BALANCED REALIZATIONS OF DISCRETE-TIME STABLE ALL-PASS SYSTEMS AND THE TANGENTIAL SCHUR ALGORITHM

R.L.M. Peeters[1], **B. Hanzon**[2], **M. Olivi**[3]

[1] *Dept. Mathematics, Universiteit Maastricht,*
P.O. Box 616, 6200 MD Maastricht, The Netherlands,
Fax: +31-43-3211889, Email: ralf.peeters@math.unimaas.nl

[2] *Dept. Econometrics, FEWEC, Vrije Universiteit,*
De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands,
Fax: +31-20-4446020, Email: bhanzon@econ.vu.nl

[3] *INRIA Sophia-Antipolis,*
B.P. 93, 06902 Sophia-Antipolis Cedex, France,
Fax: +33-4 92 38 78 58, Email: olivi@sophia.inria.fr

Keywords : All-pass systems, Balanced realizations, Overlapping parametrizations, Schur parameters, tangential Schur algorithm.

Abstract

In this paper, the connections are investigated between two different approaches towards the parametrization of multivariable stable all-pass systems in discrete-time. The first approach involves the tangential Schur algorithm, which employs linear fractional transformations. It stems from the theory of reproducing kernel Hilbert spaces and enables the direct construction of overlapping local parametrizations using Schur parameters and interpolation points. The second approach proceeds in terms of state-space realizations. In the scalar case, a balanced canonical form exists that can also be parametrized by Schur parameters. This canonical form can be constructed recursively, using unitary matrix operations. Here, this procedure is generalized to the multivariable case by establishing the connections with the first approach. It gives rise to balanced realizations and overlapping canonical forms directly in terms of the parameters used in the tangential Schur algorithm.

1 Introduction

Stable all-pass systems of finite order have several applications in linear systems theory. Within the fields of system identification, approximation and model reduction, they have been used in connection with the Douglas-Shapiro-Shields factorization, see e.g., [3, 2, 8, 5], to obtain effective algorithms for various purposes. They are one-to-one related to rational inner functions, of which the differential structure has been studied in [1]. There, a parametrization for the multivariable case has been obtained by means of a recursive procedure, the *tangential Schur algorithm*, that involves Schur parameter vectors, interpolation points and normalized direction vectors. In the scalar case, a single coordinate chart suffices to entirely describe the manifold of stable all-pass (or inner) systems of a fixed finite order. In the multivariable case, the approach leads to infinite atlases of generic charts covering these manifolds.

In another line of research, balanced state-space canonical forms have been constructed for various classes of linear systems, with special properties of these classes (such as stability) built in, see e.g., [9, 7]. Balanced realizations are well-known to have numerical advantages and are useful for model reduction purposes in conjunction with balance-and-truncate type procedures. In the constructions of [7], the case of stable all-pass systems in *continuous-time* plays a central role. In the scalar case, the resulting canonical form is balanced with a positive upper triangular reachability matrix. In the multivariable case, Kronecker indices and nice selections are used to arrive at block-balanced overlapping canonical forms. For *discrete-time* stable all-pass systems, canonical forms can be obtained from the results in continuous-time by application of a bilinear transformation. However, this destroys certain nice properties of the canonical form, e.g., truncation of state components no longer leads to reduced order systems that are balanced and in canonical form. Therefore, the ideas of [7] are directly applied in [6] to the scalar discrete-time stable all-pass case. This leads to a balanced canonical form with the desired properties, for which it turns out that it can in fact be parametrized using Schur parameters.

In this paper, the connections between these two approaches are investigated. The main technical results provide the basis for a recursive method for obtaining balanced realizations for stable all-pass systems that are parametrized directly in terms of the parameters used in the tangential Schur algorithm. This generalizes the results of [6] to the multivariable case and opens up possibilities for multivariable stable all-pass model reduction and approximation methods along the lines indicated. Due to space limitations, no proofs are given.

2 The tangential Schur algorithm and linear fractional transformations

In this section we briefly outline the use of the tangential Schur algorithm for the recursive construction of a parametrization of the space of stable all-pass systems of fixed finite order. It is derived from the method of [1, 8, 5] by relating $p \times p$ inner functions $F(z)$ to stable all-pass functions $G(z)$ via $G(z) = F(1/\overline{z})^* = F(z)^{-1}$.

By J_{2p} we denote the $2p \times 2p$ block-partitioned matrix given by

$$J_{2p} = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \tag{1}$$

with I_p the $p \times p$ identity matrix. Then a $2p \times 2p$ matrix function $\Theta(z)$ is called J_{2p}-inner if $\Theta(z) J_{2p} \Theta(z)^* \leq J_{2p}$ at all points of analyticity z inside the open unit disk, with equality at all points of analyticity on the unit circle. Likewise, a $p \times p$ function $G(z)$ is called stable all-pass if it satisfies $G(z)G(z)^* \geq I_p$ at all points of analyticity inside the open unit disk, with equality at all points of analyticity on the unit circle.

Along with each invertible $2p \times 2p$ function $\Theta(z)$ we shall associate a linear fractional transformation $\mathcal{T}_{\Theta(z)}$ that is defined to act on $p \times p$ functions $G(z)$ as follows:

$$\mathcal{T}_{\Theta(z)} : G(z) \mapsto [\Theta_4(z)G(z) + \Theta_3(z)][\Theta_2(z)G(z) + \Theta_1(z)]^{-1}, \tag{2}$$

where

$$\Theta(z) = \begin{bmatrix} \Theta_1(z) & \Theta_2(z) \\ \Theta_3(z) & \Theta_4(z) \end{bmatrix} \tag{3}$$

is block-partitioned, with each block $\Theta_i(z)$, $(i = 1, \ldots, 4)$, of size $p \times p$. Linear fractional transformations satisfy the group property $\mathcal{T}_{\Theta(z)\Phi(z)} = \mathcal{T}_{\Theta(z)} \circ \mathcal{T}_{\Phi(z)}$. If $\Theta(z)$ is J_{2p}-inner it is known that $\mathcal{T}_{\Theta(z)}$ takes stable all-pass functions again to stable all-pass functions.

If M is constant J_{2p}-unitary, its associated mapping $\mathcal{T}_M$ is a generalized Möbius transformation. It can be represented in a unique way, see [4], as

$$M = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} H(E), \tag{4}$$

where P and Q are $p \times p$ unitary matrices and $H(E)$ denotes the Halmos extension of a strictly contractive $p \times p$ matrix E (i.e., all the singular values of E are strictly less than 1). This Halmos extension $H(E)$ is the J_{2p}-unitary matrix defined by

$$H(E) = \begin{bmatrix} (I_p - EE^*)^{-\frac{1}{2}} & E(I_p - E^*E)^{-\frac{1}{2}} \\ E^*(I_p - EE^*)^{-\frac{1}{2}} & (I_p - E^*E)^{-\frac{1}{2}} \end{bmatrix} =$$
$$= \begin{bmatrix} (I_p - EE^*)^{-\frac{1}{2}} & (I_p - EE^*)^{-\frac{1}{2}}E \\ (I_p - E^*E)^{-\frac{1}{2}}E^* & (I_p - E^*E)^{-\frac{1}{2}} \end{bmatrix}. \tag{5}$$

In each recursion step of the tangential Schur algorithm, the McMillan degree of the rational stable all-pass function available, is increased by 1 by the action of a linear fractional transformation. This transformation is associated with a J_{2p}-inner matrix function $\Theta(z)$ of order 1 which has a particular form that stems from the theory of reproducing kernel Hilbert spaces:

$$\Theta(z) = \Theta(u, v, w, \xi, H; z) =$$
$$\left(I_{2p} - \frac{(1 - \overline{\xi} z)(1 - |w|^2)}{(1 - \|v\|^2)(1 - \overline{w} z)(1 - \overline{\xi} w)} \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}^* J_{2p} \right) H. \tag{6}$$

Here, the associated parameters (that may be chosen independently for each recursion step) must satisfy the following properties.
(1) $u \in \mathbf{C}^{p \times 1}$ is a normalized direction vector such that $\|u\| = 1$. (2) $v \in \mathbf{C}^{p \times 1}$ is a (generalized) Schur parameter vector satisfying $\|v\| < 1$. (3) w is an interpolation point with $|w| < 1$. (4) ξ is a point with $|\xi| = 1$. (5) H is a constant J_{2p}-unitary matrix.

At the point $z = \xi$ it holds that $\Theta(\xi) = H$. From the structure of $\Theta(z)$ it also follows that $\tilde{G}(z) = \mathcal{T}_{\Theta(z)}(G(z))$ satisfies the interpolation condition

$$\tilde{G}(\overline{w}^{-1})u = v. \tag{7}$$

In the standard case with $w = 0$ the value of $\tilde{G}(\infty)$ corresponds to the direct feedthrough term $\tilde{D}$ of any state-space realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of $\tilde{G}(z)$, so that the interpolation condition then takes the form $\tilde{D}u = v$.

The set of values for the parameters u, w, ξ and H at the first n recursion steps can serve to index a generic chart for the manifold of stable all-pass systems of order n. The Schur parameter vectors v together with an initial unitary matrix, say G_0, then provide the local coordinates for this chart. An infinite atlas of overlapping generic charts is obtained by varying the choices for u, w, ξ and H.

The following proposition plays a central role in our construction of balanced parametrizations for discrete-time stable all-pass transfer functions.

Proposition 2.1
The J_{2p}-inner matrix function $\Theta(u,v,w,\xi,H;z)$ can be factorized as:

$$\Theta(u,v,w,\xi,H;z) = H(uv^*)S_{u,w}(z)S_{u,w}(\xi)^{-1}H(uv^*)^{-1}H, \tag{8}$$

where $H(uv^*)$ denotes the J_{2p}-unitary Halmos extension of the strictly contractive matrix uv^* and where $S_{u,w}(z)$ is defined as the J_{2p}-inner matrix function

$$S_{u,w}(z) = \begin{bmatrix} I_p - \left(1 - \frac{(z-w)}{(1-\overline{w}z)}\right)uu^* & 0 \\ 0 & I_p \end{bmatrix}. \tag{9}$$

Note: for $\|u\| = 1$ and $\|v\| < 1$, the matrix uv^* is indeed strictly contractive, with Halmos extension given by:

$$H(uv^*) =$$
$$= \begin{bmatrix} I_p - (1 - \frac{1}{\sqrt{1-\|v\|^2}})uu^* & \frac{1}{\sqrt{1-\|v\|^2}}uv^* \\ \frac{1}{\sqrt{1-\|v\|^2}}vu^* & I_p - (1 - \frac{1}{\sqrt{1-\|v\|^2}})\frac{vv^*}{\|v\|^2} \end{bmatrix}. \tag{10}$$

Its inverse satisfies $H(uv^*)^{-1} = H(-uv^*)$.

Note that the linear fractional transformations associated with the constant J_{2p}-unitary matrices H, $H(uv^*)$, $H(uv^*)^{-1}$ and $S_{u,w}(\xi)^{-1}$ are all generalized Möbius transformations which do not change the order (i.e., McMillan degree) of the matrix functions on which they act. Only the transformation associated with the matrix $S_{u,w}(z)$ effectuates an order increase by 1, but it has a simple form that does not involve the Schur parameter vector v, nor ξ, nor H.

3 Recursive construction of balanced state-space realizations

We now turn towards the second approach for parametrizing the space of stable all-pass systems, now in terms of balanced state-space realizations. We start by introducing some more notation.

With each pair (U,V) of $(p+1)\times(p+1)$ matrices we associate a mapping $\mathcal{F}_{U,V}$ that is defined to act on $p \times p$ matrix functions $G(z)$ as follows.

$$\mathcal{F}_{U,V} : G(z) \mapsto F_1(z) + \frac{F_2(z)F_3(z)}{z - F_4(z)}, \tag{11}$$

with $F_1(z)$ of size $p \times p$, $F_2(z)$ of size $p \times 1$, $F_3(z)$ of size $1 \times p$ and $F_4(z)$ scalar defined by:

$$\begin{bmatrix} F_1(z) & F_2(z) \\ F_3(z) & F_4(z) \end{bmatrix} = V \begin{bmatrix} 1 & 0 \\ 0 & G(z) \end{bmatrix} U^*. \tag{12}$$

When a state-space realization (A,B,C,D) of a transfer function $G(z)$ is available, then a state-space realization of $\tilde{G}(z) = \mathcal{F}_{U,V}(G(z))$ can be obtained by working directly on the associated 'realization matrix' $\begin{bmatrix} D & C \\ B & A \end{bmatrix}$. This is the contents of the following proposition.

Proposition 3.1 *Let $G(z)$ be a $p \times p$ proper rational transfer function of finite McMillan degree, such that $\tilde{G}(z) = \mathcal{F}_{U,V}(G(z))$ is well-defined. Let (A,B,C,D) be a state-space realization of $G(z)$ with n-dimensional state-space. Then a state-space realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ of $\tilde{G}(z)$ with $(n+1)$-dimensional state-space is given by:*

$$\begin{bmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix}. \tag{13}$$

In case of a minimal balanced realization (A,B,C,D) of a $p \times p$ discrete-time stable all-pass transfer function $G(z)$, it is well-known (see [6]) that the associated realization matrix $\begin{bmatrix} D & C \\ B & A \end{bmatrix}$ is unitary. Conversely, if the realization matrix associated with a state-space realization of a transfer function $G(z)$ is unitary, then the transfer function is stable all-pass. If in addition it is minimal, then the realization is balanced.

From these observations and (13) it is immediate that for *unitary* matrices U and V, the mapping $\mathcal{F}_{U,V}$ takes stable all-pass transfer functions of order (i.e., McMillan degree) n into stable all-pass transfer functions of order $\leq n+1$.
In the scalar case, mappings of the form $\mathcal{F}_{U,V}$ with $U = I_2$ have been used in [6] to recursively construct a balanced canonical form for the space of discrete-time stable all-pass systems of finite McMillan degree. The parameters that occur in this recursion have the interpretation of Schur parameters, corresponding to the situation with interpolation points w at zero. With this connection in mind, it is the purpose of the following section to clarify the relationship between the two classes of mappings $\mathcal{T}_{\Theta(z)}$ and $\mathcal{F}_{U,V}$, with $\Theta(z)$ J_{2p}-inner of order 1 and U and V unitary.

4 Connection between the classes of mappings $\mathcal{T}_{\Theta(z)}$ and $\mathcal{F}_{U,V}$

We are interested in investigating the possibilities for representing a mapping $\mathcal{T}_{\Theta(z)}$ in terms of a corresponding mapping $\mathcal{F}_{U,V}$. This would give us balanced state-space parametrizations directly in terms of the set of parameters u, v, w, ξ and H used in the tangential Schur algorithm. Moreover, unitary matrices would be involved in the computation of these realizations, and these are known to be numerically well-conditioned.

It will prove to be be essential to introduce the J_{2p}-inner matrix function $\hat{\Theta}(u,v,w;z)$, defined by

$$\hat{\Theta}(u,v,w;z) = H(uv^*)S_{u,w}(z)H(\overline{w}uv^*). \tag{14}$$

The main result of this paper can now be stated as follows.

Theorem 4.1 *Let $u,v \in \mathbf{C}^{p\times 1}$ and $w \in \mathbf{C}$ such that $\|u\| = 1$, $\|v\| < 1$ and $|w| < 1$. Then for all $p \times p$ proper* *rational stable all-pass functions $G(z)$ of finite McMillan degree it holds that*

$$\mathcal{T}_{\hat{\Theta}(u,v,w;z)}(G(z)) = \mathcal{F}_{U,V}(G(z)), \tag{15}$$

if U and V are taken to be the unitary $(p+1)\times(p+1)$ matrices

$$U = \begin{bmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}u & I_p - (1+\frac{w\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}})uu^* \\ \frac{\overline{w}\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}u^* \end{bmatrix}, \tag{16}$$

$$V = \begin{bmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}v & I_p - (1-\frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}})\frac{vv^*}{\|v\|^2} \\ \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & -\frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}v^* \end{bmatrix}. \tag{17}$$

Note that, according to Prop. 2.1, $\Theta(u,v,w,\xi,H;z)$ is of the form $\hat{\Theta}(u,v,w;z)M$, with the matrix $M = \hat{\Theta}(u,v,w;\xi)^{-1}H$ constant and J_{2p}-unitary. As remarked before, see Eqn. (4), M can be parametrized as $M = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} H(E)$ with unique, unitary matrices P and Q and a unique strictly contractive matrix E. The following proposition indicates for which matrices M the mapping $\mathcal{T}_{\hat{\Theta}(u,v,w;z)M}(G(z))$ can be represented in the form $\mathcal{F}_{U,V}(G(z))$.

Proposition 4.2 *Let $u,v \in \mathbf{C}^{p\times 1}$ and $w \in \mathbf{C}$ such that $\|u\| = 1$, $\|v\| < 1$ and $|w| < 1$. Let $M = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} H(E)$ be J_{2p}-unitary, with P and Q $p\times p$ unitary and E $p\times p$ strictly contractive. Then the mapping $\mathcal{T}_{\hat{\Theta}(u,v,w;z)M}(G(z))$ can be represented as a mapping $\mathcal{F}_{U,V}(G(z))$ if and only if $E = 0$.*

This proposition makes clear that in general it is impossible to carry out a full recursion step of the tangential Schur algorithm by performing a mapping of the form $\mathcal{F}_{U,V}(G(z))$. However, if the action of a generalized Möbius transformation $\mathcal{T}_M$ can be carried out in terms of state-space realization matrices in some other way, then the equivalence between $\mathcal{T}_{\hat{\Theta}(u,v,w;z)}$ and $\mathcal{F}_{U,V}$ can still be very useful. This is achieved in the following theorem. It describes how the action of a generalized Möbius transformation can be carried out in terms of state-space realization matrices, with the additional property that balancedness of a realization is maintained.

Theorem 4.3 *Let $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ be a partitioned J_{2p}-unitary matrix, with blocks of size $p\times p$. Let $\mathcal{T}_M$ be its associated generalized Möbius transformation. Let $G(z)$ be a proper rational $p\times p$ discrete-time stable all-pass transfer function of finite McMillan degree n, with minimal state-space realization (A,B,C,D). Then $\tilde{G}(z) = \mathcal{T}_M(G(z))$ is well-defined, i.e., $(M_2G(z)+M_1)^{-1}$ exists. Moreover, $\tilde{G}(z)$ is again a $p\times p$ discrete-time stable all-pass transfer function of order n. A minimal state-space realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ for $\tilde{G}(z)$ is given by:*

$$\tilde{A} = A - B(M_2D+M_1)^{-1}M_2C, \tag{18}$$
$$\tilde{B} = B(M_2D+M_1)^{-1}, \tag{19}$$
$$\tilde{C} = [M_4 - (M_4D+M_3)(M_2D+M_1)^{-1}M_2]C, \tag{20}$$
$$\tilde{D} = (M_4D+M_3)(M_2D+M_1)^{-1}. \tag{21}$$

If in addition (A,B,C,D) is balanced, then $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is also balanced.

From Prop. 4.2 it follows that there exist linear fractional transformations that cannot be written as a mapping $\mathcal{F}_{U,V}$. Conversely, if U and V are unitary then it turns out to depend on the modulus of the left bottom corner entries of U and V whether the mapping $\mathcal{F}_{U,V}$ allows for a representation of the form $\mathcal{T}_{\Theta(u,v,w,\xi,H;z)}$, corresponding to the tangential Schur algorithm, or not. The precise result is as follows.

Proposition 4.4 *Let (U,V) be a pair of $(p+1)\times(p+1)$ unitary matrices. If $p>1$, then the mapping $\mathcal{F}_{U,V}$ can be represented in the form of a mapping $\mathcal{T}_{\Theta(u,v,w,\xi,H;z)}$ if and only if the modulus of the left bottom corner entry of U is strictly less than the modulus of the left bottom corner entry of V. If $p=1$ such a representation exists if and only if these two entries have different modulus.*

If (U,V) is such that $\mathcal{F}_{U,V}$ cannot be written in the form of a mapping $\mathcal{T}_{\Theta(u,v,w,\xi,H;z)}$, then in many cases there still exist linear fractional transformations that do the job. (However, they are not of the form used in the tangential Schur algorithm.) It can be shown that if (and only if) the left bottom corner entries of U and V are both zero, then the mapping $\mathcal{F}_{U,V}$ is not of the linear fractional type.

5 Concluding remarks

There are some differences between the scalar and the multivariable case; see, e.g., Prop. 4.4. We do not further go into detail about this, here. From Thms. 4.1 and 4.3 it is straightforward how the tangential Schur algorithm can be supplied with balanced parametrizations of discrete-time stable all-pass systems. By making special choices for the parameters that index the generic charts, additional properties can be obtained. If the interpolation points w are all chosen at the origin, and the direction vectors u are selected from the set of standard basis vectors in a special way, depending on the Kronecker structure of the system to be parametrized, then the balanced realization can be tailored (analogous to the continuous-time case, see [7]), e.g., for model reduction purposes of the balance-and-truncate type.

References

[1] D. Alpay, L. Baratchart and A. Gombani, On the differential structure of matrix-valued rational inner functions, *Operator Theory: Adv. and Appl.*, **73**, 30–66, (1994).

[2] L. Baratchart, On the topological structure of inner functions and its use in identification, in: B. Bonnard, B. Bride, J.P. Gauthier and I. Kupka (eds.), *Analysis of Controlled Dynamical Systems*, Progress in Systems and Control Theory, Vol. **8**, Birkhäuser, 51–59, (1991).

[3] R. Douglas, H. Shapiro and A. Shields, Cyclic vectors and invariant subspaces for the backward shift operator, *Annales de l'Institut Fourier, (Grenoble)*, **20**, 37–76, (1970).

[4] V. Dubojov, B. Fritzsche and B. Kirstein, *Matricial Version of the Classical Schur Problem*, Teubner Verlagsgesellschaft, (1992).

[5] P. Fulcheri and M. Olivi, Matrix Rational H_2 Approximation: A Gradient Algorithm Based on Schur Analysis, *SIAM J. Contr. and Optimiz.*, **36**, 2103–2127, (1998).

[6] B. Hanzon and R.L.M. Peeters, Balanced Parametrizations of Stable SISO All-Pass Systems in Discrete-Time, *To appear in MCSS.*

[7] B. Hanzon and R.J. Ober, Overlapping block-balanced canonical forms for various classes of linear systems, *Lin. Alg. Appl.*, **281**, 171–225, (1998).

[8] J. Leblond and M. Olivi, Weighted H^2 Approximation of Transfer Functions, *MCSS*, **11**, 28–39, (1998).

[9] R.J. Ober, Balanced realizations: canonical form, parametrization, model reduction, *Int. J. Control*, **46**, 643–670, (1987).

[10] V.P. Potapov, The multiplicative structure of J-contractive matrix functions, *Amer. Math. Soc. Transl.* (2), **15**, 131–243, (1960).

[11] V.P. Potapov, Linear Fractional Transformations of Matrices, *Amer. Math. Soc. Transl.* (2), **138**, 21–35, (1988).