CANONICAL LOSSLESS STATE-SPACE SYSTEMS: STAIRCASE FORMS AND THE SCHUR ALGORITHM

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Abstract: A new finite atlas of overlapping balanced canonical forms for multivariate discrete-time lossless systems is presented. The canonical forms have the property that the controllability matrix is positive upper triangular up to a suitable permutation of its columns. This is a generalization of a similar balanced canonical form for continuous-time lossless systems. It is shown that this atlas is in fact a sub-atlas of the infinite atlas of overlapping balanced canonical forms for lossless systems that is associated with the tangential Schur algorithm; such canonical forms satisfy certain interpolation conditions on a corresponding sequence of lossless transfer matrices. The connection between these balanced canonical forms for lossless systems and the tangential Schur algorithm for lossless systems is a generalization of the same connection in the SISO case that was noted before. The results are directly applicable to obtain a finite atlas of multivariate input-normal canonical forms for stable linear systems of given fixed order, which is minimal in the sense that no chart can be left out of the atlas without losing the property that the atlas covers the manifold of stable linear systems of fixed given order.

Keywords: Lossless systems, input normal forms, balanced canonical forms, model reduction, MIMO systems, tangential Schur algorithm.

1. INTRODUCTION

In linear systems theory there has been a longstanding program in developing balanced realizations and balanced canonical forms and associated parameterizations for stable linear systems and for various other classes of linear systems. The classical (Gramian based) concept of balancing as introduced by Moore, see (Moore, 1981), applies to *stable* systems and allows one to develop parameterizations in which system stability is a built-in property. One of the motivations for the interest in balancing is that it leads to a simple method for model order reduction, namely by truncation of (the last entries of) the state vector. However truncation does not always lead to a minimal system. Therefore there has been research into balanced canonical forms which have the property that truncation of the last entries in the state vector leads to a minimal system. For continuous-time systems this has led to the original balanced canonical form of Ober (cf. (Ober, 1987)) and to the new balanced canonical form of Hanzon (cf. (Hanzon, 1995); see also (Ober, 1996)). This last balanced canonical form is based on the idea that if the controllability matrix is positive upper triangular (i.e., the controllability matrix forms an upper triangular matrix with positive entries on the pivot positions), then truncation of the last entries of the state vector leads again to a system with a positive upper triangular controllability matrix, hence is controllable. Because this is in the balanced continuous-time case the controllability property implies that the resulting system is again minimal and balanced.

To use similar ideas to build overlapping balanced canonical forms is more involved. For continuous-time *lossless* systems, which form the key to these problems, a generalization of positive upper triangular matrices is used in (Hanzon and Ober, 1998). The idea used there is that it suffices if a column permutation of the controllability matrix is positive upper triangular. Under certain circumstances there will exist an associated column permutation (we also speak of a *shuffle* of columns in this context) of the so-called realization matrix, which allows one to proceed with the construction.

In the case of discrete-time systems the situation is somewhat more complicated because it is known that starting from a balanced realization, truncation of the state vector will normally not lead to a balanced state-space system. In the case of SISO lossless discrete-time systems a balanced canonical form for lossless systems, with a simple positive upper triangular controllability matrix was presented in (Hanzon and Peeters, 2000). Also the possibilities for model reduction by truncation, combined with a correction of some sort to arrive at a balanced realization of a lossless system, are discussed there.

In the current paper we treat the case of MIMO lossless discrete-time systems. We present overlapping balanced canonical forms which have the property that the corresponding controllability matrix is positive upper triangular, up to a column permutation. In this sense it is close to the corresponding results in (Hanzon and Ober, 1998); however, here a generalization is presented which we think will greatly simplify the presentation and which could also be used in the continuous-time case. The precise relation with the approach taken in (Hanzon and Ober, 1998) will be made clear.

In (Hanzon and Peeters, 2000) a connection was shown between the balanced canonical forms there presented and the Schur algorithm for lossless discrete-time transfer functions. In a recent report ((Hanzon *et al.*, 2004)) it is shown how the parameterizations for multivariable rational lossless transfer matrices by Schur parameters, based on the so-called tangential Schur algorithm, can likewise be lifted into parameterizations by Schur parameters of balanced (state-space) canonical forms of lossless systems. The main result of the current paper is to show how the atlas of overlapping balanced canonical forms presented in this paper can be interpreted as a finite sub-atlas of the infinite atlas of overlapping balanced canonical forms corresponding to the tangential Schur algorithm. In fact, a certain well-specified choice of so-called direction vectors in the tangential Schur algorithm leads to the balanced canonical forms presented here.

Although a generalization of the results of this paper to the case of complex-valued systems is straightforward, we shall restrict the discussion to the case of real-valued systems only for ease of presentation.

2. BALANCED REALIZATIONS OF LOSSLESS SYSTEMS BASED ON THE TANGENTIAL SCHUR ALGORITHM

In (Hanzon *et al.*, 2004) a new class of overlapping balanced canonical forms for MIMO discrete-time lossless systems is presented. There, each (local) balanced canonical form is characterized by (i) the choice of a number of fixed points, the so-called interpolation points, and (ii) a number of fixed vectors, the so-called direction vectors. Given the choices for the interpolation points and for the direction vectors, the choice of a sequence of parameter vectors, the so-called Schur parameter vectors, and the choice of an orthogonal matrix, together completely determine the balanced state-space realization of a lossless system. Let us now be more concrete.

Consider a linear time-invariant state-space system in discrete time with m inputs and m outputs:

$$x_{t+1} = Ax_t + Bu_t, \tag{1}$$

$$y_t = Cx_t + Du_t, \tag{2}$$

with $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$ for some nonnegative integer n (the state space dimension), $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^m$. Furthermore, the matrices A, B, C and D with real-valued entries are of compatible sizes: $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. The corresponding transfer matrix of this system is given by $G(z) = D + C(zI_n - A)^{-1}B$, which is an $m \times m$ matrix with rational functions as its entries. To any such state-space system we associate the following (square) block-partitioned matrix:

$$R = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \tag{3}$$

which we call the *realization matrix*. It will play an important role in the sequel.

Let (A, B, C, D) be some state space realization of a transfer matrix $G(z)$. If the eigenvalues of A all belong to the open unit disk, then the matrix A is called (discrete-time) asymptotically stable, and (A, B, C, D) is an asymptotically stable realization of $G(z)$. (For more details on state-space realization theory, see e.g. (Kailath, 1980).)

If (A, B, C, D) is an asymptotically stable realization, then the controllability Gramian W_c and the observability Gramian W_o are well defined as the exponentially convergent series

$$W_c = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k, \tag{4}$$

$$W_o = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k. \tag{5}$$

These Gramians are characterized as the unique (and positive semi-definite) solutions of the respective Lyapunov-Stein equations

$$W_c - A W_c A^T = B B^T, \tag{6}$$

$$W_o - A^T W_o A = C^T C. \tag{7}$$

A minimal and asymptotically stable state-space realization (A, B, C, D) of a transfer matrix is called *balanced* if its controllability and observability Gramians W_c and W_o are both diagonal and equal. Any minimal and asymptotically stable realization (A, B, C, D) is similar to a balanced realization.

A system is called *lossless* if it is stable and its $m \times m$ transfer matrix $G(z)$ is unitary for all complex z with $|z| = 1$. It is well-known (cf., e.g., Proposition 3.2 in (Hanzon *et al.*, 2004) and the references given there) that $R = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ is a balanced realization matrix of a lossless system if and only if R is an orthogonal matrix and A is asymptotically stable. For a further background on lossless systems, see e.g. (Genin *et al.*, 1983).

In (Hanzon *et al.*, 2004) an atlas of overlapping balanced canonical forms for lossless discrete-time systems of order n is presented. Each such balanced canonical form is characterized by a fixed sequence of n numbers $w_k, |w_k| < 1, k = 1, \ldots, n$, called the *interpolation points*, and a fixed sequence of n unit vectors $u_k \in \mathbb{R}^m, \|u_k\| = 1, k = 1, \ldots, n$, called the *direction vectors* (which are not to be confused with the input signal applied to a system). Here we will consider the case $w_k = 0, k = 1, \ldots, n$ hence each balanced canonical form that we consider is determined by the choice of direction vectors. Each such balanced canonical form is then parameterized by an $m \times m$ orthogonal matrix $G^{(0)}$ and a sequence of n vectors $v_k, \|v_k\| < 1, k = 1, \ldots, n$ which are called the *Schur parameter vectors*.

In fact the realization matrix can be written as a *product of matrices* of size $(m + n) \times (m + n)$:

$$R = \Gamma_n \Gamma_{n-1} \cdots \Gamma_1 \Gamma_0 \Delta_1^T \Delta_2^T \cdots \Delta_n^T, \tag{8}$$

where for $k = 1, \ldots, n$:

$$\Gamma_k = \begin{bmatrix} I_{n-k} & 0 & 0 \\ 0 & V_k & 0 \\ 0 & 0 & I_{k-1} \end{bmatrix},$$

$$\Delta_k = \begin{bmatrix} I_{n-k} & 0 & 0 \\ 0 & U_k & 0 \\ 0 & 0 & I_{k-1} \end{bmatrix}$$

with an $(m + 1) \times (m + 1)$ orthogonal matrix block V_k given by

$$V_k = \begin{bmatrix} v_k & I_m - (1 - \sqrt{1 - \|v_k\|^2}) \dfrac{v_k v_k^T}{\|v_k\|^2} \\ \sqrt{1 - \|v_k\|^2} & -v_k^T \end{bmatrix},$$

and an $(m+1) \times (m+1)$ orthogonal matrix block U_k given by

$$U_k = \begin{bmatrix} u_k & I_m - u_k u_k^T \\ 0 & u_k^T \end{bmatrix}$$

and furthermore

$$\Gamma_0 = \begin{bmatrix} I_n & 0 \\ 0 & G^{(0)} \end{bmatrix}.$$

Note that here we consider the real case with real direction vectors and real Schur parameter vectors. Note further that $\Gamma_0, \ldots, \Gamma_n$ and $\Delta_1, \ldots, \Delta_n$ are all orthogonal matrices. It is important to note and not too difficult to see that the orthogonal matrix product $\Gamma_n \Gamma_{n-1} \cdots \Gamma_1 \Gamma_0$ in fact forms a *positive m-upper Hessenberg matrix*, i.e. an $(m+n) \times (m+n)$ matrix of which the m-th sub-diagonal only has positive entries and of which the last $n - 1$ sub-diagonals are all zero. It also follows almost directly that if the direction vectors $u_1, \ldots, u_n$ are taken to be standard basis vectors, then the matrix product $\Delta_1^T \Delta_2^T \cdots \Delta_n^T$ yields a permutation matrix. Hence in that case the balanced realization matrix R is obtained as a column permutation of an orthogonal positive m-upper Hessenberg matrix.

3. BALANCED LOSSLESS SYSTEMS WITH TRIANGULAR STRUCTURE IN THE CONTROLLABILITY MATRIX

It is not difficult to see that if the realization matrix R is positive m-upper Hessenberg, then (i) the first n columns of the partitioned $n \times (m + n)$ matrix $[B|A]$ form a positive upper triangular matrix, i.e. an upper triangular matrix with only positive entries on the main diagonal, and (ii) the first n columns of the corresponding controllability matrix $[B|AB|A^2B|\ldots]$ also form a positive upper triangular matrix. Therefore the realization is controllable. In the discrete-time lossless case, if R is orthogonal this implies that A is asymptotically stable which in turn implies that the realization is minimal.

A balanced realization of a lossless systems is determined up to an arbitrary orthogonal change of basis of the state space. The effect of such a change of basis on the controllability matrix is that it is pre-multiplied with an orthogonal matrix. Now it is well-known that any non-singular square matrix can be written as a product of an orthogonal matrix and a positive upper triangular matrix in a unique way (this is the well-known QR-decomposition). If the first n columns of the controllability matrix are linearly independent then a unique orthogonal state-space isomorphism exists such that the first n columns of the controllability matrix form a positive upper triangular matrix. This determines a unique local balanced canonical form for lossless systems. In the SISO case it is in fact a global balanced canonical form and it is presented and investigated in (Hanzon and Peeters, 2000).

In the MIMO case controllability implies that the controllability matrix has at least n independent columns, but not necessarily the first n columns have to be independent. Instead we know that there has to be at least one *nice selection* of n columns from the controllability matrix that will form a square nonsingular matrix (cf., e.g., (Hanzon, 1989) and the references given there).

We now consider $[B|A]$ such that a column permutation of this matrix leads to a so-called simple positive upper triangular matrix, i.e. a matrix of which the first n columns form a square positive upper triangular matrix. If a column vector has a positive k-th entry and each j-th entry of the vector is zero for $j > k$ then we will express that by saying that the column has a *pivot* at position k.

We can now pose the following question. Which distributions of the pivots at positions $1, 2, \ldots, n$ over the columns of the $n \times (m + n)$ matrix $[B|A]$ imply that the associated controllability matrix contains a column with a pivot at position k for each $k = 1, 2, \ldots, n$?

We claim that the answer is that one of the columns of B has to contain the pivot at position 1, and that if the columns of B contain p_B pivots then the remaining $p_A = n - p_B$ pivots have to be located in the first p_A columns of A with increasing pivot positions. This implies that A has a *staircase structure*. A pivot structure of this form will be called an *admissible* pivot structure. The implication is that each choice of assigning (distinct) pivot positions to one or more of the columns of B satisfying the requirement that the pivot at position 1 has to be assigned to one of the columns of B, leads to a (local) balanced canonical form for lossless systems.

4. THE YOUNG DIAGRAM ASSOCIATED WITH AN ADMISSIBLE PIVOT STRUCTURE

We now want to describe the pivot structure of the controllability matrix. This can most easily be done by way of a numbered Young diagram $Y = (y_{i,j})$. In our case this consists of an $m \times n$ matrix with non-negative integer entries. The i-th row is associated with the i-th column of B and its images under repeated multiplication by A: in the i-th row the pivot positions of $Be_i, ABe_i, A^2Be_i, \ldots, A^{n-1}Be_i$ respectively are displayed, where as a convention we write zero if the corresponding column does not have a pivot. Here e_i denotes the i-th standard basis vector in $\mathbb{R}^m$.

The Young diagram of an admissible pivot structure has the property that if an entry of the Young diagram is zero then all the entries to the right in that same row are zero. The Young diagram has n nonzero entries, each of the numbers $1, 2, \ldots, n$ appears once. The number of nonzero entries of the i-th row of the Young diagram will be called the i-th dynamical index d_i of the pivot structure. Clearly $d_1 + d_2 + \ldots + d_m = n$. The *right-aligned* version $Y_r = (y^r_{i,j})$ of the numbered Young diagram Y is obtained by shifting the nonzero entries of each row $n - d_i$ positions to the right; i.e., $y^r_{i,j} := y_{i,j+d_i-n}$. We claim that the right-aligned version of the numbered Young diagram of an admissible pivot structure has the property that its numbering is fully determined by the permutation of the non-zero rows of Y_r (hence of Y) which make that the nonzero entries in its last column form an increasing sequence of integers. In fact we claim that the same permutation makes that each column of the resulting right-aligned matrix will have the property that the nonzero entries in the column form an increasing sequence.

Given any numbered Young diagram of this kind, we can order the corresponding n columns of the controllability matrix according to the numbering in the diagram. If the nice selection associated with the Young diagram is such that these n columns are all linearly independent (such a nice selection always exists in case of minimality) then an orthogonal change of basis can be applied such that the resulting matrix is positive upper triangular. Then the resulting controllability matrix has the required pivot structure and it can be shown that the corresponding pair (B, A) has the associated admissible pivot structure as defined above.

In (Hanzon and Ober, 1998) a similar approach was used for continuous-time lossless systems, however there with each sequence of dynamical indices $d_1, d_2, \ldots, d_m$ the unique permutation of the rows of Y was chosen which makes the corresponding permuted sequence of dynamical indices into a non-increasing sequence, while the order of the rows which have equal dynamical index is kept the same.

The finite atlas presented here is covering the manifold of lossless m-input, m-output systems of fixed McMillan degree n. The proof is analogous to the proof for the continuous-time case in (Hanzon and Ober, 1998).

In (Hanzon and Ober, 1998) one obtains one chart for each sequence of dynamical indices $d_1, \ldots, d_m$. If the same approach is taken here in the discrete time case one obtains a minimal atlas in the sense that no chart can be left out without losing the property that the atlas covers the manifold.

5. CONNECTION BETWEEN THE TWO APPROACHES

We now have two approaches to arrive at an atlas of overlapping balanced canonical forms for discrete-time lossless systems, one using the balanced realizations associated with the tangential Schur algorithm and one based on balanced realizations with an imposed pivot structure on the partitioned matrix $[B|A]$, hence on the realization matrix R. However, one of our main results is that the second approach can be obtained by making a special choice of direction vectors in the first approach. Hence the atlas of overlapping balanced canonical forms in the second approach is a sub-atlas of the atlas of overlapping balanced canonical forms in the first approach. The precise formulation is as follows.

Theorem 1. Let Y be a numbered Young diagram corresponding to an admissible pivot structure. For each $k = 1, 2, \ldots, n$ there exists a unique pair of indices $(i(k), j(k))$ such that $y_{i(k),j(k)} = k$. Choose the direction vector $u_{n+1-k} = e_{i(k)}$, the $i(k)$-th standard basis vector for each $k = 1, 2, \ldots, n$. Then for any admissible choice of the Schur parameter vectors $v_1, v_2, \ldots, v_n$ and the orthogonal matrix $G^{(0)}$, the realization matrix given by (8) has that admissible pivot structure and its controllability matrix has a pivot structure indicated by the Young diagram Y.

6. EXAMPLE

To illustrate our results we consider an example. Let the number of inputs and outputs be equal to $m = 3$, let the order of the class of lossless systems be $n = 7$, and consider the situation in which the three dynamical indices are given by $d_1 = 2$, $d_2 = 1$, $d_3 = 4$. If we use the identity permutation on the rows of the corresponding Young diagram to generate the numbered Young diagrams Y and Y^r we get the following:

$$Y = \begin{array}{|c|c|c|c|}\hline 3 & 5 & & \\ \hline 6 & & & \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array} \qquad Y_r = \begin{array}{|c|c|c|c|}\hline & & 3 & 5 \\ \hline & & & 6 \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array} \tag{9}$$

By contrast, if one uses the permutation (3,1,2), as is done effectively in the approach taken in (Hanzon and Ober, 1998), then we get the following alternative Young diagrams:

$$Y = \begin{array}{|c|c|c|c|}\hline 4 & 6 & & \\ \hline 7 & & & \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array} \qquad Y_r = \begin{array}{|c|c|c|c|}\hline & & 4 & 6 \\ \hline & & & 7 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array} \tag{10}$$

Now let us return to the numbered Young diagrams (9). Then the corresponding pivot structure of the matrix B is:

$$B = \begin{bmatrix} * & * & + \\ * & * & 0 \\ + & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding pivot structure of A is:

$$A = \begin{bmatrix} * & * & * & * & * & * & * \\ + & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & + & * & * & * & * & * \\ 0 & 0 & + & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & + & * & * & * \end{bmatrix}$$

The pivot structure of the initial part of the controllability matrix $[B|AB|A^2B|A^3B]$ is then given by:

$$\left[\begin{array}{ccc|ccc|ccc|ccc} * & * & + & * & * & * & * & * & * & * & * & * \\ * & * & 0 & * & * & + & * & * & * & * & * & * \\ + & * & 0 & * & * & 0 & * & * & * & * & * & * \\ 0 & * & 0 & * & * & 0 & * & * & + & * & * & * \\ 0 & * & 0 & + & * & 0 & * & * & 0 & * & * & * \\ 0 & + & 0 & 0 & * & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & 0 & * & * & 0 & * & * & + \end{array}\right]$$

The corresponding (reversed) sequence of direction vectors in the tangential Schur algorithm is now given by $(u_7, u_6, \ldots, u_1) = (e_3, e_3, e_1, e_3, e_1, e_2, e_3)$ where e_i denotes the i-th standard basis vector in $\mathbb{R}^3$, $i = 1, 2, 3$.

It is left to the reader to verify that with this choice of direction vectors one obtains the permutation matrix

$$\Delta_7 \Delta_6 \cdots \Delta_1 = [e_3, e_4, e_1, e_5, e_6, e_2, e_7, e_8, e_9, e_{10}]$$

where e_j denotes the j-th standard basis vector in $\mathbb{R}^{10}$, $j = 1, 2, \ldots, 10$. The transpose of this permutation matrix effectively permutes the columns of the orthogonal positive 3-upper Hessenberg matrix $\Gamma_7 \Gamma_6 \cdots \Gamma_1 \Gamma_0$ to yield the general form of the orthogonal realization matrix R which describes the associated (local) canonical form.

7. CONCLUDING REMARKS

Just as discussed in (Hanzon *et al.*, 2004) the same approach also leads to overlapping (local) canonical forms for input-normal stable systems as well as output-normal stable systems. We refer to that paper for details. Future research includes investigation of various model reduction schemes based on truncation etc., as a generalization to similar results in the SISO case presented in (Hanzon and Peeters, 2000).

REFERENCES

Genin, Y., P. Van Dooren, T. Kailath, J.-M. Delosme and M. Morf (1983). On Σ-lossless transfer functions and related questions. *Lin. Alg. Appl.* **50**, 251–275.

Hanzon, B. (1989). Identifiability, recursive identification and spaces of linear dynamical systems. CWI Tracts 63, 64, CWI Amsterdam.

Hanzon, B. (1995). A new balanced canonical form for stable multivariable systems. *IEEE Trans. Autom. Control* **AC-40**, 374–378.

Hanzon, B. and R.J. Ober (1998). Overlapping block-balanced canonical forms for various classes of linear systems. *Lin. Alg. Appl.* **281**, 171–225.

Hanzon, B. and R.L.M. Peeters (2000). Balanced parametrizations of stable SISO all-pass systems in discrete-time. *MCSS* **13**, 240–276.

Hanzon, B., M. Olivi and R.L.M. Peeters (2004). Balanced realizations of discrete-time stable all-pass systems and the tangential Schur algorithm. Rapport de recherche no. RR-5111. INRIA. Sophia-Antipolis, France.

Kailath, T. (1980). *Linear Systems*. Prentice-Hall. Englewood Cliffs, NJ.

Moore, B.C. (1981). Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Trans. Autom. Control* **AC-26**, 17–32.

Ober, R.J. (1987). Balanced realizations: canonical form, parametrization, model reduction. *Int. J. Contr.* **46**, 643–670.

Ober, R.J. (1996). Balanced canonical forms. In: *Identification, Adaptation and Learning* (S. Bittante, Ed.). Springer Verlag.