The Laplace transform in control theory.

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1 Introduction

Laplace transform is extensively used in control theory. It appears in the description of linear time-invariant systems, where it changes convolution operators into multiplication operators and allows to define the transfer function of a system. The properties of systems can be then translated into properties of the transfer function. In particular, causality implies that the transfer function must be analytic in a right half-plane. This will be explained in section 2 and a good reference for these preliminary properties and for a panel of concrete examples is [11].

Via Laplace transform, functional analysis provides a framework to formulate, discuss and solve problems in control theory. This will be sketched in section 3, in which the important notion of stability is introduced. We shall see that several kind of stability, with different physical meaning can be considered in connection with some function spaces, the Hardy spaces of the half-plane. These functions spaces provide with their norms a measure of the distance between transfer functions. This allows to translate into well-posed mathematical problems some important topics in control theory, as for example the notion of robustness. A design is robust if it works not only for the postulated model, but also for neighboring models. We may interpret closeness of models as closeness of their transfer functions.

In section 4, we review the main properties of finite order linear time-invariant (LTI) causal systems. They are described by state-space equations and their transfer function is rational. We give the definition of the McMillan degree or order of a system, which is a good measure of its complexity, and some useful factorizations of a rational transfer function, closely connected with its pole and zero structure. Then, we consider the past inputs to future outputs map, which provides a nice interpretation of the notions of controllability and observability and we define the Hankel singular values. As claimed by Glover in [6], the Hankel singular

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values are extremely informative invariants when considering system complexity and gain. For this section we refer the reader to [8] and [6].

Section 5 is concerned with system identification. In many areas of engineering, high-order linear state-space models of dynamic systems can be derived (this can already be a difficult problem). By this way, identification issues are translated into model reduction problems that can be tackle by means of rational approximation. The function spaces introduced in section 3 provide with their norms a measure of the accuracy of a model. The most popular norms are the Hankel-norm and the L^2-norm. In these two cases, the role of the Hardy space H^2 with its Hilbert space structure, is determinant in finding a solution to the model reduction problem. In the case of the Hankel norm, explicit solutions can be found [6] while in the L^2 case, local minima can be numerically computed using gradient flow methods. Note that the approximation in L^2 norm has an interpretation in stochastic identification: it minimizes the variance of the output error when the model is fed by a white noise. These approximation problems are also relevant in the design of controllers which maximize robustness with respect to uncertainty or minimize sensitivity to disturbances of sensors, and other problems from H^∞ control theory. For an introduction to these fields we refer the reader to [4].

In this paper, we are concerned with continuous-time systems for which Laplace transform is a valuable aid. The z-transform performs the same task for discrete-time systems. This is the object of [3] in the framework of stochastic systems. It must be noted that continuous-time and discrete-time systems are related through a Möbius transform which preserves the McMillan degree [6]. For some purposes, it must be easier to deal with discrete-time. In particular, the poles of stable discrete-time systems lay in a bounded domain the unit circle. Laplace transform is also considered among other transforms in [12]. This paper also provides an introduction to [2].

2 Linear time-invariant systems and their transfer functions.

Linear time-invariant systems play a fundamental role in signal and system analysis. Many physical processes possess these properties and even for nonlinear systems, linear approximations can be used for the analysis of small derivations from an equilibrium. Laplace transform has a number of properties that makes it useful for analysing LTI systems, thereby providing a set of powerful tools that form the core of signal and system analysis.

A continuous-time system is an "input-output" map

$$u(t) \to y(t),$$

from an input signal $u : \mathbb{R} \to \mathbb{C}^m$ to an output signal $y : \mathbb{R} \to \mathbb{C}^p$. It will be called linear if the map is linear and time-invariant if a time shift in the input signal results in an identical time shift in the output signal.

A linear time-invariant system can be represented by a convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau,$$

in terms of its response to a unit impulse [11]. The $p \times m$ matrix function h is called the *impulse response* of the system.

The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with a change of amplitude. Indeed, for an input of the form $u(t) = e^{st}$, the output computed through the convolution integral will be

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Assuming that the integral converges, the response to e^{st} is of the form

$$y(t) = H(s)e^{st},$$

where $H(s)$ is the Laplace transform of the impulse response $h(t)$ defined by

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

In the specific case in which $\Re\{s\} = 0$, the input is a complex integral $e^{i\omega t}$ at frequency ω and $H(i\omega)$, viewed as a function of ω, is known as the *frequency response* of the system and is given by the Fourier transform

$$H(i\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-i\omega\tau}d\tau.$$

In practice, pointwise measurements of the frequency response are often available and the classical problem of *harmonic identification* consists in finding a model for the system which reproduces these data well enough.

The *Laplace transform* of a scalar function $f(s)$

$$\mathcal{L}f(s) = \int_{-\infty}^{\infty} e^{-st}f(t)dt$$

is defined for those $s = x + i\,y$ such that

$$\int_{-\infty}^{\infty} |f(\tau)|e^{-x\tau}d\tau < \infty.$$

The range of values of s for which the integral converges is called the region of convergence. It consists of strips parallel to the imaginary axis. In particular, if $f \in L^1(\mathbb{R})$, i.e.

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty,$$

then $\mathcal{L}f$ is defined on the imaginary axis and the Laplace transform can be viewed as a generalization of the Fourier transform.

Another obvious and important property of the Laplace transform is the following. Assume that $f(t)$ is right-sided, i.e. $f(t) = 0, \quad t < T$, and that the Laplace transform of f converges for $\Re\{s\} = \sigma_0$. Then, for all s such that $\Re\{s\} = \sigma > \sigma_0$, we have that

$$\int_{-\infty}^{\infty} |f(\tau)| e^{-\sigma\tau} d\tau = \int_{T}^{\infty} |f(\tau)| e^{-\sigma\tau} d\tau \leq e^{-(\sigma-\sigma_0)T} \int_{T}^{\infty} |f(\tau)| e^{-\sigma_0\tau} d\tau,$$

and the integral converges so that Laplace transform is well defined in $\Re\{s\} \geq \sigma_0$. If $f \in L^1(\mathbb{R})$, then the Laplace transform is defined on the right half-plane and it can be proved that it is an analytic function there. It is possible that for some right-sided signal, there is no value of s for which the Laplace transform will converge. One example is the signal $h(t) = 0, \quad t < 0$ and $h(t) = e^{t^2}, \quad t \geq 0$.

The importance of Laplace transform in control theory is mainly due to the fact that it allows to express any LTI system

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau) u(\tau) d\tau$$

has a multiplication operator

$$Y(s) = H(s)U(s),$$

where

$$Y(s) = \int_{-\infty}^{\infty} y(\tau) e^{-s\tau} d\tau, \quad H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau, \quad U(s) = \int_{-\infty}^{\infty} u(\tau) e^{-s\tau} d\tau,$$

are the Laplace transforms. The $p \times m$ matrix function $H(s)$ is called the *transfer function* of the system.

Causality is a common property for a physical system. A system is causal if the output at any time depends only on the present and past values of the input. A LTI system is causal if its impulse response satisfies

$$h(t) = 0 \quad \text{for } t < 0,$$

and in this case, the output is given by the convolution integral

$$y(t) = \int_{0}^{\infty} h(\tau) u(t-\tau) d\tau = \int_{0}^{t} h(t-\tau) u(\tau) d\tau.$$

Then, the transfer function of the system is defined by the *unilateral Laplace transform*

$$H(s) = \int_0^\infty h(\tau)e^{-s\tau}d\tau, \tag{1}$$

whose region of convergence is but what precedes a right half-plane (if it is not empty). In the sequel, we shall restrict ourselves to causal systems.

Of course our signals must satisfy some conditions to ensure the existence of the Laplace transforms. There are many way to proceed. We shall require our signals to belong to some spaces of integrable functions and this is closely related to the notion of stability of a system. This will be the object of the next section. Via Laplace transform, properties of an LTI system can be expressed in terms of the transfer function and by this way, function theory brings insights in control theory.

3 Function spaces and stability.

An undesirable feature of a physical device is instability. In this section, we translate this into a statement about transfer functions. Intuitively, a stable system is one in which small inputs lead to responses that do not diverge. To give a mathematical statement, we need a measure of the size of a signal which will be provided by appropriate function spaces.

We denote by $L^q(X)$ the space of complex valued measurable functions f on X satisfying

$$\|f\|_q^q = \int_X |f(t)|^q dt < \infty, \quad \text{if } 1 \leq q < \infty,$$

$$\|f\|_\infty = \sup_X |f(t)| < \infty, \quad \text{if } q = \infty.$$

The most natural measure is the L^∞ norm. A signal will be called bounded if there is some $M > 0$ such that

$$\|u\|_\infty = \sup_{t>0} \|u(t)\| < M,$$

where $\|.\|$ denotes the Euclidean norm of a vector. We still denote by $L^\infty(0,\infty)$ the space of bounded signals, omitting to mention the vectorial dimension. A system will be called *BIBO stable* if a bounded input produces a bounded output.

We may also be interested in the energy of a system which is given by the integral

$$\|u\|_2^2 = \int_0^\infty u(t)^* u(t)dt.$$

We still denote by $L^2(0,\infty)$ the space of signal with bounded energy.

Notions of stability are associated with the requirement that the convolution operator

$$u(t) \to y(t) = h * u(t),$$

is a bounded linear operator, the input and output spaces being endowed with some (may be different) norms. This implies that the transfer functions of such stable systems belong to some spaces of analytic functions, the Hardy spaces of the right half-plane [7]. We first introduce these spaces.

3.1 Hardy spaces of the half-plane.

The Hardy space H^p is defined to be the space of functions $f(s)$ analytic in the right half-plane which satisfy

$$\|f\|_p := \sup_{0<x<\infty} \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p dy \right\}^{1/p} < \infty,$$

when $1 \le p < \infty$, and, when $p = \infty$,

$$\|f\|_\infty := \sup_{\Re\{s\}>0} |f(s)| < \infty.$$

A theorem of Fatou says that, for any $f \in H^p$, $1 \le p \le \infty$,

$$f_0(iy) = \lim_{x \to 0+} f(x+iy),$$

exits a.e. on the imaginary axis. We may identify $f \in H^p$ with $f_0 \in L^p(i\mathbb{R})$ and the identification is isometric, so that we may consider H^p as a subspace of $L^p(i\mathbb{R})$. The case $p = 2$ is of particular importance since H^2 is an Hilbert space. We denote by H^2_- the left half-plane analog of H^2 : that is $f \in H^2_-$ if and only if the function $s \to f(-s)$ is in H^2. We may also consider H^2_- as a subspace of $L^2(i\mathbb{R})$. We denote by Π^+ and Π^- the orthogonal projections from $L^2(i\mathbb{R})$ to H^2 and H^2_- respectively, and we have

$$L^2(i\mathbb{R}) = H^2 \oplus H^2_-.$$

If $f \in L^1(0,\infty)$, then $\mathcal{L}f$ is defined and analytic on the right half-plane. Moreover, we may extend the definition to functions $f \in L^2(0,\infty)$, since $L^1(0,\infty) \cup L^2(0,\infty)$ is dense in $L^2(0,\infty)$. The Laplace transform of a function $f \in L^2(0,\infty)$ is again defined and analytic on the right half-plane and we have the following theorem [13, Th.1.4.5]

Theorem 1 *The Laplace transform gives the following bijections*

$$\mathcal{L} : L^2(0,\infty) \to H^2,$$

$$\mathcal{L} : L^2(-\infty, 0) \to H^2_-,$$

and for $f \in L^2(0,\infty)$ *(resp.* $L^2(-\infty,0)$*)*

$$\|\mathcal{L}f\|_2 = \sqrt{2\Pi}\|f\|_2.$$

Since we are concerned with multi-input and multi-output systems, vectorial and matricial versions of these spaces are needed. For $p, m \in \mathbb{N}$, $H^\infty_{p\times m}$ and $H^2_{p\times m}$ are the spaces of $p \times m$ matrix functions with entries in H^∞ and H^2 respectively endowed with the norm

$$\|F\|_\infty = \sup_{-\infty<w<\infty} \|F(iw)\| \quad (2)$$

$$\|F\|_2^2 = \mathrm{Tr} \int_{-\infty}^{\infty} F(iw)^* F(iw) dw, \quad (3)$$

where $\|.\|$ denotes the Euclidean norm for a vector and for a matrix, the operator norm or spectral norm (that is the largest singular value). We shall often write H^∞, H^2 etc. for $H^\infty_{p\times m}$ and $H^2_{p\times m}$, the size of the matrix or vector functions (case $m = 1$) being understood from the context.

Remark. Note that the following inclusions hold: $H^\infty \subset H^2 \subset H^1$.

3.2 Some notions of stability.

We shall study the notions of stability which arises from the following choices of norm on the input and output function spaces:

- **stability $L^\infty \to L^\infty$ (BIBO).** A system is BIBO stable if and only if its impulse response is integrable over $(0,\infty)$. Indeed, if $h(t)$ is integrable and $\|u\|_\infty < M$, then

$$\begin{aligned} \|y(t)\| &\leq M \int_0^t \|h(t-\tau)\| d\tau \\ &= M \int_0^t \|h(\tau)\| d\tau, \\ &\leq M \int_0^\infty \|h(\tau)\| d\tau, \end{aligned}$$

  and $y(t)$ is bounded. Conversely, if $h(t)$ is not integrable, a bounded input can be constructed which produces an unbounded output (see [13] in the SISO case and [1, Prop.23.1.1] in the MIMO case).

- **stability $L^2 \to L^2$.** By Theorem 1 $\sqrt{2\Pi}\,\mathcal{L}$ is a unitary operator from $L^2(0,\infty)$ onto the Hardy space H^2. Thus a system

$$y(t) = h * u(t),$$

will be $L^2 \to L^2$ stable if its transfer function H is a bounded operator from H^2 to H^2. Now, the transfer function is a multiplication operator

$$M_H : U(s) \to Y(s),$$

whose operator norm is $\|H\|_\infty$ given by (2) and H must belong to the Hardy space H^∞.

- **stability $L^2 \to L^\infty$.** The interest of this notion of stability comes from the fact that it requires that the transfer function $H(s)$ belongs to the Hardy space H^2 which is an Hilbert space. Indeed, it can be proved that the impulse response of such a stable system must be in $L^2(0,\infty)$ and thus by Theorem 1 its transfer function must be H^2.

4 Finite order LTI systems and their rational transfer functions.

Among LTI systems, of particular interest are the systems governed by differential equations

$$\begin{array}{l} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t), \end{array} \tag{4}$$

where A, B, C, D are constant complex matrices matrices of type $n \times n$, $n \times m$, $p \times n$ and $p \times m$, and $x(t) \in \mathbb{C}^n$ is the state of the system. Assuming $x(0) = 0$, the solution is

$$\begin{array}{l} x(t) = \int_0^t e^{(t-\tau)A} Bu(\tau)d\tau, \quad t \geq 0 \\ y(t) = \int_0^t Ce^{(t-\tau)A} Bu(\tau)d\tau + Du(t), \quad t \geq 0 \end{array}$$

and the impulse response given by

$$g(t) = Ce^{At}B + D\delta_0,$$

where δ_0 is the delta function or Dirac measure at 0. Thus g is a generalized function.

As previously, we denote by the capital roman letter the Laplace transform of the function designated by the corresponding small letter. Laplace transform possesses the nice property to convert differentiation into a shift operator

$$\mathcal{L}\dot{x}(s) = sX(s).$$

so that the system (4) takes the form

$$\begin{aligned} sX(s) &= A\,X(s) + B\,U(s) \\ Y(s) &= C\,X(s) + D\,U(s), \end{aligned} \tag{5}$$

and yields

$$Y(s) = [D + C(sI - A)^{-1}B]U(s),$$

where $G(s) = D + C(sI - A)^{-1}B$ is the transfer function of the system. It is remarkable that transfer functions of LTI systems are rational.

Conversely, if the transfer function of a LTI system is rational and proper (its value at infinity is finite), then it can be written in the form (see [1])

$$G(s) = D + C(sI - A)^{-1}B.$$

We call (A, B, C, D) a realization of G and the system then admits a "state-space representation" of the form (4). A rational transfer function has many realizations. If T is a non-singular matrix, then $(TAT^{-1}, TB, T^{-1}C, D)$ is also a realization of $G(s)$. A *minimal realization* of G is a realization in which the size of A is minimal among all the realizations of G. The size n of A in a minimal realization is called the *McMillan degree* of $G(s)$. It represents the minimal number of state variables and is a measure of the complexity of the system.

For finite order systems all the notions of stability agree: a system is stable if and only if all the eigenvalues of A lie in the left half-plane.

To end with this section, we shall answer to some natural questions concerning these rational matrix functions: what is a pole? a zero? their multiplicity ? what could be a fractional representation?

Let $G(s)$ be a rational $p \times m$ matrix function. Then $G(s)$ admits the Smith form

$$G(s) = U(s)D(s)V(s),$$

where $U(s)$ and $V(s)$ are square size polynomial matrices with constant non-zero determinant and $D(s)$ is a diagonal matrix

$$D(s) = \mathrm{diag}\left(\frac{\phi_1}{\psi_1}, \frac{\phi_2}{\psi_2}, \ldots, \frac{\phi_r}{\psi_r}, 0, \ldots, 0\right)$$

in which for $i = 1, \ldots r$, ϕ_i and ψ_i are polynomials satisfying the divisibility conditions

$$\begin{aligned} &\phi_1/\phi_2/\ldots/\phi_r, \\ &\psi_r/\psi_{r-1}/\ldots/\psi_1. \end{aligned}$$

This representation exhibits the *pole-zeros structure* of a rational matrix. A zero of $G(s)$ is a zero of at least one of the polynomial ϕ_i. The multiplicity of a given zero in each of the ϕ_i is called a partial multiplicity and the sum of the partial multiplicities is the multiplicity of the zero. In the same way, the poles of $G(s)$ are the zeros of the ψ. They are also the eigenvalues of the dynamic matrix A. It must be notice that a complex number can be a pole and a zero at the same time. For more details on that Smith form, see [8]. It provides a new interpretation of the McMillan degree as the number of poles of the rational function counted with multiplicity, i.e. the degree of $\psi = \psi_1\psi_2\cdots\psi_r$.

The Smith form also allows to write a left coprime *polynomial factorization* (see [1, Chap.11] or [8]) of the form

$$G(s) = D(s)^{-1}N(s),$$

where $D(s)$ and $N(s)$ are left coprime polynomial matrices, i.e.

$$D(s)E_1(s) + N(s)E_2(s) = I, \quad s \in \mathbb{C},$$

for some polynomial matrices $E_1(s)$ and $E_2(s)$. In this factorization the matrix $D(s)$ brings the pole structure of $G(s)$ and the matrix $N(s)$ its zero structure.

This representation is very useful in control theory. In our function spaces context another factorization is more natural. It is the *inner-unstable or Douglas-Shapiro-Shields factorization*

$$G(s) = Q(s)P(s),$$

where $Q(s)$ is an inner function in H^∞, i.e. such that

$$Q(iw)^*Q(iw) = I, \quad w \in \mathbb{R},$$

and $P(s)$ is unstable (analytic in the left half-plane). We shall also require this factorization to be minimal. It is then unique up to a common left constant unitary matrix and the McMillan degree of Q is the McMillan degree of G. The existence of such a factorization follows from Beurling theorem on shift invariant subspaces of H^2 [5]. Here again, the inner factor brings the pole structure of the transfer function and the unstable factor the zero structure. In many approximation problems this factorization allows to reduce the number of optimization parameters, since the unstable factor can often be computed from the inner one. This makes the interest of inner function together with the fact that inner functions are the transfer function of conservative systems.

4.1 Controllability, observability and associated gramians.

The notions of controllability and observability are central to the state-space description of dynamical systems. Controllability is a measure for the ability to use a system's external inputs to manipulate its internal state. Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs.

The following facts are well-known [8]. A system described by a state-space realization (A, B, C, D) is controllable if the pair (A, B) is controllable, i.e. the matrix

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

has rank n, and the pair (C, A) observable, i.e. the matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n. A realization is minimal if and only if it is both controllable and observable. Note that the matrix D play no role in this context.

We now give an alternative description of these notions which is more adapted to our functional framework [6, Sect.2]. If the eigenvalues of A are assumed to be strictly in the left half-plane, then we can define the controllability gramian as

$$P = \int_0^\infty e^{At} BB^* e^{A^*t} dt,$$

and the observability gramian as

$$Q = \int_0^\infty e^{A^*t} C^* C e^{At} dt.$$

It is easily verified that P and Q satisfy the following Lyapunov equations

$$\begin{aligned} AP + PA^* + B^*B = 0, \\ A^*Q + QA + C^*C = 0. \end{aligned}$$

A standard result is that the pair (A, B) is controllable if and only if P is positive definite and the pair (C, A) observable if and only if Q is positive definite.

These gramians can be illustrated by considering the mapping from the past inputs to the future outputs, $\gamma_g : L^2(-\infty, 0) \to L^2(0, \infty)$, given by

$$(\gamma_g u)(t) = \int_{-\infty}^0 Ce^{A(t-\tau)} Bu(\tau) d\tau = \int_0^\infty Ce^{A(t+\tau)} Bv(\tau) d\tau, \tag{6}$$

where $v(t) = u(-t)$ is in $L^2(0, \infty)$. The mapping γ_g can be view as a composition of two mappings:

$$u(t) \to x(0) = \int_0^\infty e^{A\tau} Bu(-\tau) d\tau,$$

and

$$x(0) \to y(t) = Ce^{At}x(0),$$

where $x(0)$ is the state at time $t = 0$. Now, consider the following minimum energy problem

$$\min_{u \in L^2(-\infty,0)} \|u\|_2^2 \quad \text{subject to} \quad x(0) = x_0.$$

Since x_0 is a linear function of $u(t)$, the solution $\hat{u}$ exists provided that P is positive definite and is given by the pseudo-inverse

$$\hat{u}(t) = B^* e^{-A^* t} P^{-1} x_0.$$

It satisfies

$$\|\hat{u}\|_2^2 = x_0^* P^{-1} x_0.$$

If P^{-1} is large, there will be some state that can only be reached if a large input energy is used. If the system is realized from $x(0) = x_0$ with $u(t) = 0, \quad t \geq 0$ then

$$\|y\|_2^2 = x_0^* Q x_0,$$

so that, if the observability gramian Q is nearly singular then some initial conditions will have little effect on the output.

4.2 Hankel singular values and Hankel operator.

We now introduce the Hankel singular values which turn out to be fundamental invariants of a linear system related to both gain and complexity [6]. The link with complexity will be further illustrated in section 5.1.

The problem of approximating a matrix by a matrix of lower rank was one of the earliest application of the singular-value decomposition ([10], see [6, Prop.2.2] for a proof).

Proposition 1 *Let $M \in \mathbb{C}^{p \times m}$ have singular value decomposition given by*

$$M = UDV,$$

where U and V are square unitary and $D = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}$, $D_r = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r)$, where $\alpha_1 \geq \alpha_2 \ldots \geq \alpha_r > 0$ are the singular values of M. Then,

$$\inf_{\operatorname{rank} \hat{M} \leq k} \|M - \hat{M}\| = \alpha_{k+1},$$

and the bound is achieved by

$$\hat{D}_k = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}, \quad D_k = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_k).$$

This result can be generalized to the case of a bounded linear operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ from an Hilbert space $\mathcal{H}$, to another, $\mathcal{K}$. For $k = 0, 1, 2, \ldots$, the kth *singular value* $\sigma_k(T)$ of T is defined by

$$\sigma_k(T) = \inf\{\|T - R\|, R \in \mathcal{L}(\mathcal{H}, \mathcal{K}), \quad \operatorname{rank} R \leq k\}.$$

Thus $\sigma_0(T) = \|T\|$ and

$$\sigma_0(T) \geq \sigma_1(T) \geq \sigma_2(T) \geq \cdots \geq 0.$$

When T is compact, it can be proved that $\sigma_k(T)$ is an eigenvalue of T^*T [15, Th.16.4]. Any corresponding eigenvector of T^*T is called a *Schmidt vector* of T corresponding to the singular value $\sigma_k(T)$. A *Schmidt pair* is a pair of vectors $x \in \mathcal{H}$ and $y \in \mathcal{K}$ such that

$$Tx = \sigma_k(T)y, \quad Ty = \sigma_k(T)x.$$

The past inputs to future outputs mapping γ_g associated with a LTI system by (6) is a compact operator from $L^2(-\infty, 0)$ to $L^2(0, \infty)$. The *Hankel singular values* of a LTI system are defined to be the singular values of γ_g. Via the Laplace transform, we may associate with γ_g, the Hankel operator

$$\Gamma_G : H^2_- \to H^2,$$

whose symbol G is the Laplace transform of g. It is defined by

$$\Gamma_G(x) = \Pi_+(Gx), x \in H^2_-.$$

Since γ_g and Γ_G are unitarily equivalent via the Laplace transform, they share the same set of singular values

$$\sigma_0(G) \geq \sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq 0.$$

The Hankel norm is defined to be the operator norm of Γ_G , which turns out to be its largest singular value $\sigma_0(G)$:

$$\|G\|_H = \|\Gamma_G\| = \sigma_0(G).$$

Note that

$$\|G\|_H = \sup_{u \in L^2(-\infty,0)} \frac{\|y\|_{L^2(0,\infty)}}{\|u\|_{L^2(-\infty,0)}},$$

so that the Hankel norm gives the L^2 gain from past inputs to future outputs.

If the LTI system has finite order, then its Hankel singular values correspond to the singular values of the matrix PQ, where P is controllability gramian and Q the observability gramian. Indeed, let σ be a singular value of γ_g with u the corresponding eigenvector of $\gamma_g^*\gamma_g$: $(\gamma_g^*\gamma_g u)(t) = \sigma^2 u(t)$. Then, since the adjoint operator γ_g^* is given by

$$(\gamma_g^* y)(t) = \int_0^\infty B^* e^{A^*(-t+\tau)} C^* y(\tau) d\tau,$$

we have that

$$(\gamma_g^*\gamma_g u)(t) = (\gamma_g^* y)(t) = B^* e^{-A^* t} Q x_0,$$

so that

$$u(t) = \sigma^{-2} B^* e^{-A^* t} Q x_0 \tag{7}$$

Now,

$$\sigma^2 x_0 = \int_0^\infty e^{(A\tau)} B \sigma^2 u(-\tau) d\tau = PQx_0,$$

and σ^2 is an eigenvalue of PQ associated with the eigenvector x_0. Conversely, if σ^2 is an eigenvalue of PQ associated with the eigenvector x_0, then σ is a singular value of γ_g with corresponding eigenvector of $\gamma_g^* \gamma_g$ given by (7). A useful state-space realization in this respect is the balanced realization for which $P = Q = \mathrm{diag}(\sigma_0, \sigma_1, \ldots, \sigma_{\mathrm{n}-1})$.

Remark. The Hankel norm of a finite order LTI system doesn't depend on its 'D matrix'.

5 Identification and approximation.

The identification problem is to find an accurate model of an observed system from measured data. This definition covers many different approaches depending on the class of models we choose and on the data we have at hand. We shall pay more attention on harmonic identification. The data are then pointwise values of the frequency response in some bandwidth and the models are finite order linear time-invariant (LTI) systems. A robust way to proceed is to interpolate the data on the bandwidth into a high order transfer function, possibly unstable. A first step consists in approximating the unstable transfer function by a stable one. This can be done by solving bounded extremal problems (see [2]).

For computational reasons, it is desirable if such a high-order model can be replaced by a reduced-order model without incurring to much error. This can be stated as follows:

Model reduction problem: given a $p \times m$ stable rational matrix function $G(z)$ of McMillan degree N, find $\hat{G}$ stable of McMillan degree $n < N$ which minimizes

$$|||G - \hat{G}|||. \tag{8}$$

The choice of the norm $|||.|||$ is influenced by what norms can be minimized with reasonable computational efforts and whether the chosen norm is an appropriate measure of error. The most natural norm from a physical viewpoint is the norm $\|.\|_\infty$. But this is an unresolved problem : there is no known numerical method which is guaranteed to converge. In Banach spaces other than Hilbert spaces, best approximation problems are usually difficult. There are two cases in which the situation is easier since they involve the Hardy space H^2 which is an Hilbert space: the L^2-norm and the Hankel norm, since the Hankel operator acts on H^2. In this last case an explicit solution can be computed.

5.1 Hankel-norm approximation

In the seventies, it was realized that the recent results on L^∞ approximation problems, such as Nehari's theorem and the result of Adamjan, Arov and Krein on the Nehari-Takagi problem, were relevant to the current problems of some engineers in control theory. In the context of LTI systems, they have led to efficient new methods of model reduction.

A first step in solving the model reduction problem in Hankel-norm is provided by Nehari's theorem. Translated in the control theory framework, it states that if one wishes to approximate a causal function $G(s)$ by an anticausal function, then the smallest error norm that can be achieved is precisely the Hankel-norm of $G(s)$.

Theorem 2 *For $G \in H^\infty$*

$$\sigma_0(G) = \|G\|_H = \inf_{F \in H_-^\infty} \|G - F\|_\infty.$$

The model reduction problem, known under the name of Nehari-Takagi, was first solved by Adamjan, Arov and Krein for SISO systems and Kung and Lin for MIMO discrete-time systems. In our continuous-time framework, it can be stated as follow:

Theorem 3 *Given a stable, rational transfer function $G(s)$ then*

$$\sigma_k(G) = \inf_{\hat{G} \in H^\infty} \|G - \hat{G}\|_H, \qquad \text{McMillan degree of } \hat{G} \leq k.$$

The fact that $\|G - \hat{G}\|_H \geq \sigma_k(G)$, for all $\hat{G}(s)$ stable and of McMillan degree $\leq k$, is no more than a continuous-time version of Proposition 1 [6, Lemma 7.1]. This famous paper [6] gives a beautiful solution of the computational problem using state-space methods. An explicit construction of a solution $\hat{G}(s)$ is presented which makes use of a balanced realization of $G(s)$ [6, Th.6.3]. Moreover, in [6] all the optimal Hankel norm approximations are characterized in state-space form.

Since,

$$\|G - \hat{G}\|_H = \inf_{F \in H_-^\infty} \|G - \hat{G} - F\|_\infty,$$

the Hankel norm approximation $\hat{G}(s)$ can be a rather bad approximant in L^∞ norm. However, the choice of the 'D matrix' for the approximation is arbitrary, since the Hankel-norm doesn't depend on D, while $\|G - \hat{G}\|_\infty$ does depend on D. In [6, Sect.9, Sect.10.2] a particular choice of D is suggested which ensures that

$$\|G - \hat{G}\|_\infty \leq \sigma_k(G) + \sum_{j>k} \sigma_j(G).$$

It is often the case in practical applications that Γ_G has a few sizable singular values and the remaining ones tail away very quickly to zero. In that case the right hand-side can be made very small, and one is assured that an optimal Hankel norm approximant is also good with respect to the L^∞ norm.

5.2 L^2-norm approximation

In the case of the L^2 norm, an explicit solution of the model reduction problem cannot be computed. However, the L^2 norm being differentiable we may think of using a gradient flow method. The main difficulty in this problem is to describe the set of approximants, i.e. of rational stable functions of McMillan degree n. The approaches than can be found in the literature mainly differ from the choice of a parametrization to describe this set of approximants. These parametrizations often arise from realization theory and the parameters are some entries of the matrices (A, B, C, D). To cope with their inherent complexity, some approaches choose to relax a constraint : stability or fixed McMillan degree. They often run into difficulties since smoothness can be lost or an undesirable approximant reached.

Another approach can be proposed. The number of optimization parameters can be reduced using the inner-unstable factorization (see section 4) and the projection property of an Hilbert space. Let $\hat{G}$ be a best L^2 approximant of G, with inner-unstable factorization

$$\hat{G} = QP,$$

where Q is the inner factor and P the unstable one. Then, H^2 being an Hilbert space, $\hat{G}$ must be the projection of G onto the space $H(Q)$ of matrix functions of degree n whose left inner factor is Q. We shall denote this projection by $\hat{G}(Q)$ and the problem consists now in minimizing

$$Q \to \|G - \hat{G}(Q)\|_2,$$

over the set of inner functions of McMillan degree n.

Then, more efficient parametrizations can be used which arise from the manifold structure of this set. It consists to work with an atlas of charts, that is a collection of local coordinate maps (the charts) which cover the manifold and such that changing from one map to another is a smooth operation. Such a parametrization present the advantages to ensure identifiability, stability of the result and the nice behavior of the optimization process. The optimization is run over the set as a whole changing from one chart to another when necessary. Parametrizations of this type are available either from realization theory or from interpolation theory in which the parameters are interpolation values. Their description overcomes the aim of this paper, and we refer the reader to [9] and the bibliography therein for more informations on this approach.

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