum_{i=0}^{n-1} \lambda_i \tilde{q} \nu_i = 0,$$

or

$$\sum_{i=0}^{n-1} \left(\frac{\partial}{\partial q_i} \left(\frac{L(q)}{q} \right) \right) \lambda_i = 0.$$

This yields

$$q \sum_{i=0}^{n-1} \lambda_i \frac{\partial L(q)}{\partial q_i} - L(q) \sum_{i=0}^{n-1} \lambda_i z^i = 0$$

and the polynomial q must divide the sum $\sum_{i=0}^{n-1} \lambda_i z^i$ which is of degree $n-1$. As q is of degree n, we get that

$$\sum_{i=0}^{n-1} \lambda_i z^i = 0$$

and each real number λ_i is zero which proves the independance of the polynomials ν_i.

Now, we can evaluate the variation of the criterion at the critical point q, using the Hessian matrix H whose entries are by definition:

$$\frac{\partial^2}{\partial q_i \partial q_j} < f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >$$

that we compute as

$$-2 \frac{\partial}{\partial q_i} < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_j} \frac{L(q)}{q} >$$

$$= -2 \left[- < \frac{\partial}{\partial q_i} \left(\frac{L(q)}{q} \right), \frac{\partial}{\partial q_j} \left(\frac{L(q)}{q} \right) > + < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_j} \left(\frac{L(q)}{q} \right) > \right]$$

$$= 2 \left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > - < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i} \left(\frac{\tilde{q}}{q^2} \nu_j \right) > \right]$$

$$= 2 \left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > - < f - \frac{L(q)}{q}, \frac{\tilde{q}}{q^2} \frac{\partial \nu_j}{\partial q_i} > - < f - \frac{L(q)}{q}, \nu_j \frac{\partial}{\partial q_i} \left(\frac{\tilde{q}}{q^2} \right) > \right]$$

and using (18), we get

$$2 \left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > - < f - \frac{L(q)}{q}, \nu_j \frac{\partial}{\partial q_i} \left(\frac{\tilde{q}}{q^2} \right) > \right].$$

But

$$\frac{\partial}{\partial q_i} \left(\frac{\tilde{q}}{q^2} \right) = \frac{1}{q^4} [q^2 z^{n-i} - 2 \tilde{q} q z^i] = \frac{z^{n-i}}{q^2} - 2 \tilde{q} \frac{z^i}{q^3},$$

and using (18) again, we obtain

$$\frac{\partial^2}{\partial q_i \partial q_j} < f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >= 2\left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > + < f - \frac{L(q)}{q}, 2\frac{\tilde{q} z^i \nu_j}{q^3} > \right].$$

Now, the variation of the criterion in a neighbourhood of the critical point q following a direction given by the real vector $(\lambda_0, ..., \lambda_{n-1})$ in the space $\mathbf{R}[z]_n^-$ is

$$\Delta_q(\lambda_0, ..., \lambda_{n-1}) = (\lambda_0, ..., \lambda_{n-1}) H \begin{pmatrix} \lambda_0 \\ . \\ . \\ . \\ \lambda_{n-1} \end{pmatrix}.$$

As the family of polynomials $(\nu_i)_{i=0,..,n-1}$ is independent, we choose the numbers λ_i such that

$$\sum_{i=0}^{n-1} \lambda_i \nu_i = L(q).$$

The value of $(\frac{1}{2})\Delta_q(\lambda_0, ..., \lambda_{n-1})$ becomes

$$\sum_{i,j=0}^{n-1} \lambda_i \lambda_j \left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > + < f - \frac{L(q)}{q}, 2\frac{\tilde{q} z^i \nu_j}{q^3} > \right],$$

or

$$< \frac{L(q)}{q}, \frac{L(q)}{q} > + < f - \frac{L(q)}{q}, 2\frac{\tilde{q}}{q^3} L(q) \left[\sum_{i=0}^{n-1} \lambda_i z^i \right] > .$$

On the other hand

$$\left(\frac{\partial L(q)}{\partial q_i} \right) q = z^i L(q) + \tilde{q} \nu_i.$$

Using this equality together with (18) gives the following expression for $(\frac{1}{2})\Delta_q$:

$$\begin{aligned} &\left\| \frac{L(q)}{q} \right\|^2 + < f - \frac{L(q)}{q}, -2\frac{\tilde{q}^2}{q^3} \sum_{i=0}^{n-1} \lambda_i \nu_i > \\ &= \left\| \frac{L(q)}{q} \right\|^2 - 2 < f - \frac{L(q)}{q}, \frac{\tilde{q}^2}{q^3} L(q) > \\ &\geq \left\| \frac{L(q)}{q} \right\|^2 - 2 \left\| f - \frac{L(q)}{q} \right\| \left\| \frac{L(q)}{q} \right\|. \end{aligned}$$

As the order of q increases, $\left\| f - \frac{L(q)}{q} \right\|$ tends to zero, and the variation following the chosen direction becomes positive which means that the critical point q may not be a maximum.

Suppose now that q is reducible. We shall get our result by using the more general fact that in any case, i.e. at every order, a reducible point may not be either a maximum or a minimum. For a minimum, this is proposition 1; the proof is easy to modify in the case of a maximum, and this we leave to the reader.

Theorem 3 *Let f, be a function as in theorem 1. Then, critical points which are local maxima can only appear for a finite range of orders.*

Note, in particular, that theorem 3 implies that the global minimum can be attained from some (unfortunately unknown!) initial condition on $\partial\Delta_n$ by integrating the vector field $-\nabla\psi_n$ provided n is large enought.

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