On a recursive state-space method for discrete-time H_2-approximation

R.L.M. Peeters[1], **M. Olivi**[2], **B. Hanzon**[3]

[1] *Dept. Mathematics, Universiteit Maastricht,*
P.O. Box 616, 6200 MD Maastricht, The Netherlands,
Fax: +31-43-3884910, Email: ralf.peeters@math.unimaas.nl

[2] *INRIA Sophia-Antipolis,*
B.P. 93, 06902 Sophia-Antipolis Cedex, France,
Fax: +33-4 92 38 78 58, Email: olivi@sophia.inria.fr

[3] *Dept. Econometrics, FEWEC, Vrije Universiteit,*
De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands,
Fax: +31-20-4446020, Email: bhanzon@econ.vu.nl

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1 Introduction

In [4] a gradient method for discrete-time H_2-approximation was developed which proceeds recursively with respect to the order n of the approximants. Here, the Douglas-Shapiro-Shields factorization (or inner-unstable factorization, see [3]) is utilized to reduce the H_2-approximation problem into an optimization problem over inner (or, equivalently, stable all-pass) functions of order n, by optimizing analytically with respect to the unstable factor. To implement this approach, the space of inner functions of order n is parametrized by means of an atlas of overlapping generic charts, obtained from application of the tangential Schur algorithm in the spirit of [1]. This construction supports the possibility to embed any given approximant of order $n-1$ into the boundary of a chart of approximants of order n. If an approximant of order $n-1$ constitutes a local minimum for the "concentrated criterion function" in the space of order $n-1$ approximants, then the corresponding embedded boundary point of a chart of systems of order n is taken as the starting point for the next iteration run of the gradient-based approximation algorithm.

It can be shown that: (i) a boundary point does not constitute a local minimum if the given system to be approximated is at least of order n; (ii) if the gradient of the "concentrated" criterion is well-defined – which may depend on the precise choice of parameters in this approach but can easily be achieved by making certain natural choices – then it is orthogonal to the boundary of the chart; if in addition it is nonzero then it points outwards.

The result (ii) is in some sense not very surprising, because, due to normalization within the tangential Schur algorithm which underlies the construction at hand, the boundary of each generic chart consists entirely of lower order systems, over which optimization has just taken place in the previous iteration run. Improvement of the criterion should therefore initially be looked for in a direction orthogonal to the boundary.

In the present paper we focus on a state-space implementation of the same idea. In this set-up, the "concentration step" corresponds to analytic optimization of the (C, D) pair for a fixed reachable input pair (A, B), which without loss of generality can be assumed input normal. The space of input normal pairs (A, B) is parametrized using overlapping charts constructed with $\mathcal{F}_{U,V}$-mappings defined in Section 3. This construction employs products of unitary matrices, having distinguished numerical advantages. As before, it remains possible to embed any given approximant of order $n-1$ into the boundary of a chart of input normal pairs of order n. However, the boundary of a chart in this construction, in contrast to the tangential Schur approach described above, does not consist exclusively of input normal pairs of order $n-1$, but instead generically of input normal pairs of order n. Nevertheless, it can be shown that properties (i) and (ii) also apply to the present construction, establishing feasibility of this state-space analogue of the recursive gradient-based algorithm for H_2-approximation.

2 A state-space approach to the discrete-time H_2-approximation problem

In this paper we consider stable linear time-invariant causal systems of finite dimension in discrete-time. Such systems are studied from an input-output point of view, and they are therefore identified with their associated transfer function matrices, which are (complex) proper rational matrices of finite McMillan degree. Discrete-time stability is defined as BIBO stability, which comes down to the property that all the poles of the transfer function are strictly inside the open complex unit disk. We shall be pursuing a state-space approach. From realization theory it is well known that a $q \times p$ proper rational matrix $\hat{G}(z)$ of McMillan degree $\hat{n}$ always admits a minimal state-space realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \mathbb{C}^{\hat{n}\times\hat{n}} \times \mathbb{C}^{\hat{n}\times p} \times \mathbb{C}^{q\times\hat{n}} \times \mathbb{C}^{q\times p}$, so that it holds that $\hat{G}(z) = \hat{D} + \hat{C}(zI_{\hat{n}} - \hat{A})^{-1}\hat{B}$. Minimality in this context is equivalent to the state-space realization being both reachable and observable, i.e., the reachability matrix $\hat{\mathcal{R}} = \begin{pmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{\hat{n}-1}\hat{B} \end{pmatrix}$ has full row rank $\hat{n}$ and the observability matrix $\hat{\mathcal{O}} = \begin{pmatrix} \hat{C}^* & \hat{A}^*\hat{C}^* & \ldots & (\hat{A}^*)^{\hat{n}-1}\hat{C}^* \end{pmatrix}^*$ has full column rank $\hat{n}$. From a given minimal realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ the space of all minimal realizations of $\hat{G}(z)$ is obtained by the action of the general linear group, representing the freedom to choose a basis for the state space, which generates the minimal quadruples $(T\hat{A}T^{-1}, T\hat{B}, \hat{C}T^{-1}, \hat{D})$ for invertible $T \in \mathbb{C}^{\hat{n}\times\hat{n}}$. This freedom is commonly exploited to impose additional structure on the matrix quadruple of the state-space realization, e.g., when designing a canonical form. Stability of a minimal state-space realization is equivalent to asymptotic stability of the dynamical matrix $\hat{A}$, since its eigenvalues then coincide with the poles of $\hat{G}(z)$. See also [6].

A discrete-time stable proper rational transfer function $\hat{G}(z)$ admits a Laurent series expansion about $z = \infty$, denoted by $\hat{G}(z) = \hat{G}_0 + \hat{G}_1 z^{-1} + \hat{G}_2 z^{-2} + \ldots$, for which it holds that the sequence of Markov matrices $\{\hat{G}_0, \hat{G}_1, \hat{G}_2, \ldots\}$ is defined by $\hat{G}_0 = \hat{D}$, $\hat{G}_k = \hat{C}\hat{A}^{k-1}\hat{B}$ (for $k = 1, 2, \ldots$), converging exponentially to zero. The H_2-distance between two stable transfer functions $\hat{G}(z)$ and $G(z)$ can now be introduced as follows.

Definition 2.1 *Let $\hat{G}(z)$ and $G(z)$ be two $q \times p$ discrete-time stable proper rational transfer function matrices, of which the Laurent series expansions about $z = \infty$ are denoted by $\hat{G}(z) = \hat{G}_0 + \hat{G}_1 z^{-1} + \hat{G}_2 z^{-2} + \ldots$ and $G(z) = G_0 + G_1 z^{-1} + G_2 z^{-2} + \ldots$, respectively. Then the (squared) H_2-distance between $\hat{G}(z)$ and $G(z)$ is defined as*

$$\|\hat{G}(z) - G(z)\|_{H_2}^2 = \operatorname{tr}\left\{\sum_{k=0}^{\infty}(\hat{G}_k - G_k)(\hat{G}_k - G_k)^*\right\}, \quad (1)$$

where $\operatorname{tr}\{\cdot\}$ denotes the trace operator, and * denotes Hermitian transposition (i.e., the joint action of complex conjugation and matrix transposition).

Here, convergence of the infinite sum follows from the stability assumptions on $\hat{G}(z)$ and $G(z)$.

The H_2-approximation problem can now be stated as the problem of finding a stable approximant $G(z)$ of McMillan degree $\leq n$ of a given stable transfer function $\hat{G}(z)$ of McMillan degree $\hat{n}$, which minimizes the H_2-distance between $\hat{G}(z)$ and $G(z)$.

Obviously, as in the definitions above, one may study this problem in the function theoretic language of the frequency (transfer function) domain, but a state-space approach is also possible. In the latter case, with obvious notation, the H_2-criterion expressing the squared H_2-distance to be minimized, assumes the form

$$\begin{aligned} V(A,B,C,D) &= \operatorname{tr}\left\{(\hat{D} - D)(\hat{D} - D)^*\right\} + \quad (2) \\ &+ \operatorname{tr}\left\{\sum_{k=1}^{\infty}(\hat{C}\hat{A}^{k-1}\hat{B} - CA^{k-1}B)(\hat{C}\hat{A}^{k-1}\hat{B} - CA^{k-1}B)^*\right\}. \end{aligned}$$

For our purposes, the following definition will be useful.

Definition 2.2 *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p}$ be an input pair. Then (A, B) is called* weakly input normal *if it holds that $AA^* + BB^* = I_n$. If in addition (A, B) is reachable, it is called* input normal.

From the literature it well known that if $n \leq \hat{n}$, then any local minimizer of V corresponds to a quadruple (A, B, C, D) of McMillan degree n, see [2]. Therefore, one may impose minimality of (A, B, C, D) without loss of generality. It is also clear that state-space basis transformations do not affect the value of the H_2-criterion. Any reachable input pair (A, B) with an asymptotically stable matrix A can always be brought into input normal form (TAT^{-1}, TB), e.g., by choosing $T = P^{-1/2}$ where P is the (unique, positive definite, Hermitian) solution to the discrete-time Lyapunov-Stein equation $P - APA^* = BB^*$. Therefore, one may also impose input normality of (A, B) without loss of generality. For our considerations it will be essential, however, to relax this condition somewhat and also to admit weakly input normal pairs (A, B), as this will enable us to embed approximants of order $n - 1$ into boundaries of charts of approximants of order n and to study this in a consistent way. The following lemma characterizes weakly input normal pairs to some extent.

Lemma 2.3 *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p}$ be an input pair which is weakly input normal. Then there exists a unitary matrix Q for which the transformed matrices $\tilde{A} := QAQ^*$ and $\tilde{B} := QB$ admit the block-partitions*

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}, \quad (3)$$

for which $\tilde{A}_1$ is asymptotically stable of size $r \times r$, $\tilde{A}_2$ is unitary of size $(n - r) \times (n - r)$ and $\tilde{B}_1$ is of size

$r \times p$ *such that* $(\tilde{A}_1, \tilde{B}_1)$ *is (discrete-time) input normal, where* r *denotes the rank of the controllability matrix* $\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$.

We are now in a position to characterize all the optimal choices of C and D globally minimizing V for a fixed weakly input normal pair (A, B), and to give an expression of the corresponding "concentrated" criterion value.

Proposition 2.4 *Let* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *be a minimal state-space representation of a* $q \times p$ *discrete-time stable proper rational transfer function* $\hat{G}(z)$ *of finite McMillan degree* $\hat{n}$. *Let* $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p}$ *be a given, fixed, weakly input normal pair and consider the associated linear space of* $q\times p$ *proper rational transfer functions* $G(z) = D + C(zI_n - A)^{-1}B$ *of McMillan degree* $\leq n$. *Then there is a unique transfer function* $G_{\rm opt}(z)$ *within this space which minimizes the discrete-time* H_2*-distance to* $\hat{G}(z)$. *The associated set of corresponding optimal state-space realizations* $(A, B, C_{\rm opt}, D_{\rm opt})$ *is given by* $D_{\rm opt} = \hat{D}$ *and* $C_{\rm opt} = \hat{C}P_2 + \Gamma$, *where* P_2 *denotes the unique solution to the discrete-time Sylvester equation* $P_2 - \hat{A}P_2A^* = \hat{B}B^*$ *and* Γ *is any* $q \times n$ *matrix of which all the rows are in the left kernel of the reachability matrix associated with* (A, B). *The corresponding value of the (squared)* H_2*-distance between* $\hat{G}(z)$ *and* $G_{\rm opt}(z)$ *is then given in terms of the weakly input normal pair* (A, B) *by*

$$\begin{aligned} V_c(A,B) &= \|\hat{G}(z) - G_{\rm opt}(z)\|^2_{H_2} = \\ &= \operatorname{tr}\left\{\hat{C}P_1\hat{C}^*\right\} - \operatorname{tr}\left\{\hat{C}P_2P_2^*\hat{C}^*\right\}, \end{aligned} \tag{4}$$

where P_1 *denotes the unique solution to the discrete-time Lyapunov-Stein equation* $P_1 - \hat{A}P_1\hat{A}^* = \hat{B}\hat{B}^*$.

This proposition makes clear how the H_2-approximation problem can be rephrased in state-space terms as a minimization problem over the space of weakly input normal pairs $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p}$, using the 'concentrated H_2-criterion' $V_c(A, B)$.

3 Schur parametrization of balanced realizations of stable all-pass systems and of input normal pairs

A practical implementation of the state-space approach to the H_2-approximation problem still needs to be supplied with a suitable parametrization of the space of input normal pairs (A, B) of order n. These may be derived from balanced state-space realizations of (multi-input multi-output) discrete-time stable all-pass systems (see also [8, 5]). The parametrizations studied in the present paper are based on the constructions of [9] and employ 'realization matrices' of stable all-pass systems which are constructed as products of structured unitary matrices. The construction of the charts in the atlas described here, employs mappings $\mathcal{F}_{U,V}$ acting on $p \times p$ proper rational discrete-time stable all-pass transfer functions $G(z)$ as follows:

$$\mathcal{F}_{U,V}(G(z)) = F_1(z) + \frac{F_2(z)F_3(z)}{z - F_4(z)}, \tag{5}$$

where

$$F(z) = \begin{pmatrix} F_1(z) & F_2(z) \\ F_3(z) & F_4(z) \end{pmatrix} = V \begin{pmatrix} 1 & 0 \\ 0 & G(z) \end{pmatrix} U^*, \tag{6}$$

with U and V unitary matrices of size $(p+1) \times (p+1)$ and with $F(z)$ partitioned such that $F_4(z)$ is scalar.

Each of the mappings $\mathcal{F}_{U,V}$ takes the set of rational stable all-pass transfer functions into itself. In state-space terms it holds that if (A, B, C, D) is a state-space realization of $G(z)$ with A of size $n \times n$, then a state-space realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of $\tilde{G}(z) = \mathcal{F}_{U,V}(G(z))$, with $\tilde{A}$ of size $(n+1) \times (n+1)$, is given by

$$\begin{pmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & I_n \end{pmatrix}. \tag{7}$$

This demonstrates that if the McMillan degree of $G(z)$ is equal to n, then the McMillan degree of $\tilde{G}(z)$ is $\leq n+1$. It can be established (see [9]) that if the left bottom corner entries of U and V have different modulus, then the McMillan degree of $\tilde{G}(z)$ is $n+1$. In that case the mapping $\mathcal{F}_{U,V}$ can also be rewritten into the form of a linear fractional transformation associated with a particular J-inner matrix of McMillan degree 1, as employed in the tangential Schur algorithm. A particular choice for the unitary matrices U and V which makes this connection with the tangential Schur algorithm explicit and which provides $\tilde{G}(z)$ with a balanced state-space realization when starting from a balanced state-space realization of $G(z)$, is given by

$$U = \begin{pmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}u & I_p - (1 + \frac{w\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}})uu^* \\ \frac{\bar{w}\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}u^* \end{pmatrix}, \tag{8}$$

$$V = \begin{pmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}v & I_p - (1 - \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}})\frac{vv^*}{\|v\|^2} \\ \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & -\frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}v^* \end{pmatrix}, \tag{9}$$

where $w \in \mathbb{C}$ with $|w| < 1$ is an interpolation point in the open unit disk, $u \in \mathbb{C}^{p\times 1}$ with $\|u\| = 1$ is a normalized direction vector, and $v \in \mathbb{C}^{p\times 1}$ with $\|v\| < 1$ is a Schur vector, through which the actual parametrization of a corresponding chart of stable all-pass systems takes place.

Each chart in the atlas constructed recursively along these lines, is then indexed by a fixed set of n interpolation points $w_1, \ldots, w_n$ and n normalized direction vectors $u_1, \ldots, u_n$, while the local coordinates are specified through the set of n Schur vectors $v_1, \ldots, v_n$ and an initial (constant) unitary matrix D_0 of size $p \times p$. When using this approach just to generate charts for the manifold of reachable input normal pairs (A, B) of order n, the freedom of the unitary group is factored out naturally, by choosing $D_0 = I_p$. This leads to the following parametrization of a chart in this atlas of reachable input normal pairs (A, B) of order n:

$$\begin{pmatrix} B & A \end{pmatrix} = \begin{pmatrix} 0 & I_n \end{pmatrix} \begin{pmatrix} V_n & 0 \\ 0 & I_{n-1} \end{pmatrix} \cdots \tag{10}$$
$$\cdots \begin{pmatrix} I_{n-1} & 0 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ 0 & U_1^* \end{pmatrix} \cdots \begin{pmatrix} U_n^* & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

where the matrix blocks U_k and V_k are of the form described by Eqns. (8)–(9) with $w = w_k$, $u = u_k$ and $v = v_k$ (for $k = 1, 2 \ldots, n$).

4 Embedding of a lower order approximant into the boundary of a generic chart

In the construction above, a generic chart of reachable input normal pairs (A, B) of order n requires Schur vectors $v_1, \ldots, v_n$ of norm *strictly* less than 1. Points on the boundary of such a chart are obtained when one or more of these Schur vectors have norm equal to 1. For $\|v\| = 1$, the unitary matrices U and V given by Eqns. (8)–(9) attain the form

$$U = \begin{pmatrix} u & I_p - uu^* \\ 0 & u^* \end{pmatrix}, \qquad V = \begin{pmatrix} v & I_p - vv^* \\ 0 & -v^* \end{pmatrix}, \tag{11}$$

and w no longer plays a role. The following lemma establishes a necessary and sufficient condition for a boundary point of the chart to correspond to a stable all-pass system of McMillan degree n.

Lemma 4.1 *For a given vector u of norm 1, a given vector v of norm 1 and a given stable all-pass function $G(z)$ of McMillan degree $n-1$, the stable all-pass function $\tilde{G}(z) = \mathcal{F}_{U,V}(G(z))$ is of McMillan degree n if and only if there does not exists a scalar λ of modulus 1 such that $v = -\lambda^{-1} G(\lambda) u$.*

For given u and stable all-pass $G(z)$, the set $\{v \,|\, v = -\lambda^{-1} G(\lambda) u, |\lambda| = 1\}$ is obviously non-empty, so that there are always stable all-pass systems of order $< n$ on the boundary of a chart of stable all-pass systems of order n. For any vector u, any vector v of norm 1 may give rise to some $\tilde{G}(z)$ of McMillan degree $< n$, depending on the specific choice of $G(z)$. In case $n = 1$, however, when using the prescribed initialization $G(z) = D_0 = I_p$, the choice of $G(z)$ is restricted so that $\tilde{G}(z)$ is again of McMillan degree 0 if and only if $v = -\lambda^{-1} u$ for some $|\lambda| = 1$. In that case it is easily computed that $\tilde{G}(z) = I_p$ if and only if $v = u$.

It follows in the *scalar* case $p = 1$ for fixed stable all-pass $G(z)$ of McMillan degree $n-1$ and fixed (scalar) u and v of modulus 1, that $\mathcal{F}_{U,V}(G(z)) = uv^*$ is a constant unimodular scalar, because $I_p - uu^* = I_p - vv^* = 0$, so that the McMillan degree is 0. In the *multivariable case* $p > 1$, however, the set $\{v \,|\, v = -\lambda^{-1} G(\lambda) u, |\lambda| = 1\}$ constitutes a manifold of real dimension 1, whereas the boundary set $\{v \,|\, \|v\| = 1\}$ has real dimension $2p - 1 > 1$. This implies that the boundary of the chart in that case generically consists of stable all-pass systems of McMillan degree n, while on the other hand a 'thin' subset of lower order stable all-pass systems does always occur.

The construction procedure above for a state-space parametrization of stable all-pass systems of McMillan degree n with the help of the mappings $\mathcal{F}_{U_k,V_k}$ can be extended in a simple way to embed an arbitrary fixed stable all-pass system of McMillan degree n into the boundary of a corresponding chart of stable all-pass systems of McMillan degree $n+1$. This is achieved by the application of an extra initial mapping $\mathcal{F}_{U_0,V_0}$ which takes $D_0 = I_p$ to itself. We have already indicated that $\mathcal{F}_{U_0,V_0}(I_p) = I_p$ if and only if $v_0 = u_0$. In this case, the state-space realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of $\tilde{G}(z) = G(z)$ becomes non-minimal and attains the form

$$\begin{pmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{pmatrix} = \left(\begin{array}{c|cc} D & C & 0 \\ \hline B & A & 0 \\ 0 & 0 & -1 \end{array} \right). \tag{12}$$

The construction gives rise to an extra state component, which is obviously both uncontrollable and unobservable and may simply be removed by mere truncation of the state vector.

When using the above way of embedding a stable all-pass system of McMillan degree n into the boundary of a chart of stable all-pass systems of order $n+1$ within the context of H_2-approximation, where we have to focus on the associated input pairs (A, B), the concentrated H_2-criterion takes the form

$$\tilde{V}_c(\tilde{A}, \tilde{B}) = \mathrm{tr}\left\{\hat{C} P_1 \hat{C}^*\right\} - \mathrm{tr}\left\{\hat{C} \tilde{P}_2 \tilde{P}_2^* \hat{C}^*\right\}, \tag{13}$$

where $P_1 - \hat{A} P_1 \hat{A}^* = \hat{B}\hat{B}^*$ and $\tilde{P}_2 - \hat{A}\tilde{P}_2\tilde{A}^* = \hat{B}\tilde{B}^*$. This is an immediate consequence of Prop. 2.4, upon noting that all boundary points of the chart of input normal pairs (A, B) are weakly input normal because of continuity. Partitioning $\tilde{P}_2 = \begin{pmatrix} \tilde{P}_{21} & \tilde{P}_{22} \end{pmatrix}$ with $\tilde{P}_{22}$ consisting of a single column, leads to

$$\begin{pmatrix} \tilde{P}_{21} & \tilde{P}_{22} \end{pmatrix} - \hat{A} \begin{pmatrix} \tilde{P}_{21} & \tilde{P}_{22} \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & -1 \end{pmatrix} = \hat{B} \begin{pmatrix} B^* & 0 \end{pmatrix}. \tag{14}$$

Working out the partition, it is obtained that $\tilde{P}_{21} - \hat{A}\tilde{P}_{21} A^* = \hat{B} B^*$ and $\tilde{P}_{22} + \hat{A}\tilde{P}_{22} = 0$, from which it follows that $\tilde{P}_2 = \begin{pmatrix} P_2 & 0 \end{pmatrix}$ yields the unique solution. As a consequence $\tilde{V}_c(\tilde{A},\tilde{B}) = \mathrm{tr}\left\{\hat{C}P_1\hat{C}^*\right\} - \mathrm{tr}\left\{\hat{C}\tilde{P}_2\tilde{P}_2^*\hat{C}^*\right\} = \mathrm{tr}\left\{\hat{C}P_1\hat{C}^*\right\} - \mathrm{tr}\left\{\hat{C}P_2P_2^*\hat{C}^*\right\} = V_c(A,B)$, which shows that the embedding is well-behaved with respect to the current context of H_2-approximation.

5 The gradient of the concentrated H_2-criterion at a lower order embedded approximant

The foregoing exposition has made clear that the value of the concentrated H_2-criterion $V_c(A,B)$ at a weakly input normal pair (A,B) inside or on the boundary of one of the charts in our construction, can be computed as

$$V_c(A,B) = \mathrm{tr}\left\{\hat{C}P_1\hat{C}^*\right\} - \mathrm{tr}\left\{\hat{C}P_2P_2^*\hat{C}^*\right\} \tag{15}$$

where $P_1 - \hat{A}P_1\hat{A}^* = \hat{B}\hat{B}^*$ does not involve (A,B), and $P_2 - \hat{A}P_2A^* = \hat{B}B^*$ does. Any directional derivative $\dot{V}_c(A,B)$ is therefore given by

$$\dot{V}_c(A,B) = -\mathrm{tr}\left\{\hat{C}\dot{P}_2P_2^*\hat{C}^* + \hat{C}P_2\dot{P}_2^*\hat{C}^*\right\} \tag{16}$$

where $\dot{P}_2$ can be computed from the discrete-time Sylvester equation

$$\dot{P}_2 - \hat{A}\dot{P}_2A^* = \hat{A}P_2\dot{A}^* + \hat{B}\dot{B}^*. \tag{17}$$

Note that the structure of the left-hand side of this linear matrix equation is similar to that of the Sylvester equation determining P_2, so that a unique solution for $\dot{P}_2$ exists provided that the right-hand side matrix is well-defined. Since this involves directional derivatives of A and B, the particular parametrization of (A,B) at hand plays a crucial role.

When the parameters are chosen as the entries of the Schur vectors v_k (for all $k=1,2,\ldots,n$), it follows that no problems of differentiability emerge as long as $\|v_k\| < 1$ for all k, i.e., inside the open charts. However, at the boundaries of the charts the entries in U_k and V_k involving the expression $\sqrt{1-\|v_k\|^2}$ will cause differentiability problems. These can be cured, in general, by employing local reparametrizations of the Schur vectors v_k which approach the boundary at a slow enough rate. One instance of such a local reparametrization is offered by writing each Schur vector as $v_k = \cos(r_k)\tilde{v}_k$, with $r_k \geq 0$ and with $\tilde{v}_k$ a smoothly parametrized vector of norm 1 (using any convenient smooth local parametrization of the unit sphere in $\mathbb{C}^p$), since then $\sqrt{1-\|v_k\|^2} = \sin(r_k)$ is a smooth function of r_k exhibiting no differentiability problems at $r_k = 0$.

Above we have also seen that if (A,B) constitutes an input normal pair underlying an approximant $G(z)$ of order n to the given transfer function $\hat{G}(z)$, and (A,B) is embedded into the boundary of a chart of input normal pairs $(\tilde{A},\tilde{B})$ of order $n+1$ by the application of an extra initial mapping $\mathcal{F}_{U_0,V_0}$, then (A,B) is represented for $v_0 = u_0$ by the extended weakly input normal pair $(\tilde{A},\tilde{B})$ given by

$$(\tilde{A},\tilde{B}) = (\begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix}). \tag{18}$$

Then the associated matrix $\tilde{P}_2$ which uniquely solves the discrete-time Sylvester equation $\tilde{P}_2 - \hat{A}\tilde{P}_2\tilde{A}^* = \hat{B}\tilde{B}^*$ is of the form $\tilde{P}_2 = \begin{pmatrix} P_2 & 0 \end{pmatrix}$, with the zeros constituting a single column and with P_2 uniquely solving the discrete-time Sylvester equation $P_2 - \hat{A}P_2A^* = \hat{B}B^*$.

We are interested in computing directional derivatives of $\tilde{V}_c(\tilde{A},\tilde{B})$ at the given boundary point. If the local parametrization is such that $\dot{\tilde{A}}$ and $\dot{\tilde{B}}$ are well-defined (i.e., having finite values), then also $\dot{\tilde{P}}_2$ is well-defined and we have, upon partitioning $\dot{\tilde{P}}_2 = \begin{pmatrix} \dot{\tilde{P}}_{21} & \dot{\tilde{P}}_{22} \end{pmatrix}$ where $\dot{\tilde{P}}_{22}$ consists of a single column:

$$\dot{\tilde{V}}_c(\tilde{A},\tilde{B}) = -\mathrm{tr}\left\{\hat{C}\dot{\tilde{P}}_{21}P_2^*\hat{C}^* + \hat{C}P_2\dot{\tilde{P}}_{21}^*\hat{C}^*\right\} \tag{19}$$

where $\dot{\tilde{P}}_{21}$ satisfies the equation

$$\dot{\tilde{P}}_{21} - \hat{A}\dot{\tilde{P}}_{21}A^* = \hat{A}P_2\dot{\tilde{A}}_{11}^* + \hat{B}\dot{\tilde{B}}_1^*, \tag{20}$$

with $\dot{\tilde{A}}_{11}$ and $\dot{\tilde{B}}_1$ denoting the directional derivative of the $n\times n$ left upper block of $\tilde{A}$ and of the $n\times p$ upper part of $\tilde{B}$, respectively.

Now suppose that (A,B) of order n constitutes a *stationary point* of $V_c(A,B)$, then all the directional derivatives of $V_c(A,B)$ with respect to the parameters that compose the Schur vectors $v_1,\ldots,v_n$, are zero. As a consequence, since $\tilde{V}_c(\tilde{A},\tilde{B}) = V_c(A,B)$ and since the choice $v_0 = u_0$ makes that the additional $(n+1)$-st state component is uncontrollable and can be truncated, also the corresponding directional derivatives of $\tilde{V}_c(\tilde{A},\tilde{B})$ with respect to the parameters in $v_1,\ldots,v_n$ at the boundary point $(\tilde{A},\tilde{B})$, are zero.

On the other hand, if we consider directional derivatives involving the Schur vector v_0 only, under the restriction that $\|v_0\| = 1$, i.e., along the boundary of the chart (or more precisely, along the boundary of the subchart obtained by keeping $v_1,\ldots,v_n$ fixed and varying only v_0), then the following proposition makes clear that these directional derivatives can alternatively be obtained by jointly varying $v_1,\ldots,v_n$ in a specific way and keeping $v_0 = u_0$ fixed.

Proposition 5.1 *For a given $q\times p$ discrete-time stable proper rational transfer function $\hat{G}(z)$, with minimal state-space realization $(\hat{A},\hat{B},\hat{C},\hat{D})$ of order $\hat{n}$, consider the associated 'concentrated H_2-criterion' $\tilde{V}_c(\tilde{A},\tilde{B})$ on a parametrized chart of weakly input normal pairs of order $n+1$, constructed in conjunction with a sequence of $n+1$ mappings $\mathcal{F}_{U_k,V_k}$, with w_k and u_k indexing the chart and v_k containing the parameters as in Eqns. (8)–(9), such that $\|v_k\| \leq 1$ for $k=0,1,2,\ldots,n$.*

On the boundary of this chart, consider a parametrized curve of points, with Schur vectors $v_k(t)$ given by $v_0(t) = e^{tX}u_0$ and $v_k(t) = e^{tX}v_k^0$ for $k = 1, 2, \ldots, n$, where X is a constant skew-Hermitian $p \times p$ matrix satisfying $(I_p - u_0u_0^*)X(I_p - u_0u_0^*) = 0$. Then the function $\tilde{V}_c(\tilde{A}, \tilde{B})$ is differentiable along this curve, having a stationary point at the lower order embedded input pair on the boundary occurring for $t = 0$.

Note that for an arbitrary vector ν such that $\nu^*u_0 + u_0^*\nu = 0$, it holds that $X = -u_0\nu^* + \nu u_0^* + (\nu^*u_0)u_0u_0^*$ is skew-Hermitian and satisfies the condition $(I_p - u_0u_0^*)X(I_p - u_0u_0^*) = 0$, while $\dot{v}_0(0) = Xu_0 = \nu$. As a result, if the embedded boundary point corresponds to a *stationary point of order* n, then also all the directional derivatives with respect to v_0 'along the boundary of the subchart' are zero. It therefore follows that the gradient of $\tilde{V}_c(\tilde{A}, \tilde{B})$ is orthogonal to the boundary of the chart at the embedded lower order stationary point of $V_c(A, B)$.

6 Conclusions

The H_2-approximation problem may be reduced, in state space, to an optimization problem over input normal pairs (A, B). These input normal pairs can be parametrized by means of sparse products of unitary matrices, facilitating efficient numerical computation, as described by [9]. A gradient-based approach may be followed along the lines of [4], proceeding recursively with respect to the order n of the approximant. Starting the iteration run for order n at an approximant constituting a local minimum for order $n - 1$, embedded at the boundary of a chart in this atlas, corresponds to a gradient which is orthogonal to that boundary. This method was implemented recently and is found to perform satisfactorily. It is intended to present an example in the final paper.

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