Inner-unstable factorization of stable rational transfer functions.

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1 Introduction

We develop in this paper a factorization involving inner functions for linear constant stable dynamical systems, which is different from the classical inner-outer factorization. In fact, we merely develop further in the rational case a construction which is given in greater generality (strictly noncyclic systems) in [1]. This provides one with an alternative to classical polynomial factorizations, which may be of interest for parametrisation purposes.

2 Linear systems and Fuhrmann's realization.

In this section, we recall some basic results concerning discrete linear dynamical systems, namely Fuhrmann's realization theory. This should help stressing the link between the inner-unstable factorization that we shall develop in the sequel, which pertains to the analytical side of the theory, and the classical polynomial approach. Proofs will be sketched only when necessary for a better understanding. A complete treatment of these questions is given in [1][2].

To describe discrete-time dynamical systems (in short: systems), it is customary to represent a sequence of inputs $(u_k)_{k \geq k_0}$, $u_k \in \mathbf{R}^m$,

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$k_0 \in \mathbf{Z}$, where the intput u_k has been applied at time $t = k$, by the formal series

$$u(\lambda) = \sum_{k \geq k_0} u_k \lambda^{-k};$$

and the associated sequence of outputs $(y_k)_{k \geq k_0}$, $y_k \in \mathbf{R}^p$, $k_0 \in \mathbf{Z}$, where the output y_k occurs at time $t = k$, by the formal series

$$y(\lambda) = \sum_{k \geq k_0} y_k \lambda^{-k}.$$

The time axis, which is the set of integers $\mathbf{Z}$, is mapped in a one-to-one way onto the set of powers of the indeterminate λ by the rule $k \to \lambda^{-k}$.

In this way, a linear system may be described by a $\mathbf{R}$-linear map

$$\begin{array}{cccc} \sigma : & \mathbf{R}^m((1/\lambda)) & \to & \mathbf{R}^p((1/\lambda)) \\ & u & \to & y \end{array},$$

where $\mathbf{R}^n((1/\lambda))$ is the set of all formal series of the form $\sum_{k \geq k_0} a_k \lambda^{-k}$, $a_k \in \mathbf{R}^n$, and is a module over the ring $\mathbf{R}((1/\lambda))$ obviously isomorphic to $[\mathbf{R}((1/\lambda))]^n$.

It is easy to give the mathematical meaning of the following classical properties of the system. The system is stationnary if and only if σ is $\mathbf{R}(\lambda)$-linear, where $\mathbf{R}(\lambda)$ denotes the set of rational functions. If moreover it is causal, then σ is $\mathbf{R}((1/\lambda))$-linear. In this case, the system may be represented in the canonical basis by a product of matrices

$$\sigma : U \to T\ U$$

where U is the column vector associated with the input in $\mathbf{R}((1/\lambda))^m$, and T is a matrix whose coefficients, by causality again, lie in $1/\lambda\ \mathbf{R}[[1/\lambda]]$. The matrix T is the transfer function of the system.

2.1 Nerode's representation.

We also have another description of the system, the internal one, which we are going to deduce from the previous one.

The polynomial module $\mathbf{R}^n[\lambda]$ of polynomials with coefficients in $\mathbf{R}^n$ is a $\mathbf{R}[\lambda]$ submodule of $\mathbf{R}^n((1/\lambda))$, and the quotient module $\mathbf{R}^n((1/\lambda))/\mathbf{R}^n[\lambda]$ can be identified with $1/\lambda\ \mathbf{R}^n[[1/\lambda]]$, where

$\mathbf{R}^n[[1/\lambda]]$ denotes the module of all formal power series in $1/\lambda$. In other words, we have an $\mathbf{R}[\lambda]$-module isomorphism

$$\mathbf{R}^n((1/\lambda)) \approx \mathbf{R}^n[\lambda] \oplus 1/\lambda \ \mathbf{R}^n[[1/\lambda]]. \tag{1}$$

This decomposition corresponds to the splitting of the time axis between past and future.

Let us consider the restricted input-output map

$$f : \mathbf{R}^m[\lambda] \to 1/\lambda \ \mathbf{R}^p[[1/\lambda]],$$

defined by

$$f = \pi^- \circ \sigma_{|\mathbf{R}^m[\lambda]},$$

where π^- is the projection onto $\mathbf{R}^n((1/\lambda))/\mathbf{R}^n[\lambda]$ and $\sigma_{|\mathbf{R}^m[\lambda]}$ is the restriction of σ to $\mathbf{R}^m[\lambda]$.
From the $\mathbf{R}((1/\lambda))$-linearity of σ, we deduce that f is a $\mathbf{R}[\lambda]$-homomorphism.
We can factor f as follows

$$\begin{tikzcd}
\mathbf{R}^m[\lambda] \arrow[rr, "f"] \arrow[dr, "\pi"'] & & 1/\lambda \ \mathbf{R}^p[[1/\lambda]] \\
& \mathbf{R}^m[\lambda]/ker f \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

The map $\bar{f}$ is the observability map and is injective, while the map π is the reachability map and is surjective. The space $\mathbf{R}^m[\lambda]/ker f$ define the state space up to an isomorphism. Indeed, when you know the "state" of the system, you have all the informations you need to deduce the forthcomming output if the input stops.

Now try to compute the output step by step. Let $x_0 = 0$ be the state of the system at time $t = 0$, and let $u(\lambda) = \sum_{k \geq 0} u_k \lambda^{-k}$ be some input. The state of the system at time $t = n + 1$ will be

$$\begin{array}{rcl}
x_{n+1} & = & \pi(u_0 \ \lambda^n + ... + u_n) \\
& = & \pi(\lambda \ (u_0 \ \lambda^{n-1} + ... + u_{n-1}) + u_n) \\
& = & \pi(\lambda \ (u_0 \ \lambda^{n-1} + ... + u_{n-1})) + \pi(u_n)
\end{array} .$$

Since f is a $\mathbf{R}[\lambda]$-homomorphism $\pi(\lambda\ (u_0\ \lambda^{n-1}+...+u_{n-1}))$ depends only on the state $x_n=\pi(u_0\ \lambda^{n-1}+...+u_{n-1})$.
Let

$$F: \begin{array}{ccc} X & \rightarrow & X \\ \pi(u) & \rightarrow & \pi(\lambda u) \end{array},$$

and $G=\pi_{|\mathbf{R}^m}$, then we have

$$x_{n+1}=F(x_n)+G(u_n).$$

Now $\bar{f}(x_{n+1})=f(u_0\ \lambda^n+...+u_n)=y_{n+1}\ 1/\lambda+y_{n+2}\ 1/\lambda^2+...$, and thus, y_{n+1} is the Taylor coefficient of $1/\lambda$ in the expansion of $\bar{f}(x_{n+1})$. We denote this $\mathbf{R}$-homomorphism by H and we finally find back the classical dynamical equations

$$\left\{\begin{array}{lcl} x_{n+1} & = & F(x_n)+G(u_n) \\ y_{n+1} & = & H(x_{n+1}) \end{array}\right. \tag{2}$$

Untill now, we did not assume the state space to be finite dimensional. A classical result claims that the state space X is a finite dimensional vector space over $\mathbf{R}$ if and only if the transfer function of the system, T, has rational coefficients. In this case the dimension of X is called the Mac-Millan degree of the transfer function. Moreover, the functions $F,\ G,\ H$, become linear maps on finite dimensional vector spaces and may be represented in some basis by matrices. The triple (F,G,H) is a minimal realization of the system, that is it corresponds to a minimal size of F.

This is for the theoretic viewpoint and explains the interest of a factorization of f for the realization problem (i.e. find a triple (F,G,H) satisfying (2)). In fact, it is not difficult to see that the two problems are completely equivalent. In what follows, we shall be interested with finding an effective (i.e. computable) factorization of f.

2.2 Fuhrmann's factorization.

In the sequel, we shall consider a rational transfer function of Mac-Millan degree n.
The rationality of T may be expressed in the following way:

$$\exists p\in(\mathbf{R}[\lambda]),\text{ such that } pT\in(\mathbf{R}[\lambda])^{p\times m},$$

where $(\mathbf{R}[\lambda])^{p\times m}$ is the ring of matrices with coefficients in $\mathbf{R}[\lambda]$. Consider the set

$$\mathcal{J} = \{M \in (\mathbf{R}[\lambda])^{p\times p} / MT \in (\mathbf{R}[\lambda])^{p\times m}\}.$$

It is a left ideal in $(\mathbf{R}[\lambda])^{p\times p}$, and $pI \in \mathcal{J}$. Moreover, we have

Lemma 1 *There exists a nonsingular $D \in (\mathbf{R}[\lambda])^{p\times p}$ such that*

$$\mathcal{J} = (\mathbf{R}[\lambda])^{p\times p} D.$$

Proof. The ideal structure in $\mathbf{R}[\lambda]^{p\times p}$ is well-known, and follows from the existence of a greatest common divisor between polynomial matrices. More precisely, let $A \in (\mathbf{R}[\lambda])^{p\times k}$ and $B \in (\mathbf{R}[\lambda])^{p\times l}$ be two matrices with the same number of rows, then they have a greatest common left divisor. Indeed, the set

$$\mathcal{M} = A\ \mathbf{R}[\lambda]^k + B\ \mathbf{R}[\lambda]^l,$$

is a $\mathbf{R}[\lambda]$ submodule of $\mathbf{R}[\lambda]^p$. This module is free of rank $r \leq p$, as a submodule of a free module over a principal ring. Thus

$$\mathcal{M} = C\ \mathbf{R}[\lambda]^r,$$

where the columns of $C \in \mathbf{R}[\lambda]^{n\times r}$ constitute a basis for $\mathcal{M}$. It is easily proved that C is a greatest common left divisor to A and B, and is unique up to a right invertible factor in $\mathbf{R}[\lambda]^{r\times r}$.
In the same way, two matrices with the same number of columns have a greatest right common divisor.
This last result combined with the fact that every ideal in $\mathbf{R}[\lambda]^{p\times p}$ is finitely generated, proves that every left ideal $\mathcal{I}$ in $\mathbf{R}[\lambda]^{p\times p}$ writes $\mathcal{I} = \mathbf{R}[\lambda]^{p\times p} D$, for some D in $\mathbf{R}[\lambda]^{p\times p}$. In particular this result holds for $\mathcal{J}$, and since pI belongs to $\mathcal{J}$, D is non singular. Q.E.D.

Putting $N = D\ T$, we get a factorization of T. More precisely

Theorem 1 *A rational transfer function T has the representation*

$$T = D^{-1} N,$$

where $D \in (\mathbf{R}[\lambda])^{p\times p},\ detD \neq 0,\ N \in (\mathbf{R}[\lambda])^{p\times m}$.
If D and N are left coprime, then they are unique up to a common left invertible factor in $\mathbf{R}[\lambda]^{p\times p}$.

This factorization induces a factorization of the reduced input-output map f.
We have

$$\begin{array}{rcl} kerf & = & \{P \in \mathbf{R}^m[\lambda],\ TP \in \mathbf{R}^p[\lambda]\} \\ & = & \{P \in \mathbf{R}^m[\lambda],\ NP \in D\mathbf{R}^p[\lambda]\} \end{array}.$$

Now denote by K_D the quotient module $\mathbf{R}^p[\lambda]/D\mathbf{R}^p[\lambda]$ and by π_D the canonical projection.
Consider the map $\mathcal{R} : P \to \pi_D(NP)$. We have $ker\mathcal{R} = kerf$, and since the matrices D and N are left coprime, $Im\mathcal{R} = Im\pi_D = K_D$, so that $\mathcal{R}$ induces an isomorphism

$$\mathbf{R}^m[\lambda]/kerf \approx K_D,$$

and we can take K_D as a model for the state space. In particular, since the dimension of K_D over $\mathbf{R}$ is given by the degree of $detD$ (this is easily deduced from the Smith-Mac-Millan form of D), we have the

Theorem 2 *The Mac-Millan degree of T is equal to the degree of the polynomial $detD$.*

Moreover, f factors through K_D. Let us make this precise.

The factorization $T = D^{-1}N$ induces a factorization of f:

$$\begin{array}{ccccc} \underset{P}{\mathbf{R}^m[\lambda]} & & \xrightarrow{f} & & \underset{\pi_-(D^{-1}P')}{1/\lambda\ \mathbf{R}^p[[1/\lambda]]} \\ & \overset{\phi}{\searrow} & & \overset{\psi}{\nearrow} & \\ & & \underset{P'=NP}{\mathbf{R}^p[\lambda]} & & \end{array}.$$

Let $P' \in \mathbf{R}^p[\lambda]$. From (1), $D^{-1}P'$ decomposes as decomposition

$$D^{-1}P' = P'' + S, \quad P'' \in \mathbf{R}^p[\lambda], \quad S \in 1/\lambda\ \mathbf{R}^p[[1/\lambda]] \qquad (3)$$

and $\pi_-(D^{-1}P') = S$. Therefore, $ker\psi = D\ \mathbf{R}^p[\lambda] = ker\pi_D$, and ψ factor through K_D:

$$\begin{tikzcd}
\mathbf{R}^p[\lambda] \arrow[rr, "\psi"] \arrow[dr, "\pi_D"'] & & 1/\lambda\ \mathbf{R}^p[[1/\lambda]] \\
& K_D \arrow[ur, "\mathcal{O}"'] &
\end{tikzcd} .$$

Now, (3) shows that the polynomials P' and DS are in the same class in K_D. Moreover, from (1) there is in each class a unique element of the form DS, with $S \in 1/\lambda\ \mathbf{R}^p[[1/\lambda]]$.
Identifying K_D with the set of polynomials of this form, we can see that $\mathcal{O}$ is nothing else than multiplication by D^{-1}.

Finally, f factors through K_D as follows:

$$\begin{tikzcd}
\mathbf{R}^m[\lambda] \arrow[rr, "f"] \arrow[dr, "\mathcal{R}"'] & & 1/\lambda\ \mathbf{R}^p[[1/\lambda]] \\
& K_D \arrow[ur, "\mathcal{O}"'] &
\end{tikzcd} .$$

The main inconvenience of this factorization comes from the fact that the matrices D and N are seriously non-unique and, moreover, the degrees of the polynomials in D and N are not *a priori* bounded, whereas, in practice, you couldn't work with an infinite number of coefficients! In the next section, we are going to enrich the algebraic context to a topological one. To this end, we shall introduce some convergence conditions in order to convert formal series into complex functions.

3 Transfer functions in Hardy spaces .

The relevant spaces of complex functions here will be the Hardy spaces. Let us recall some facts about them. We denote by L^q the Banach space of all complex functions defined on the unit circle T whose

qth power is integrable with respect to the normalized Lebesgue measure, and L^∞ the space of all essentially bounded functions.
Thus we have for $1 \le q \le q' \le \infty$ that $L^1 \supset L^q \supset L^{q'} \supset L^\infty$. Each $f \in L^1$ has well-defined Fourier coefficients given by

$$a_n = \frac{1}{2\pi} \int f(e^{it}) e^{-int} dt$$

We define for $1 \le q \le \infty$ the Hardy space H^q to be the closed subspace of L^q consisting of all functions for which $a_n = 0$ when $n < 0$. We shall also use the Hardy space $\bar{H}^q$ of functions for which $a_n = 0$ when $n > 0$.
It is well-known that functions in H^q turn out to be restrictions to the unit circle of holomorphic function on the unit disk satisfying growth conditions at the boundary, while functions in $\bar{H}^q$ come from holomorphic functions outside the unit disk ([1]).

For $q = 2$, the spaces H^2 and $\bar{H}^2$ are subspaces of the Hilbert space L^2. Recall ([3]) that the scalar product in L^2 is given by

$$< f, g >= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g}(e^{it}) dt,$$

and if $f = \sum_k a_k z^k$ and $g = \sum_k b_k z^k$, Parseval's equality yields

$$< f, g >= \sum_k a_k \bar{b}_k.$$

Moreover, we have the orthogonal decomposition

$$L^2 = H^2 \oplus 1/z \; \bar{H}^2. \tag{4}$$

Let us close this section with the Beurling-Lax theorem. A subspace $\mathcal{X} \in H^2$ is called invariant if it is an invariant subspace of the so-called "shift operator" S, that is if $\mathcal{X}$ is a closed subspace such that

$$S(\mathcal{X}) \subset \mathcal{X},$$

where S is defined by

$$\forall f \in H^2, \quad Sf(z) = zf(z).$$

A celebrated theorem of Beurling ([3]) describes the invariant subspaces of H^2 by mean of inner functions, namely functions

$f \in H^\infty$ for which $|f| = 1$ a.e. on the unit circle T.
The Beurling-Lax theorem extends this result to vector-valued Hardy spaces, which are a generalization of Hardy spaces to complex functions taking their values in a Hilbert space. As it is sufficient for our purpose, we shall give the statement in the particular case of a finite dimensional range, and for full-range subspaces only. Consider the space $(H^q)^p$ of complex functions with values in $\mathbf{C}^p$ such that each component function belongs to H^q.
An invariant subspace of $(H^2)^p$ is said to be of full range if, a.e. on the unit circle, $\{f(e^{it}), f \in \mathcal{X}\}$ spans $\mathbf{C}^p$. An inner function will be now a $p \times p$ matrix Q with entries in H^∞ such that $Q(e^{it})$ is unitary a.e. on the unit circle, that is

$$Q(e^{it})^t \overline{Q(e^{it})} = I_p, \tag{5}$$

where I_p the identity matrix of order p.

Theorem 3 *(Beurling-Lax) Let $\mathcal{X}$ be an invariant subspace of full range of $(H^2)^p$. Then there exists an inner function $Q \in (H^\infty)^{p \times p}$ such that*

$$\mathcal{X} = Q\ (H^2)^p.$$

Moreover, Q is unique up to right multiplication by some unitary matrix.

Proof. Let us sketch a proof.
Since $\mathcal{X}$ doesn't reduce to 0, there exists a smallest integer n_0 such that $\mathcal{X}$ contains a function f of the form

$$f(z) = \sum_{n \geq n_0} c_n z^n, \quad c_n \in \mathbf{C}^p.$$

But $f \notin z\mathcal{X}$, and thus $z\mathcal{X}$ is a proper subspace of $\mathcal{X}$. Let L be its orthogonal complement

$$\mathcal{X} = L \oplus z\mathcal{X},$$

and let $(\Phi_\alpha)_{\alpha \in \Omega}$ be an orthonormal system in L.
It can be proven that $(\Phi_\alpha z^n)_{\alpha \in \Omega, n \geq 0}$ form an orthonormal system in $\mathcal{X}$ and this implies that, for almost every t, the family $(\Phi_\alpha(e^{it}))$ is orthogonal in $\mathbf{C}^p$. Thus (Φ_α) has at most p elements.

Now the condition to be of full range implies that the family $(\Phi_\alpha(e^{it}))_{\alpha\in\Omega}$ spans $\mathbf{C}^p$ for almost every t, and thus (Φ_α) has at least p elements.
Finaly, $dim L = p$.
Now, define the $p \times p$ matrix $Q(e^{it})$, whose columns are the $\Phi_k(e^{it})$.
The function Q satisfies the conclusion the theorem. Q.E.D.

For system-theoretic applications, we have to restrict ourselves to real subspaces consisting of functions whose Fourier coefficients are real. We shall indicate this restriction by the subscript r, and write L^q_r, H^q_r , $\bar{H}^q_r$ for the real Hardy spaces.
Now, we need a real version of the Beurling-Lax theorem. It is not difficult to see that the previous proof works for real spaces. In this case, it yields a function $Q \in (H^\infty_r)^{p\times p}$.

Corollary 1 *(Beurling-Lax, real version) Let $\mathcal{X}$ be a real invariant subspace of full range of $(H^2_r)^p$. Then there exists an inner function $Q \in (H^\infty_r)^{p\times p}$ such that*

$$\mathcal{X} = Q\ (H^2_r)^p.$$

Moreover, Q is unique up to right multiplication by some orthogonal matrix.

We want to consider transfer functions as functions of the complex variable z. Since our transfer functions are series of negative powers of the variable z, they must be holomorphic outside the unit disk. Thus, they must be stable, namely with poles in the unit disk only.
The transfer functions involved in the sequel will be stable rational and therefore elements of $(1/z\ \bar{H}^\infty_r)^{p\times m}$.
It will be also necessary to consider the input and the output as complex functions. This will be done by assuming l^2-convergence for our series. Indeed, $1/z\ \bar{H}^\infty_r$ acts on L^2_r by multiplication, and we may view our system as a function from $(L^2_r)^m$ to $(L^2_r)^p$.
Restricted input-output maps are now defined between Hardy spaces

$$f : (H^2_r)^m \rightarrow 1/z\ (\bar{H}^2_r)^p.$$

With this in mind, let us come back to the factorization problem.

4 The inner-unstable factorization.

Let T be some stable rational transfer function. Consider the set

$$\mathcal{V} = \{M \in (H_r^2)^{p\times p} / MT \in (H_r^2)^{p\times m}\}.$$

It is a real subspace of $(H_r^2)^{p\times p}$, and we have

Lemma 2 *There exists an inner function $Q \in (H_r^\infty)^{p\times p}$ such that $\mathcal{V} = (H_r^2)^{p\times p}Q$. The matrix Q is unique up to an orthogonal left factor.*

Proof. Let

$$\mathcal{E} = \{m \in (H_r^2)^p / {}^t mT \in (H_r^2)^m\}.$$

Clearly, $\mathcal{E}$ is an invariant real subspace of $(H_r^2)^p$. Now, let $\mathcal{F}(e^{it})$ be the family $\{f(e^{it}), f \in \mathcal{E}\}$, where e^{it} is fixed on the unit circle. If $(e_1, ..., e_n)$ denotes the canonical basis in $\mathbf{C}^p$, and since

$$\exists p \in (\mathbf{R}[z]), \text{ such that } pT \in (\mathbf{R}[z])^{p\times m},$$

the family $\{p(e^{it})e_1, ..., p(e^{it})e_n\} \subset \mathcal{F}(e^{it})$ spans $\mathbf{C}^p$ if and only if $p(e^{it}) \neq 0$. Thus, $\mathcal{F}(e^{it})$ spans $\mathbf{C}^p$ a.e. on the unit circle. So, $\mathcal{E}$ is an invariant real subspace of full range of (H_r^2), and the real version of the Beurling-Lax theorem applies: there exists an inner function $Q \in (H_r^\infty)^{p\times p}$ such that

$$\mathcal{E} = Q\ (H_r^2)^p.$$

Since the rows of $M \in \mathcal{V}$ belong to $\mathcal{E}$, we have

$${}^tM \in Q\ (H_r^2)^{p\times p}.$$

Our result follows at once, replacing Q by ${}^tQ$. Q.E.D.

Let $C = QT$, then we have a new factorization of T of the form

$$T = Q^{-1}C,$$

where $C \in (H_r^2)^{p\times m}$. Moreover, since Q and T have their elements in L_r^∞, the same holds for C. But $L_r^\infty \bigcap H_r^2 = H_r^\infty$, so that C belongs to $(H_r^\infty)^{p\times m}$.

We are going to stress the link between this factorization and Fuhrmann's factorization $T = D^{-1}N$, used in Theorem 1.

Lemma 3 *The matrix Q is rational, and we have the representation*

$$Q^{-1} = D^{-1}\Lambda,$$

where Λ is a polynomial matrix of order p, D and Λ being left coprime. Moreover, Λ is invertible in $(H_r^\infty)^{p\times p}$ and we have

$$\mathcal{V} = (H_r^2)^{p\times p} D.$$

Proof. Since $D \in \mathcal{V}$, there exists $\Lambda \in (H_r^2)^{p\times p}$ such that

$$D = \Lambda Q.$$

We have $\Lambda = DQ^{-1}$, where D is polynomial and the elements of Q^{-1} belongs to $\bar{H}_r^\infty$. Thus Λ is the sum of a polynomial matrix and a matrix with elements in $\bar{H}_r^\infty$. Since $\Lambda \in (H_r^2)^{p\times p}$, it must be polynomial.
As

$$Q^{-1} = D^{-1}\Lambda,$$

it follows that Q is rational.

Now, there exists $A \in (\mathbf{R}[z])^{p\times p}$ and $B \in (\mathbf{R}[z])^{m\times p}$ such that

$$DA + NB = I_p. \tag{6}$$

Multiplying by D^{-1} gives

$$D^{-1} = A + TB,$$

and

$$\Lambda^{-1} = QD^{-1} = QA + CB \in (H_r^\infty)^{p\times p}.$$

Hence, Λ is invertible in $(H_r^\infty)^{p\times p}$.

Since $\Lambda^{-1} \in (H_r^\infty)^{p\times p}$, it follows that $det\Lambda$ has all its roots outside the unit disk, while $detD$ has all its roots inside. Consequently D and Λ must be left coprime.

Moreover the invertibility of Λ implies

$$(H_r^2)^{p\times p}\Lambda = (H_r^2)^{p\times p},$$

and thus

$$\mathcal{V} = (H_r^2)^{p\times p} D.$$

In fact, we have proven that the set $\mathcal{J}$ of Lemma 1 is dense in $\mathcal{V}$. Q.E.D.

Remark. The relation

$$D = \Lambda Q,$$

is nothing else than the inner-outer factorization of D ([1]).

Theorem 4 *(Inner-Unstable factorization for rational functions) A rational transfer function $T \in 1/z\ (\bar{H}_r^\infty)^{p\times p}$ can be represented as*

$$T = Q^{-1}C,$$

where $Q \in (H_r^\infty)^{p\times p}$ is inner, C belongs to $(H_r^\infty)^{p\times m}$, and where Q and C are left coprime. With this condition, the decomposition is unique up to a common left orthogonal factor. The Mac-Millan degree of T is equal to the Mac-Millan degree of Q^{-1}.

Proof. The first assertion has been already proven.

Before proving the left coprimeness of our matrices let us make precise what sort of condition we can expect. Since H_r^∞ is a ring, divisibility makes sense for matrices with elements in H_r^∞. Although H_r^∞ is no longer principal, every finitely generated ideal is, and everything works as for matrices with entries in $\mathbf{R}[z]$ (cf. proof of lemma1). The greatest common left divisor C of two matrices $A \in (H_r^\infty)^{p\times k}$ and $B \in (H_r^\infty)^{p\times l}$ having the same number of rows exists, and is determined by

$$A\ (H_r^\infty)^k + B\ (H_r^\infty)^l = C\ (H_r^\infty)^r,$$

up to some right invertible factor in $(H_r^\infty)^{r\times r}$.
One can notice, at least if A or B is regular since we did not appeal to a more general version of the Beurling-Lax theorem which deals with non full-range subspaces as well ([1]), that corollary 1 as applied to the invariant subspace

$$A\ (H_r^\infty)^k + B\ (H_r^\infty)^l$$

yields an inner matrix which provides a somewhat unique representant of the greatest common left divisor since it is defined up to some orthogonal matrix.

We now proceed with the proof. Condition (6) of left coprimeness between D and N gives

$$\Lambda^{-1}(DA + NB)\Lambda = I_p,$$

that is

$$Q(A\Lambda) + C(B\Lambda) = I_p,$$

which ensures the left coprimeness of Q and C. Now, if $T = Q'^{-1}C'$, where $Q' \in (H_r^\infty)^{p\times p}$ is inner and $C' \in ((H_r^\infty))^{p\times m}$, then $Q' \in \mathcal{V}$ and there exists $U \in (H_r^\infty)^{p\times p}$ such that $Q' = UQ$ and $C' = UC$. Since Q' and C' are supposed to be left coprime, then U is invertible in $(H_r^\infty)^{p\times p}$, and we have $\mathcal{V} = (H_r^2)^{p\times p}Q'$. By the uniqueness part of the Beurling-Lax theorem, we are done. Q.E.D.

This new factorization is of some interest in practice. Indeed, it allows us to point out another type of pair (D, N) than the coprime one in Fuhrmann's factorization, for which a boundedness condition on the degree of polynomials holds. This is stated in the next result.

Proposition 1 *Let $q = detD$ be of degree n. Then $detQ = \pm q/\tilde{q}$, where $\tilde{q}$ is the reciprocal polynomial $\tilde{q}(z) = z^n q(1/z)$. The matrices $D_Q = \tilde{q}Q$ and $N_Q = \tilde{q}C$ are polynomial matrices. Moreover, the degree of the entries does not exceed n in D_Q and $n-1$ in C_Q.*

Proof. The proof is quite easy. The relation $D = \Lambda Q$ implies

$$detD = det\Lambda \ detQ.$$

where $detQ$ is a scalar inner function and $det\Lambda$ is invertible in H_r^2. In other words, this is the inner-outer factorization in H^2 ([3]). Since such a factorization is unique up to a complex factor of modulus one, it must be

$$q = u\tilde{q} \ \bar{u}q/\tilde{q}, \quad u \in \mathbf{T}.$$

The matrix Q having real coefficients, u must be ± 1, so that

$$detQ = \pm q/\tilde{q}.$$

Now, since Q is rational the inner condition (5) extends to all complex numbers as

$$Q^{-1}(z) = {}^tQ(1/z),$$

and thus, $Q^{-1} \in (\bar{H}_r^\infty)^{p\times p}$.
Now, by Cramer formula,

$$Q^{-1} = {}^t com(Q)/det(Q),$$

or else

$$qQ^{-1} = \pm\ {}^t com(Q)\ \tilde{q}.$$

Now, the right hand-side lies in $(H_r^\infty)^{p\times p}$, while the left hand-side, being the product of the polynomial q by $Q^{-1} \in (\bar{H}_r^\infty)^{p\times p}$, writes as the sum of a polynomial matrix and a matrix in $1/z\ (\bar{H}_r^\infty)^{p\times p}$. Thus, qQ^{-1} is a polynomial matrix and its degree does not exceed the degree of q, that is n.
The same holds true for $\tilde{q}Q$ since

$$\tilde{q}(z)Q(z) = z^n q(1/z)Q^{-1}(1/z).$$

A similar argument gives now the desired conclusion for the matrix $\tilde{q}C = (\tilde{q}Q)\ T$. Q.E.D.

5 Application to realization theory.

Following the first section, we are going to factor the restricted input-output map

$$f : (H_r^2)^m \to 1/z\ (\bar{H}_r^2)^p.$$

The factorization $T = Q^{-1}C$ induces a factorization of f:

$$\begin{tikzcd}
(H_r^2)^m \arrow[rr, "f"] \arrow[dr, "\bar{\phi}"'] & & 1/z\ (\bar{H}_r^2)^p \\
& (H_r^2)^p \arrow[ur, "\bar{\psi}"'] &
\end{tikzcd}\quad .$$

where π_- in the definition of $\bar{\psi}$ is now the projection onto $1/z\ (\bar{H}_r^2)^p$.

Let $g \in (H_r^2)^p$. From (4), $Q^{-1}g$ can be decomposed as

$$Q^{-1}g = h_+ + h_-, \quad h_+ \in (H_r^2)^p, \quad h_- \in 1/z\ (\bar{H}_r^2)^p$$

and $\pi_-(Q^{-1}g) = h_-$. Therefore, $ker\bar{\psi} = Q\ (H_r^2)^p$. Denote by K_Q the quotient vector space $(H_r^2)^p/Q\ (H_r^2)^p$, and by π_Q the projection onto K_Q Then $\bar{\psi}$ factors through K_Q:

$$\begin{tikzcd} (H_r^2)^p \arrow[rr, "\bar{\psi}"] \arrow[dr, "\pi_Q"] & & 1/z\ (\bar{H}_r^2)^p \\ & K_Q \arrow[ur, "\mathcal{O}"] & \end{tikzcd} .$$

Moreover, K_Q is isomorphic to a subspace of $(H_r^2)^p$ by the orthogonal Hilbert space decomposition

$$(H_r^2)^p = Q(H_r^2)^p \oplus K_Q.$$

Now compare this decomposition with the vector space decomposition of $(\mathbf{R}[z])^p$, we had in the first section

$$(\mathbf{R}[z])^p \approx D(\mathbf{R}[z])^p \oplus K_D.$$

Since, $(\mathbf{R}[z])^p$ is dense in $(H_r^2)^p$ and $D(\mathbf{R}[z])^p$ is dense in $Q(H_r^2)^p$ (cf. lemma 3), K_D must be dense in K_Q. Since K_D is a finite dimensional vector space we have a $\mathbf{R}[z]-module$ isomorphism:

$$K_Q \approx K_D.$$

This is capsulized in the next statement.

Theorem 5 *Let $T = Q^{-1}C$ be the inner-unstable factorization of the rational transfer function T. Then the state space is isomorphic to the space K_Q in the orthogonal decomposition*

$$(H_r^2)^p = Q(H_r^2)^p \oplus K_Q.$$

The function f factors through K_Q as follows

$$\begin{tikzcd}
(H_r^2)^m \arrow[rr, "f"] \arrow[dr, "\pi_Q(C.)"'] & & 1/z\ (H_r^2)^p \\
& K_Q \arrow[ur, "Q^{-1}"'] &
\end{tikzcd}.$$

6 Application to parametrization problems.

Another respect in which the inner-unstable factorization may be useful, is bounded parametrisation problems in system-theory. Consider the smooth manifold $\mathcal{S}_n^{p,m}$ of stable $p \times m$ transfer functions of fixed Mac-Millan degree n, as imbedded in $(\bar{H}_r^2)^{p\times m}$ [4]). Consider further the set $\mathcal{I}_n^p$ of inner matrices of size $p \times p$ and of Mac-Millan degree n, and let $\mathcal{C}_Q$ be the subspace of $(H_r^2)^{p\times m}$ consisting of those C such that $Q^{-1}C$ belongs to $(\bar{H}_r^2)^{p\times m}$. If $\mathcal{P}$ denotes the subset of $(H_r^2)^{p\times p} \times (H_r^2)^{p\times m}$ consisting of pairs (Q, C) where $C \in \mathcal{C}_Q$ is coprime to Q, then $\mathcal{P}$ endowed with the map

$$\Gamma :\ \mathcal{P} \to \mathcal{S}_n^{p,m}$$

given by $\Gamma(Q, C) = Q^{-1}C$ is a fibered space with compact base over $\mathcal{S}_n^{p,m}$. This is of importance in several rational approximation problems, where the criterion can be brought down to the compact set $\mathcal{I}_n^p$. [5].

7 Conclusion.

Among the many parametrizations for stable transfer functions, the one presented here exhibits in some sense the maximal number of bounded parameters, because $Q^{-1}C$, for fixed Q and $C \in \mathcal{C}_Q$ is a vector space included in $\mathcal{S}_n^{p,m}$. We expect this form to be of interest in identification problems where numerical optimisation on $\mathcal{S}_n^{p,m}$ has to be performed. The set of inners, which has been recognized for a long time to be of importance in system parametrization, plays here again an interesting role, as "carrying" the bounded part of the model.

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