Linear Fractional Transformations and Balanced Realization of Discrete-Time Stable All-Pass Systems

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Abstract

The tangential Schur algorithm provides a means of constructing the class of $p \times p$ discrete-time stable all-pass transfer functions of a prescribed McMillan degree n. In each of its n iteration steps a linear fractional transformation is employed which is associated with a J-inner rational matrix of McMillan degree 1 involving certain parameters. In this set-up, the issue of generating corresponding state-space realizations in terms of these parameters is not addressed. In the present contribution we present a unified framework in which linear fractional transformations on transfer functions are represented by corresponding linear fractional transformations on state-space realization matrices. When applied to the case of the tangential Schur algorithm, minimal balanced realizations of stable all-pass systems in terms of the parameters used are obtained. The balanced state-space approach of [9] for SISO stable all-pass systems is incorporated as a special case.

1 Introduction

Stable all-pass systems of finite order have several applications in linear systems theory. Within the fields of system identification, approximation and model reduction, they have been used in connection with the Douglas-Shapiro-Shields factorization, see e.g., [3, 2, 13, 7], to obtain effective algorithms for various purposes. The differential structure of the one-to-one related class of inner functions has been studied in [1]. There, a parametrization has been obtained in the multivariable case by means of the *tangential Schur algorithm* which involves Schur parameter vectors, interpolation points and normalized direction vectors. In the scalar case, a single coordinate chart suffices to entirely describe the manifold of stable all-pass (or inner) systems of a fixed finite order. In the multivariable case this no longer can be achieved, and the approach leads to infinite atlases of generic charts covering these manifolds.

In another line of research, balanced state-space canonical forms have been constructed for various classes of linear systems; see e.g., [15, 10]. Balanced realizations allegedly have numerical advantages and are useful for model reduction purposes in conjunction with the popular 'balance-and-truncate' approach. In the constructions of [10], the case of stable all-pass systems in *continuous-time* plays a central role. However, when those results are carried over to the discrete-time case by means of a bilinear transformation, several nice properties of the realizations are not preserved, e.g., truncation of state components no longer leads to reduced order systems that are balanced and in canonical form. Therefore, the ideas of [10] have been applied directly in [9] to the SISO discrete-time stable all-pass case. This gave rise to a balanced canonical form which could in fact be parametrized with Schur parameters.

In this paper, a unified framework based on linear fractional transformations is presented which clarifies the connections between these two approaches. It encompasses and extends the preliminary results on this subject communicated in [16]. The main result in Theorem 3.5 provides the basis for a recursive method for obtaining balanced realizations for stable all-pass systems which are parametrized directly in terms of the parameters used in the (reversed) tangential Schur algorithm. This generalizes the results of [9] to the multivariable case and opens up possibilities for multivariable stable all-pass model reduction and approximation. Proofs are only given for all the new results in this paper.

2 Preliminaries

In this section we present some background material on the three topics that come together in the research of this paper: linear fractional transformations, state-space realization theory and Σ-lossless functions. We introduce the definitions and notation used in subsequent sections of the paper and we present a number of relevant results known from the literature.

2.1 Linear fractional transformations

We shall be considering linear fractional transformations of two different types that are associated with an invertible block-partitioned rational matrix $\Theta = \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix} \in \mathbb{K}^{2p\times 2p}(z)$ in the indeterminate z, where $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$. Most of the results in this section can be found in [17, 18].

Definition 2.1 *The linear fractional transformations $\mathcal{T}_\Theta$ and $\widehat{\mathcal{T}}_\Theta$ are defined by:*

$$\mathcal{T}_\Theta : \mathcal{M}_\Theta \to \mathcal{M}_{\Theta^{-1}}, \qquad G \mapsto (\Theta_4 G + \Theta_3)(\Theta_2 G + \Theta_1)^{-1}, \tag{1}$$

$$\widehat{\mathcal{T}}_\Theta : \widehat{\mathcal{M}}_\Theta \to \widehat{\mathcal{M}}_{\Theta^{-1}}, \qquad G \mapsto (G\Theta_2 + \Theta_4)^{-1}(G\Theta_1 + \Theta_3). \tag{2}$$

with their domains and co-domains specified by

$$\mathcal{M}_\Theta := \{G \in \mathbb{K}^{p\times p}(z) \,|\, \mathrm{rk}(\Theta_2 G + \Theta_1) = p\}, \tag{3}$$

$$\widehat{\mathcal{M}}_\Theta := \{G \in \mathbb{K}^{p\times p}(z) \,|\, \mathrm{rk}(G\Theta_2 + \Theta_4) = p\}. \tag{4}$$

It then holds that $\mathcal{T}_\Theta$ and $\widehat{\mathcal{T}}_\Theta$ are bijections with $\mathcal{T}_\Theta^{-1} = \mathcal{T}_{\Theta^{-1}}$ and $\widehat{\mathcal{T}}_\Theta^{-1} = \widehat{\mathcal{T}}_{\Theta^{-1}}$. In fact, these identities constitute special instances of the following well known group properties which hold for the composition of such mappings:

$$\mathcal{T}_\Theta \circ \mathcal{T}_\Psi = \mathcal{T}_{\Theta\Psi} \qquad \text{on } \mathcal{M}_\Psi \cap \mathcal{M}_{\Theta\Psi}, \tag{5}$$

$$\widehat{\mathcal{T}}_\Theta \circ \widehat{\mathcal{T}}_\Psi = \widehat{\mathcal{T}}_{\Psi\Theta} \qquad \text{on } \widehat{\mathcal{M}}_\Psi \cap \widehat{\mathcal{M}}_{\Psi\Theta}, \tag{6}$$

with $\Theta, \Psi \in \mathbb{K}^{2p\times 2p}(z)$ both invertible. (Note the order in which the matrices Θ and Ψ are multiplied.)

Each linear fractional transformation can be represented in both ways. Denoting

$$J_p = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \tag{7}$$

it holds that $\mathcal{T}_\Theta = \widehat{\mathcal{T}}_{J_p\Theta^{-1}J_p}$ (and equivalently $\widehat{\mathcal{T}}_\Theta = \mathcal{T}_{J_p\Theta^{-1}J_p}$), where $\mathcal{M}_\Theta = \widehat{\mathcal{M}}_{J_p\Theta^{-1}J_p}$ (and equivalently $\widehat{\mathcal{M}}_\Theta = \mathcal{M}_{J_p\Theta^{-1}J_p}$).

For two invertible rational matrices $\Theta, \Psi \in \mathbb{K}^{2p\times 2p}(z)$ it is well known that $\mathcal{T}_\Theta = \mathcal{T}_\Psi$ (or equivalently $\widehat{\mathcal{T}}_\Theta = \widehat{\mathcal{T}}_\Psi$) if and only if there exists a scalar rational function $\lambda(z) \in \mathbb{K}(z)$, not identically zero, such that $\Psi = \lambda\Theta$. Thus, a linear fractional transformation determines the associated (invertible) matrix Θ up to a (nonzero) scalar function λ.

2.2 State-space realization of transfer functions

In this section we present a review of a number of well known results from realization theory which shall be of importance in the sequel. See also [11].

Let $G(z)$ be a $p \times m$ rational matrix in $\mathbb{K}^{p\times m}(z)$. A matrix quadruple $(A, B, C, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ is called a state-space realization of $G(z)$ if it holds that

$$G(z) = D + C(zI_n - A)^{-1}B. \tag{8}$$

From this formula it is obvious that state-space realizations may only exist for proper rational matrices. Conversely, it is well known that every proper rational matrix $G(z)$ admits a state-space realization (A, B, C, D) for some suitably chosen $n \in \mathbb{N}$.

The associated (discrete-time) state-space system is described by the set of equations

$$x_{t+1} = Ax_t + Bu_t, \tag{9}$$
$$y_t = Cx_t + Du_t. \tag{10}$$

Here x_t denotes the n-dimensional state vector, u_t the m-dimensional input (or control) vector and y_t the p-dimensional output vector, all at the time instant $t \in \mathbb{Z}$. When the (two-sided) z-transformation is applied to this system of equations, it is found that $G(z)$ appears as the transfer function of this system, relating the z-transform $U(z)$ of the input sequence to the z-transform $Y(z)$ of the output sequence in a linear fashion: $Y(z) = G(z)U(z)$. Thus, in the absence of initial conditions, $G(z)$ provides a description of the state-space system regarded as a mechanism which establishes an input-output mapping from sequences $\{u_t\}$ to sequences $\{y_t\}$.

A state-space realization (A, B, C, D) of a transfer function $G(z)$ is called minimal if the associated state-space dimension n is as small as possible among all possible state-space realizations. A minimal state-space realization (A, B, C, D) of a proper rational transfer function $G(z)$ always exists, and in fact it is well known that a state-space realization (A, B, C, D) of $G(z)$ is minimal if and only if the associated state-space dimension n is equal to the McMillan degree of $G(z)$.

If (A, B, C, D) is a minimal n-dimensional state-space realization of $G(z)$, then (A', B', C', D') is another minimal state-space realization of $G(z)$ if and only if there exists a (unique) nonsingular matrix $T \in \mathbb{K}^{n\times n}$ such that $(A', B', C', D') = (TAT^{-1}, TB, CT^{-1}, D)$. Thus, the freedom in choosing a minimal state-space realization for a given transfer function is that of the general linear group $G\ell_n(\mathbb{K})$, corresponding to a change of basis of the state space. Here the states x'_t for the state-space realization (A', B', C', D') are related to the states x_t for (A, B, C, D) by means of the linear transformation $x'_t = Tx_t$. One common way of constructing canonical forms for (sub)classes of linear systems is by fixing a particular choice of T such as to impose a special structure on the matrices A, B, C and D.

Two important system theoretic concepts are that of controllability and observability. A state-space realization (A, B, C, D) happens to be controllable if and only if the associated controllability matrix $\mathcal{C}_n := \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$ has full rank n. Likewise, it is observable if and only if the observability matrix $\mathcal{O}_n := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ has full rank n. It is well known that a state-space realization is minimal if and only if it is both controllable and observable.

As is well known, a transfer function $G(z)$ of McMillan degree n has exactly n poles when counting algebraic multiplicities. If $G(z)$ is not proper, some of its poles are located at infinity. From the definition of a state-space realization (A, B, C, D) of a proper rational matrix function $G(z)$ it follows that the poles of $G(z)$ are contained within the set of eigenvalues of the dynamical matrix A. In case the realization is minimal the poles of $G(z)$ coincide with the eigenvalues of the dynamical matrix A, including algebraic multiplicities.

A transfer function $G(z)$ is called asymptotically stable if all its poles are within the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. In the same spirit, a matrix $A \in \mathbb{K}$ is called asymptotically stable if all its eigenvalues are within the open unit disk and state-space realization (A, B, C, D) is called asymptotically stable if the dynamical matrix A is asymptotically stable. With regard to mere stability, there exist a few slightly different concepts in the literature. These are of importance only in situations where all poles (or eigenvalues) are inside the closed unit disk while some of them are on the unit circle. As we shall see, in case of a stable all-pass system none of its poles lie on the unit circle so it is in fact asymptotically stable.

For an asymptotically stable state-space realization (A, B, C, D) one often considers the following two associated discrete-time Lyapunov equations:

$$W_c - AW_cA^* = BB^*, \tag{11}$$
$$W_o - A^*W_oA = C^*C. \tag{12}$$

Because of the asymptotic stability of A it follows that these equations admit unique solutions W_c and W_o, respectively, which are positive semi-definite and can be expressed as exponentially converging infinite sums:

$$W_c = \sum_{k=0}^{\infty} A^k BB^*(A^*)^k = \mathcal{C}_\infty \mathcal{C}_\infty^*, \tag{13}$$
$$W_o = \sum_{k=0}^{\infty} (A^*)^k C^*CA^k = \mathcal{O}_\infty^* \mathcal{O}_\infty. \tag{14}$$

If the realization is controllable then W_c is positive definite and W_c is called the controllability Gramian. Likewise, if the realization is observable then W_o is positive definite and W_o is called the observability Gramian. One speaks of a balanced realization if it is minimal (both controllable and observable) and such that the two Gramians W_c and W_o are identical diagonal matrices. It is well known that an asymptotically stable transfer function $G(z)$ always admits a balanced realization (A, B, C, D). Balanced realizations are well known for their allegedly good numerical properties and they lie at the heart of the well-known 'balance and truncate' procedure for state-space model reduction.

2.3 Σ-lossless rational matrix functions and state-space realization

A $p \times p$ matrix Σ is called a signature matrix if it is diagonal with all its main diagonal entries equal to ± 1. In the spirit of [8] we employ the following (slightly generalized) definition of Σ-lossless rational matrix functions.

Definition 2.2 *Let Σ be a $p \times p$ signature matrix. In discrete time, a $p \times p$ rational matrix function $G(z) \in \mathbb{K}^{p\times p}(z)$ in the complex variable z is called Σ-lossless if the following three conditions are satisfied at all points of analyticity z:*

$$\Sigma - G(z)\Sigma G(z)^* \geq 0 \qquad \text{for } |z| > 1, \tag{15}$$
$$\Sigma - G(z)\Sigma G(z)^* = 0 \qquad \text{for } |z| = 1, \tag{16}$$
$$\Sigma - G(z)\Sigma G(z)^* \leq 0 \qquad \text{for } |z| < 1. \tag{17}$$

In addition, a matrix function $G(z)$ is called Σ-unitary if condition (16) is satisfied.

In line with the terminology used in various parts of the literature (see [8, 5, 9, 12, 7, 1]) the following terminology is adopted: I_r-lossless is also called lossless, or stable all-pass; $(-I_r)$-lossless is also called I_r-inner, or inner, or anti-stable all-pass; more generally, $(-\Sigma)$-lossless is also called Σ-inner; finally, I_r-unitary is also called all-pass.

If $G(z)$ is Σ-unitary then it is invertible with its inverse given by $G^{-1}(z) = \Sigma G^*(z^{-1})\Sigma$. Here $G^*(z)$ is obtained from $G(z)$ by transposition and subsequent complex conjugation of all the coefficients in the rational expressions that constitute its entries (but not of the variable z): $G^*(z) = G(\bar{z})^*$ for all $z \in \mathbb{C}$. Note that the class of Σ-lossless matrix functions is a subset of the class of Σ-unitary matrices.

If $G(z)$ is Σ-lossless, then $G^*(z)$ is also Σ-lossless, while $G^{-1}(z)$ and $G(z^{-1})$ are both $(-\Sigma)$-lossless (or Σ-inner). As a matter of fact, it is well known (see [8]) that every two of the three conditions (15)-(17) from the definition of the Σ-lossless property implies the remaining third. Also, throughout the conditions (15)-(17) one may simultaneously replace all expressions $\Sigma - G(z)\Sigma G(z)^*$ appearing on the left-hand side by the expression $\Sigma - G(z)^*\Sigma G(z)$ without affecting validity of the definition.

It is well known that a $p \times p$ transfer function $G(z)$ of McMillan degree n is discrete-time stable all-pass, if and only if for each balanced state-space realization (A, B, C, D) the associated Gramians W_c and W_o are both equal to the identity matrix I_n. In that case, one has that (A', B', C', D') is also a balanced state-space realization of $G(z)$ if and only if there exists a *unitary* matrix Q such that $(A', B', C', D') = (QAQ^*, QB, CQ^*, D)$. Since all the diagonal entries of the two Gramians coincide, the class of stable all-pass systems actually constitutes a case in which the degree of freedom for choosing balanced state-space representations is maximal.

We now make the following preparatory definitions, which set up a link between state-space realizations and transfer functions by means of a linear fractional transformations and sub-matrix selection.

Definition 2.3 *The state-space realization matrix R associated with an n-dimensional state-space realization (A, B, C, D) of a transfer function $G(z) \in \mathbb{K}^{p\times p}(z)$, is defined as the block-partitioned $(p + n) \times (p + n)$ matrix*

$$R = \begin{pmatrix} D & C \\ B & A \end{pmatrix}. \tag{18}$$

Definition 2.4 *Let $p \geq 1$ and $n \geq 0$ be integers. The mapping $\mathcal{R}_{p,n} : \mathcal{M}_{p,n} \to \mathcal{M}_{p,n}$ is defined by*

$$\mathcal{R}_{p,n}(X(z)) := \begin{pmatrix} I_p & 0 \end{pmatrix} \mathcal{T}_{R(z)}(X(z)) \begin{pmatrix} I_p \\ 0 \end{pmatrix} \tag{19}$$

for all rational $(p + n) \times (p + n)$ matrices $X(z)$ in the domain $\mathcal{M}_{p,n} := \mathcal{M}_{R(z)}$, where $R(z) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -zI_n & 0 & I_n \\ 0 & 0 & I_p & 0 \\ 0 & I_n & 0 & 0 \end{pmatrix}$.

Alternatively, the same mapping is given by $\mathcal{R}_{p,n}(X(z)) = \begin{pmatrix} I_p & 0 \end{pmatrix} \widehat{\mathcal{T}}_{\widehat{R}(z)}(X(z)) \begin{pmatrix} I_p \\ 0 \end{pmatrix}$ where $\widehat{R}(z) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & I_p & 0 \\ 0 & -I_n & 0 & zI_n \end{pmatrix}$. Also, it is easily verified that if R denotes the realization matrix associated with an n-dimensional state-space realization (A, B, C, D) of a proper rational transfer function $G(z) \in \mathbb{K}^{p\times p}(z)$, then $R \in \mathcal{M}_{p,n}$ and $\mathcal{R}_{p,n}(R) = G(z)$.

When the mapping $\mathcal{R}_{p,n}$ is applied to a $\begin{pmatrix} \Sigma & 0 \\ 0 & I_n \end{pmatrix}$-lossless matrix function, the result is Σ-lossless. This is the content of the following theorem, of which the proof involves a generalization of Eqn. (22) of [8].

Theorem 2.5 *Let Σ be a $p \times p$ signature matrix and let $G(z)$ be $\begin{pmatrix} \Sigma & 0 \\ 0 & I_n \end{pmatrix}$-lossless, of size $(p+n) \times (p+n)$. Then $H(z) = \mathcal{R}_{p,n}(G(z))$ is Σ-lossless, of size $p \times p$.*

Proof. Let $G(z) = \begin{pmatrix} G_1(z) & G_2(z) \\ G_3(z) & G_4(z) \end{pmatrix}$ be block-partitioned conformably with $\begin{pmatrix} \Sigma & 0 \\ 0 & I_n \end{pmatrix}$. Then the expression $\Sigma - G(z)\Sigma G(z)^*$ can be cast into the form:

$$\begin{pmatrix} I_p & G_2(z)(zI_n - G_4(z))^{-1} \end{pmatrix} \left[Q(z) + (|z|^2 - 1) \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} \right] \begin{pmatrix} I_p \\ (\overline{z}I_n - G_4(z)^*)^{-1} G_2(z)^* \end{pmatrix}, \tag{20}$$

with $Q(z) = \begin{pmatrix} \Sigma & 0 \\ 0 & I_n \end{pmatrix} - G(z) \begin{pmatrix} \Sigma & 0 \\ 0 & I_n \end{pmatrix} G(z)^*$. Therefore it follows from the assumption of $G(z)$ being $\begin{pmatrix} \Sigma & 0 \\ 0 & I_n \end{pmatrix}$-lossless and the fact that also the inertia of $(|z|^2 - 1) \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}$ depends entirely on the sign of $(|z|^2 - 1)$, that $G(z)$ is indeed Σ-lossless. □

In particular, if $G(z)$ is lossless then also $H(z) = \mathcal{R}_{p,n}(G(z))$ is lossless. When $G(z)$ is a constant unitary matrix, it therefore acts as a realization matrix for a lossless transfer function $H(z)$. This well known result is addressed in the following two theorems; see also [9, 8, 14].

Theorem 2.6 *Let $R = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ be a unitary block-partitioned realization matrix of size $(p+n) \times (p+n)$. Then the associated $p \times p$ transfer function $G(z) = D + C(zI_n - A)^{-1}B$ is stable all-pass (lossless) of McMillan degree $\leq n$. The state-space realization (A, B, C, D) is minimal if and only if A is asymptotically stable, in which case the realization is balanced.*

Conversely, it can be established that every lossless transfer function admits a unitary realization matrix corresponding to a minimal balanced realization.

Theorem 2.7 *Let $G(z)$ be a $p \times p$ rational lossless transfer function of McMillan degree equal to n. Then it is proper and it admits a minimal balanced realization (A, B, C, D). In that case A is asymptotically stable of size $n \times n$, the Gramians W_c and W_o are both equal to I_n, and the realization matrix $R = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ is unitary.*

The results above can actually be generalized to the case of proper Σ-lossless transfer functions in the following way. As in [8, 14] a state-space realization (A, B, C, D) is called Σ-balanced if it is minimal and the solutions W_c' and W_o' of the generalized discrete-time Lyapunov equations

$$W_c' - AW_c'A^* = B\Sigma B^*, \tag{21}$$

$$W_o' - A^*W_o'A = C^*\Sigma C, \tag{22}$$

exist, are unique, and are identical diagonal matrices. Then if $G(z)$ is a $p \times p$ proper Σ-lossless transfer function of McMillan degree n, it admits a Σ-balanced state-space realization (A, B, C, D) with $W_c' = W_o' = \Sigma$, while conversely such a Σ-balanced state-space realization with $W_c' = W_o' = \Sigma$ corresponds to a proper Σ-lossless transfer function.

To conclude this section we state the following proposition.

Proposition 2.8 *Let Σ be a $p \times p$ signature matrix.*
(i) If $\Theta(z) \in \mathbb{K}^{2p\times 2p}(z)$ is $\begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix}$-lossless and $G(z) \in \mathcal{M}_{\Theta(z)}$ is lossless, then $\widetilde{G}(z) := \mathcal{T}_{\Theta(z)}(G(z))$ is Σ-lossless.
(ii) If $\Theta(z) \in \mathbb{K}^{2p\times 2p}(z)$ is $\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}$-lossless and $G(z) \in \widehat{\mathcal{M}}_{\Theta(z)}$ is lossless, then $\widetilde{G}(z) := \widehat{\mathcal{T}}_{\Theta(z)}(G(z))$ is Σ-lossless.

Proof. We first address part (ii). Let Θ be partitioned into $p \times p$ blocks, as usual. Then $\widetilde{G} = (G\Theta_2 + \Theta_4)^{-1}(G\Theta_1 + \Theta_3)$ and the expression $\Sigma - \widetilde{G}\Sigma\widetilde{G}^*$ can be seen to attain the form $(G\Theta_2 + \Theta_4)^{-1} \begin{pmatrix} G & I_p \end{pmatrix} \left[\left[\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} - \Theta \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \Theta^*\right] - \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}\right] \begin{pmatrix} G^* \\ I_p \end{pmatrix} (G\Theta_2 + \Theta_4)^{-*}$. From the assumptions on Θ and G it now follows directly that this expression is positive semi-definite for $|z| > 1$, zero for $|z| = 1$ and negative semi-definite for $|z| < 1$, whence $\widetilde{G}(z)$ is indeed Σ-lossless.

To see part (i), note that this follows from part (ii) because $\mathcal{T}_\Theta = \widehat{\mathcal{T}}_{J_p\Theta^{-1}J_p}$ with now $J_p\Theta^{-1}J_p$ being $\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}$-lossless. $\square$

3 A state-space framework for linear fractional transformations

In this section we establish a framework in which the linear fractional transformations $\mathcal{T}_{\Theta(z)}(G(z)$ and $\widehat{\mathcal{T}}_{\Theta(z)}(G(z)$ are represented, respectively, as the linear fractional transformations $\mathcal{T}_{\widetilde{\Phi}}(\widetilde{R})$ and $\widehat{\mathcal{T}}_{\widetilde{\Psi}}(\widetilde{R})$, in which $\widetilde{\Phi}$, $\widetilde{\Psi}$ and $\widetilde{R}$ denote constant block-partitioned matrices involving state-space realizations of $\Theta(z^{-1})$, $\Theta(z)$ and $G(z)$. More precisely, the matrix $\widetilde{R}$ denotes a (non-minimal, extended) realization matrix for $G(z)$ and the outcomes $\mathcal{T}_{\widetilde{\Phi}}(\widetilde{R})$ and $\widehat{\mathcal{T}}_{\widetilde{\Psi}}(\widetilde{R})$ constitute realization matrices for $\mathcal{T}_{\Theta(z)}(G(z)$ and $\widehat{\mathcal{T}}_{\Theta(z)}(G(z)$, respectively. This makes it possible to supply a sequence of transfer functions generated by iterative application of linear fractional transformations, with corresponding state-space realizations. The importance of this issue lies in the fact that the availability of state-space realizations enables the application of many tools from linear algebra and modern control theory, enabling the development of new theory and proofs along different lines as well as achieving a decrease of the gap between theory and practical applications.

We first consider the following four lemmas. The first lemma will come as no surprise, since a change of basis of the state space leaves the transfer function of a system unchanged.

Lemma 3.1 *Let $S = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & T \end{pmatrix}$ where T is a rational invertible matrix of size $n \times n$. It then holds that*

$$\mathcal{R}_{p,n} = \mathcal{R}_{p,n} \circ \mathcal{T}_S = \mathcal{R}_{p,n} \circ \widehat{\mathcal{T}}_S. \tag{23}$$

Proof. Note that S commutes with the matrices $R(z)$ and $\widehat{R}(z)$ in the definition of the mapping $\mathcal{R}_{p,n}$ (or its alternative form). Note also that $\mathcal{T}_S$ and $\widehat{\mathcal{T}}_S$ leave the $p \times p$ block in the left upper corner of their arguments unchanged. $\square$

The next lemma, like in the definition of the mapping $\mathcal{R}_{p,n}$, combines the action of sub-matrix selection with the action of a linear fractional transformation.

Lemma 3.2 *Let $X \in \mathbb{K}^{(p+n)\times(p+n)}(z)$ and $\Theta = \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{pmatrix}$ a block-partitioned invertible rational matrix of size $2p \times 2p$ with blocks Θ_i $(i = 1, \ldots, 4)$ of size $p \times p$. Denote $\widetilde{\Theta} := \begin{pmatrix} \Theta_1 & 0 & \Theta_2 & 0 \\ 0 & I_n & 0 & 0 \\ \Theta_3 & 0 & \Theta_4 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}$. Then:*

$$\mathcal{T}_\Theta(\begin{pmatrix} I_p & 0 \end{pmatrix} X \begin{pmatrix} I_p \\ 0 \end{pmatrix}) = \begin{pmatrix} I_p & 0 \end{pmatrix} \mathcal{T}_{\widetilde{\Theta}}(X) \begin{pmatrix} I_p \\ 0 \end{pmatrix} \tag{24}$$

if $\mathcal{T}_{\widetilde{\Theta}}(X)$ is well-defined. Likewise:

$$\widehat{\mathcal{T}}_\Theta(\begin{pmatrix} I_p & 0 \end{pmatrix} X \begin{pmatrix} I_p \\ 0 \end{pmatrix}) = \begin{pmatrix} I_p & 0 \end{pmatrix} \widehat{\mathcal{T}}_{\widetilde{\Theta}}(X) \begin{pmatrix} I_p \\ 0 \end{pmatrix} \tag{25}$$

if $\widehat{\mathcal{T}}_{\widetilde{\Theta}}(X)$ is well-defined.

Proof. Let X be block-partitioned as $\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ with X_1 of size $p \times p$. Consider $\mathcal{T}_{\widetilde{\Theta}}(X) = \begin{pmatrix} \Theta_4 X_1 + \Theta_3 & \Theta_4 X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} \Theta_2 X_1 + \Theta_1 & \Theta_2 X_2 \\ 0 & I_n \end{pmatrix}^{-1}$, from which it follows that the $p \times p$ block in the left upper corner is given by $(\Theta_4 X_1 + \Theta_3)(\Theta_2 X_1 + \Theta_1)^{-1} = \mathcal{T}_\Theta(X_1)$. This proves the first identity; the second one is proved likewise. $\square$

The following lemma presents a result on the composition of a linear fractional transformation with the mapping $\mathcal{R}_{p,n}$.

Lemma 3.3 *Let $\Theta(z) = \begin{pmatrix} \Theta_1(z) & \Theta_2(z) \\ \Theta_3(z) & \Theta_4(z) \end{pmatrix}$ be a block-partitioned rational matrix in $\mathbb{K}^{2p\times 2p}(z)$ with blocks $\Theta_i(z)$ $(i = 1, \ldots, 4)$ of size $p \times p$. Denote $\widetilde{\Theta}(z) := \begin{pmatrix} \Theta_1(z) & 0 & \Theta_2(z) & 0 \\ 0 & I_n & 0 & 0 \\ \Theta_3(z) & 0 & \Theta_4(z) & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}$. Then:*

$$\mathcal{R}_{p,n} \circ \mathcal{T}_{\widetilde{\Theta}(z)} = \mathcal{T}_{\Theta(z)} \circ \mathcal{R}_{p,n}. \tag{26}$$

Likewise:

$$\mathcal{R}_{p,n} \circ \widehat{\mathcal{T}}_{\widetilde{\Theta}(z)} = \widehat{\mathcal{T}}_{\Theta(z)} \circ \mathcal{R}_{p,n}. \tag{27}$$

Proof. To see the first identity, note that (for all rational $X(z)$ for which the mappings are well-defined) one may write $\mathcal{R}_{p,n} \circ \mathcal{T}_{\widetilde{\Theta}(z)}(X(z)) = \begin{pmatrix} I_p & 0 \end{pmatrix} \mathcal{T}_{R(z)} \circ \mathcal{T}_{\widetilde{\Theta}(z)}(X(z)) \begin{pmatrix} I_p \\ 0 \end{pmatrix}$ where $R(z) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -zI_n & 0 & I_n \\ 0 & 0 & I_p & 0 \\ 0 & I_n & 0 & 0 \end{pmatrix}$. From the structure of $\widetilde{\Theta}(z)$ it is clear that it commutes with

$R(z)$. Application of Lemma 3.2 now yields the first stated result. The second identity is proved in an analogous fashion. $\Box$

Finally, we have an associativity property for the composition of (compatible) mappings of the form $\mathcal{R}_{p,n}$.

Lemma 3.4 *Let $p > 0$, $n \geq 0$ and $m \geq 0$ be integers. Then*

$$\mathcal{R}_{p,n} \circ \mathcal{R}_{p+n,m} = \mathcal{R}_{p,n+m} \tag{28}$$

on the whole domain of rational matrices of size $(p+n+m) \times (p+n+m)$ for which the composition of mappings on the left-hand side is well-defined.

Proof. For $X \in \mathbb{K}^{2(p+n+m)\times 2(p+n+m)}(z)$ consider the expression $\mathcal{R}_{p,n} \circ \mathcal{R}_{p+n,m}(X)$ (provided it is well-defined); this is equal to $\begin{pmatrix} I_p & 0 \end{pmatrix} \mathcal{T}_{R_1(z)}(\begin{pmatrix} I_{p+n} & 0 \end{pmatrix} \mathcal{T}_{R_2(z)}(X) \begin{pmatrix} I_{p+n} \\ 0 \end{pmatrix}) \begin{pmatrix} I_p \\ 0 \end{pmatrix}$, where $R_1(z) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -zI_n & 0 & I_n \\ 0 & 0 & I_p & 0 \\ 0 & I_n & 0 & 0 \end{pmatrix}$ and $R_2(z) = \begin{pmatrix} I_{p+n} & 0 & 0 & 0 \\ 0 & -zI_m & 0 & I_m \\ 0 & 0 & I_{p+n} & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix}$. Using Lemma 3.2 this can be rewritten as $\begin{pmatrix} I_p & 0 \end{pmatrix} \begin{pmatrix} I_{p+n} & 0 \end{pmatrix} \mathcal{T}_{\widetilde{R}_1(z)} \circ \mathcal{T}_{R_2(z)}(X) \begin{pmatrix} I_{p+n} \\ 0 \end{pmatrix} \begin{pmatrix} I_p \\ 0 \end{pmatrix}$ with $\widetilde{R}_1(z) = \begin{pmatrix} I_p & 0 & 0 & 0 & 0 & 0 \\ 0 & -zI_n & 0 & 0 & I_n & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_p & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m \end{pmatrix}$. Note that $\widetilde{R}_1(z) R_2(z) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -zI_{n+m} & 0 & I_{n+m} \\ 0 & 0 & I_p & 0 \\ 0 & I_{n+m} & 0 & 0 \end{pmatrix}$, from which the lemma follows. $\Box$

We now come to the main result of this section.

Theorem 3.5 *Let (A, B, C, D) be an n-dimensional state-space realization of a $p \times p$ proper rational transfer function $G(z)$ and let $R := \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ be its associated realization matrix. Also, let $(\mathcal{A}, \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{pmatrix}, \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}, \begin{pmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ \mathcal{D}_3 & \mathcal{D}_4 \end{pmatrix})$ be an m-dimensional state-space realization of a $2p \times 2p$ proper rational transfer function $\Theta(z) = \begin{pmatrix} \Theta_1(z) & \Theta_2(z) \\ \Theta_3(z) & \Theta_4(z) \end{pmatrix}$, conformably block-partitioned with blocks $\Theta_i(z)$ $(i = 1, \ldots, 4)$ of size $p \times p$. In addition, denote $\widetilde{R} := \begin{pmatrix} R & 0 \\ 0 & I_m \end{pmatrix}$, $\widetilde{G}(z) := \begin{pmatrix} G(z) & 0 \\ 0 & I_m \end{pmatrix}$, $\widetilde{\Theta}(z) := \begin{pmatrix} \Theta_1(z) & 0 & \Theta_2(z) & 0 \\ 0 & I_n & 0 & 0 \\ \Theta_3(z) & 0 & \Theta_4(z) & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}$, $\Phi := \begin{pmatrix} \mathcal{D}_1 & \mathcal{C}_1 & \mathcal{D}_2 & 0 \\ \mathcal{B}_1 & \mathcal{A} & \mathcal{B}_2 & 0 \\ \mathcal{D}_3 & \mathcal{C}_2 & \mathcal{D}_4 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}$ and $\widetilde{\Phi} := \begin{pmatrix} \mathcal{D}_1 & 0 & \mathcal{C}_1 & \mathcal{D}_2 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ \mathcal{B}_1 & 0 & \mathcal{A} & \mathcal{B}_2 & 0 & 0 \\ \mathcal{D}_3 & 0 & \mathcal{C}_2 & \mathcal{D}_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & I_m \end{pmatrix}$. Then:*

$$\mathcal{T}_{\Theta(z^{-1})}(G(z)) = \mathcal{R}_{p,n} \circ \mathcal{T}_{\widetilde{\Theta}(z^{-1})}(R) = \mathcal{R}_{p,m} \circ \mathcal{T}_{\Phi}(\widetilde{G}(z)) = \mathcal{R}_{p,n+m} \circ \mathcal{T}_{\widetilde{\Phi}}(\widetilde{R}) \tag{29}$$

provided that $\widetilde{R} \in \mathcal{M}_{\widetilde{\Phi}}$. *Likewise:*

$$\widehat{\mathcal{T}}_{\Theta(z)}(G(z)) = \mathcal{R}_{p,n} \circ \widehat{\mathcal{T}}_{\widetilde{\Theta}(z)}(R) = \mathcal{R}_{p,m} \circ \widehat{\mathcal{T}}_{\Phi}(\widetilde{G}(z)) = \mathcal{R}_{p,n+m} \circ \widehat{\mathcal{T}}_{\widetilde{\Phi}}(\widetilde{R}) \tag{30}$$

if $\widetilde{R} \in \widehat{\mathcal{M}}_{\widetilde{\Phi}}$.

Proof. We concentrate on the first series of identities (29), involving linear fractional transformations of the type $\mathcal{T}_{\Theta}$. The second series of identities (30) can be handled in a similar fashion.

To see the first equality in (29) we simply invoke Lemma 3.3 with the mappings applied to the realization matrix R.

To prove the second equality in (29), we elaborate on the expression $\mathcal{R}_{p,m} \circ \mathcal{T}_{\Phi}(\widetilde{G}(z))$ which is first rewritten into the form $\begin{pmatrix} I_p & 0 \end{pmatrix} \mathcal{T}_{V(z)} \circ \mathcal{T}_{\Phi}(\widetilde{G}(z)) \begin{pmatrix} I_p \\ 0 \end{pmatrix}$ with $V(z) = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -zI_m & 0 & I_m \\ 0 & 0 & I_p & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix}$. The matrix product $W(z) := V(z)\Phi$ takes the form $W(z) = \begin{pmatrix} \mathcal{D}_1 & \mathcal{C}_1 & \mathcal{D}_2 & 0 \\ -z\mathcal{B}_1 & -z\mathcal{A} & -z\mathcal{B}_2 & I_m \\ \mathcal{D}_3 & \mathcal{C}_2 & \mathcal{D}_4 & 0 \\ \mathcal{B}_1 & \mathcal{A} & \mathcal{B}_2 & 0 \end{pmatrix}$. Working out the structure of $\widetilde{G}(z)$ the expression $\mathcal{T}_{W(z)}(\widetilde{G}(z))$ is equivalent to $\begin{pmatrix} \mathcal{D}_4 G(z) + \mathcal{D}_3 & \mathcal{C}_2 \\ \mathcal{B}_2 G(z) + \mathcal{B}_1 & \mathcal{A} \end{pmatrix} \begin{pmatrix} \mathcal{D}_2 G(z) + \mathcal{D}_1 & \mathcal{C}_1 \\ -z(\mathcal{B}_2 G(z) + \mathcal{B}_1) & I_m - z\mathcal{A} \end{pmatrix}^{-1}$. We are looking for the left upper $p \times p$ block of this matrix, which is given by the product $\begin{pmatrix} \mathcal{D}_4 G(z) + \mathcal{D}_3 & \mathcal{C}_2 \end{pmatrix} \begin{pmatrix} I_p \\ (z^{-1}I_m - \mathcal{A})^{-1}(\mathcal{B}_2 G(z) + \mathcal{B}_1) \end{pmatrix} (\mathcal{D}_2 G(z) + \mathcal{D}_1 + \mathcal{C}_1(z^{-1}I_m - \mathcal{A})^{-1}(\mathcal{B}_2 G(z) + \mathcal{B}_1))^{-1}$ using a well known result on the inversion of block-partitioned matrices; see, e.g., [19, App. A.1]. From the definition of the state-space realization of $\Theta(z)$ it then follows that this is equal to $(\Theta_4(z^{-1})G(z) + \Theta_3(z^{-1}))(\Theta_2(z^{-1})G(z) + \Theta_1(z^{-1}))^{-1} = \mathcal{T}_{\Theta(z^{-1})}(G(z))$.

To prove the third equality in (29), note that $\mathcal{R}_{p,n+m} \circ \mathcal{T}_{\widetilde{\Phi}}(\widetilde{R})$ can be written as $\mathcal{R}_{p,n+m} \circ \mathcal{T}_{\widetilde{\Pi}} \circ \mathcal{T}_{\widetilde{\Pi}^*} \circ \mathcal{T}_{\widetilde{\Phi}} \circ \mathcal{T}_{\widetilde{\Pi}} \circ \mathcal{T}_{\widetilde{\Pi}^*}(\widetilde{R})$ where $\widetilde{\Pi}$ denotes the block-partitioned permutation matrix $\widetilde{\Pi} := \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & \Pi \end{pmatrix}$, in which Π denotes the permutation matrix $\Pi := \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$. Note that according to Lemma 3.1 it holds that $\mathcal{R}_{p,n+m} \circ \mathcal{T}_{\widetilde{\Pi}} = \mathcal{R}_{p,n+m}$, which can be rewritten using Lemma 3.4 as $\mathcal{R}_{p,m} \circ \mathcal{R}_{p+m,n}$. Also, $\mathcal{T}_{\widetilde{\Pi}^*} \circ \mathcal{T}_{\widetilde{\Phi}} \circ \mathcal{T}_{\widetilde{\Pi}} = \mathcal{T}_{\widehat{\Phi}}$ with $\widehat{\Phi} := \begin{pmatrix} \Phi_1 & 0 & \Phi_2 & 0 \\ 0 & I_n & 0 & 0 \\ \Phi_3 & 0 & \Phi_4 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}$, where Φ_i $(i = 1, \ldots, 4)$ denote the four blocks of size $(p+m) \times (p+m)$ of a corresponding block-partition of Φ. Moreover, $\mathcal{T}_{\widetilde{\Pi}^*}(\widetilde{R}) = \begin{pmatrix} D & 0 & C \\ 0 & I_m & 0 \\ B & 0 & A \end{pmatrix}$. Taken together this yields the expression $\mathcal{R}_{p,m} \circ \mathcal{R}_{p+m,n} \circ \mathcal{T}_{\widehat{\Phi}}(\begin{pmatrix} D & 0 & C \\ 0 & I_m & 0 \\ B & 0 & A \end{pmatrix})$. Application of Lemma 3.3 then yields the equivalent expression $\mathcal{R}_{p,m} \circ \mathcal{T}_{\Phi} \circ \mathcal{R}_{p+m,n}(\begin{pmatrix} D & 0 & C \\ 0 & I_m & 0 \\ B & 0 & A \end{pmatrix}) = \mathcal{R}_{p,m} \circ \mathcal{T}_{\Phi}(\widetilde{G}(z))$. □

These representations of linear fractional transformations entirely in terms of associated state-space realizations as expressed by the right-most expressions in (29) and (30), have the computational advantage that only constant matrices are involved, which are conveniently handled by standard methods from numerical linear algebra. The theorem makes clear that the resulting realization matrices on the right-hand sides are of size $(p+n+m) \times (p+n+m)$ whence the resulting transfer function matrices on the left-hand sides are of McMillan degree at most $n+m$. In contrast, the linear fractional transformations on the left-hand sides involve rational matrices in the symbolic variable z, initially giving rise to common denominator polynomials of degree $2(n+m)$. The cancellation of common factors which is guaranteed to take place by theory, presents one with the nontrivial task of adequately detecting and removing these common factors in this mixed numerical and symbolic context.

4 Balanced realization of the class of discrete-time stable all-pass functions

We now turn to the subject of constructing parametrizations of the class of discrete-time stable all-pass systems. We shall discuss two different approaches. The first approach (see [7, 1]) proceeds entirely on the level of transfer functions by means of the (reversed) tangential Schur algorithm which employs linear fractional transformations. The second approach (see [9]) instead acts on the level of balanced state-space realizations by means of unitary matrix operations and also happens to involve Schur parameters. Using the tools developed in the previous sections these two approaches are captured in a unified framework which helps to clarify their interrelationships and enables the construction of balanced state-space realizations associated with the tangential Schur algorithm. The material presented here yields a significant extension of the preliminary results on this topic in [16].

4.1 The tangential Schur algorithm

In each recursion step of the (reversed) tangential Schur algorithm, the McMillan degree of the rational stable all-pass function at hand is increased by 1 by the action of a linear fractional transformation. This transformation is associated with a J_p-inner matrix function $\Theta(z)$ of McMillan degree 1 which has a particular form that stems from the theory of reproducing kernel Hilbert spaces (see also [5, 6, 7]):

$$\Theta(z) = \Theta(u, v, w, \xi, H; z) = \left(I_{2p} - \frac{(1-\overline{\xi}z)(1-|w|^2)}{(1-\|v\|^2)(1-\overline{w}z)(1-\overline{\xi}w)} \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}^* J_p \right) H. \tag{31}$$

Here, the associated parameters (that may be chosen independently for each recursion step) are required to satisfy the following properties. (1) $u \in \mathbb{C}^{p \times 1}$ is a normalized direction vector such that $\|u\| = 1$. (2) $v \in \mathbb{C}^{p \times 1}$ is a (generalized) Schur parameter vector satisfying $\|v\| < 1$. (3) w is an interpolation point with $|w| < 1$. (4) ξ is a point with $|\xi| = 1$. (5) H is a constant J_p-unitary matrix.

At the point $z = \xi$ it holds that $\Theta(\xi) = H$. From the structure of $\Theta(z)$ it also follows that $\widetilde{G}(z) = \mathcal{T}_{\Theta(z)}(G(z))$ satisfies the interpolation condition

$$\widetilde{G}(\overline{w}^{-1})u = v. \tag{32}$$

In the standard case with $w = 0$ the value of $\widetilde{G}(\infty)$ corresponds to the direct feedthrough term $\widetilde{D}$ of any state-space realization $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ of $\widetilde{G}(z)$, so that the interpolation condition then takes the form $\widetilde{D}u = v$.

If $G(z)$ is stable all-pass of McMillan degree n then (for each set of admissible choices of the parameters) it is well known that it belongs to the domain $\mathcal{M}_{\Theta(u,v,w,\xi,H;z)}$ of the linear fractional transformation involved; moreover its image $\widetilde{G}(z) = \mathcal{T}_{\Theta(u,v,w,\xi,H;z)}(G(z))$ is also stable all-pass and of McMillan degree $n+1$.

Conversely, if $\widetilde{G}(z)$ is stable all-pass of McMillan degree $n+1$ and $\Theta(u,v,w,\xi,H;z)$ is a J_p-inner function (stemming from an admissible set of choices of the parameters) then there exists a stable all-pass function $G(z)$ of McMillan degree n such that $\widetilde{G}(z) = \mathcal{T}_{\Theta(u,v,w,\xi,H;z)}(G(z))$ if and only if u, v and w are such that $\widetilde{G}(z)$ satisfies the interpolation condition (32). Note that for all w in the open unit disk it holds that $\widetilde{G}(\overline{w}^{-1})\widetilde{G}(\overline{w}^{-1})^* \leq I_p$; therefore, if $\widetilde{G}(z)$ is not constant unitary, one can indeed always find a vector u of norm 1 for which $v := \widetilde{G}(\overline{w}^{-1})u$ has norm strictly less than 1.

It then can be shown that each set of admissible values for the parameters u, w, ξ and H at the first n recursion steps may serve to index a generic chart for the manifold of stable all-pass systems of order n. The Schur parameter vectors v together with an initial unitary matrix, say G_0, provide the local coordinates for this chart. An infinite atlas of overlapping generic charts is obtained by varying the choices for u, w, ξ and H, but atlases containing less charts may also be obtained by limiting the freedom for choosing these parameters.

The following proposition (see [16]) shows how a linear fractional transformation in an iteration step of the tangential Schur algorithm can be written as a composition of a number of more simple linear fractional transformations.

Proposition 4.1 *The J_p-inner matrix function $\Theta(u,v,w,\xi,H;z)$ can be factored as:*

$$\Theta(u,v,w,\xi,H;z) = H(uv^*)S_{u,w}(z)S_{u,w}(\xi)^{-1}H(uv^*)^{-1}H, \tag{33}$$

where $H(uv^*)$ denotes the J_p-unitary Halmos extension of the strictly contractive matrix uv^* and where $S_{u,w}(z)$ is defined as the J_p-inner matrix function

$$S_{u,w}(z) = \begin{pmatrix} I_p - \left(1 - \frac{(z-w)}{(1-\overline{w}z)}\right)uu^* & 0 \\ 0 & I_p \end{pmatrix}. \tag{34}$$

Here, the Halmos extension $H(E)$ of a strictly contractive matrix E (i.e., having all its singular values strictly less than 1) is the J_p-unitary matrix defined by

$$H(E) = \begin{pmatrix} (I_p - EE^*)^{-\frac{1}{2}} & E(I_p - E^*E)^{-\frac{1}{2}} \\ E^*(I_p - EE^*)^{-\frac{1}{2}} & (I_p - E^*E)^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} (I_p - EE^*)^{-\frac{1}{2}} & (I_p - EE^*)^{-\frac{1}{2}}E \\ (I_p - E^*E)^{-\frac{1}{2}}E^* & (I_p - E^*E)^{-\frac{1}{2}} \end{pmatrix}. \tag{35}$$

For $\|u\| = 1$ and $\|v\| < 1$, the matrix uv^* is indeed strictly contractive, with Halmos extension given by

$$H(uv^*) = \begin{pmatrix} I_p - (1 - \frac{1}{\sqrt{1-\|v\|^2}})uu^* & \frac{1}{\sqrt{1-\|v\|^2}}uv^* \\ \frac{1}{\sqrt{1-\|v\|^2}}vu^* & I_p - (1 - \frac{1}{\sqrt{1-\|v\|^2}})\frac{vv^*}{\|v\|^2} \end{pmatrix}. \tag{36}$$

Its inverse satisfies $H(uv^*)^{-1} = H(-uv^*)$.

Note that the linear fractional transformations associated with the constant J_p-unitary matrices H, $H(uv^*)$, $H(uv^*)^{-1}$ and $S_{u,w}(\xi)^{-1}$ are all generalized Möbius transformations; they do not change the McMillan degree of a matrix function on which they act. Only the transformation associated with the matrix $S_{u,w}(z)$ effectuates an order increase by 1, but it has a simple form that does not involve the Schur parameter vector v, nor ξ, nor H.

For reference we mention that if M is a constant J_p-unitary matrix it can be represented in a unique way, see [4], as

$$M = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} H(E), \tag{37}$$

where P and Q are $p \times p$ unitary matrices and $H(E)$ denotes the Halmos extension of a strictly contractive $p \times p$ matrix E.

4.2 The balanced state-space approach

We now turn towards the second approach for parametrizing the space of stable all-pass systems, in terms of balanced state-space realizations. We start by introducing some more notation, reformulating concepts previously introduced in [16] in the language of the present framework of linear fractional transformations.

With each pair (U, V) of constant $(p+1) \times (p+1)$ matrices we associate a mapping $\mathcal{F}_{U,V}$ that is defined to act on $p \times p$ rational matrix functions $G(z)$ as follows:

$$\mathcal{F}_{U,V}(G(z)) := \mathcal{R}_{p,1} \circ \mathcal{T}_{\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}}\left(\begin{pmatrix} 1 & 0 \\ 0 & G(z) \end{pmatrix}\right) \tag{38}$$

for all $G(z)$ for which the right-hand side expression is well-defined. Let (A, B, C, D) be an n-dimensional state-space realization of $G(z)$. Then an $(n+1)$-dimensional state-space realization $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ of $\widetilde{G}(z) := \mathcal{F}_{U,V}(G(z))$ is given by:

$$\begin{pmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & C \\ 0 & B & A \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & I_n \end{pmatrix}. \tag{39}$$

From the earlier observations on balanced state-space realization of stable all-pass functions and Eqn. (39) it is immediate that for *unitary* matrices U and V, the mapping $\mathcal{F}_{U,V}$ takes stable all-pass transfer functions of McMillan degree n into stable all-pass transfer functions of McMillan degree $\leq n+1$.

In the SISO case, mappings of the form $\mathcal{F}_{U,V}$ with $U = I_2$ have been used in [9] to recursively construct a balanced canonical form for the space of discrete-time stable all-pass systems of finite McMillan degree. The parameters that occur in this recursion have the interpretation of Schur parameters, corresponding to the situation with interpolation points w at zero. Note that on the level of state-space realization matrices, a composition of mappings of the form $\mathcal{F}_{U,V}$ is implemented as a product of unitary matrices, which has several corresponding numerical advantages. The question thus arises whether it is possible to represent a linear fractional transformation $\mathcal{T}_{\Theta(u,v,w,\xi,H;z)}$ from an iteration step of the tangential Schur algorithm as a mapping of the form $\mathcal{F}_{U,V}$. This would give us balanced state-space parametrizations directly in terms of the set of parameters u, v, w, ξ and H used in the tangential Schur algorithm. In [16] the conditions under which this can be achieved have been completely analyzed.

Let the J_p-inner matrix function $\hat{\Theta}(u, v, w; z)$ be introduced as

$$\hat{\Theta}(u, v, w; z) = H(uv^*) S_{u,w}(z) H(\overline{w}uv^*). \tag{40}$$

The main result of [16] then reads as follows.

Theorem 4.2 *Let u, $v \in \mathbb{C}^{p\times 1}$ and $w \in \mathbb{C}$ such that $\|u\| = 1$, $\|v\| < 1$ and $|w| < 1$. Then for all $p \times p$ proper rational stable all-pass functions $G(z)$ of finite McMillan degree it holds that*

$$\mathcal{T}_{\hat{\Theta}(u,v,w;z)}(G(z)) = \mathcal{F}_{U,V}(G(z)), \tag{41}$$

if U and V are taken to be the unitary $(p+1) \times (p+1)$ matrices

$$U = \begin{pmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}u & I_p - \left(1 + \frac{w\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}}\right)uu^* \\ \frac{\overline{w}\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}u^* \end{pmatrix}, \tag{42}$$

$$V = \begin{pmatrix} \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}v & I_p - \left(1 - \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}}\right)\frac{vv^*}{\|v\|^2} \\ \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}} & -\frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}v^* \end{pmatrix}. \tag{43}$$

According to Proposition 4.1 the matrix $\Theta(u, v, w, \xi, H; z)$ is of the form $\hat{\Theta}(u, v, w; z)M$, with the matrix $M = \hat{\Theta}(u, v, w; \xi)^{-1}H$ constant and J_p-unitary. Thus M can be parametrized as $M = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} H(E)$ for unique, unitary matrices P and Q and a unique strictly contractive matrix E. The following proposition indicates for which matrices M the mapping $\mathcal{T}_{\hat{\Theta}(u,v,w;z)M}(G(z))$ can be represented in the form $\mathcal{F}_{U,V}(G(z))$.

Proposition 4.3 *Let u, $v \in \mathbb{C}^{p\times 1}$ and $w \in \mathbb{C}$ such that $\|u\| = 1$, $\|v\| < 1$ and $|w| < 1$. Let $M = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} H(E)$ be J_p-unitary, with P and Q $p \times p$ unitary and E $p \times p$ strictly contractive. Then the mapping $\mathcal{T}_{\hat{\Theta}(u,v,w;z)M}(G(z))$ can be represented as a mapping $\mathcal{F}_{U,V}(G(z))$ if and only if $E = 0$.*

This proposition makes clear that in general it is impossible to carry out a full recursion step of the tangential Schur algorithm by performing a mapping of the form $\mathcal{F}_{U,V}(G(z))$. However, if one is willing to give up some of the freedom available for choosing the constant J_p-unitary right multiplier matrix H, it can always be achieved that $E = 0$, e.g., by putting $H = \hat{\Theta}(u, v, w; \xi)$. On the other hand, as remarked in [16], one may also still construct a sequence of balanced realizations corresponding to the tangential Schur algorithm with the most general choices of H, by alternating application of mappings of the form $\mathcal{F}_{U,V}$ relating to the linear fractional transformations $\mathcal{T}_{\hat{\Theta}(u,v,w;z)}$, and Möbius transformations $\mathcal{T}_M$ implemented with the help of Lemma 3.3. Since M is J_p-inner, balancedness is then maintained by virtue of Proposition 2.8 and Theorem 2.5.

4.3 A unified framework based on linear fractional transformations

In addition to the connections described above between the two approaches towards the parametrization of discrete-time stable all-pass systems we make the following remarks.

The representation of linear fractional transformations on transfer functions in terms of associated state-space realizations, as described in Theorem 3.5, can also be applied directly to the tangential Schur algorithm. Note that from the expression (31) an explicit 1-dimensional state-space realization of $\Theta(u, v, w, \xi, H; z)$ is readily computed. In fact, with $\Theta(u, v, w, \xi, H; z)$ being J_p-inner it also is not difficult to compute a $(-J_p)$-balanced realization in terms of the parameters used. From Proposition 2.8 and Theorem 2.5 it then follows that application of Theorem 3.5 gives rise to a sequence of balanced realizations for the sequence of all-pass functions encountered in the tangential Schur algorithm.

It should be noted that the balanced realizations obtained in this way are usually different from the ones described in the previous subsection. Indeed, the matrices $\widetilde{\Phi}$ employed in Theorem 3.5 are in general not block-diagonal with blocks of size $(p + n + m) \times (p + n + m)$, so that the associated linear fractional transformations are not of the form $\mathcal{F}_{U,V}$. This is due the fact that the extended non-minimal realization matrices $\widetilde{R}$ to which the mappings $\mathcal{T}_{\widetilde{\Phi}}$ are applied, are structured, containing zeros and ones in specific locations. Therefore, in contrast to the relatively simple uniqueness remarks about linear fractional transformations at the end of Section 2.1, there exists a non-trivial class of equivalent linear fractional transformations with respect to their action on matrices of the form $\widetilde{R}$.

5 Conclusions

In this paper we have presented a unified framework, based on linear fractional transformations, in which two different approaches towards the parametrization of the class of discrete-time stable all-pass systems come together. It enables the construction of atlases of balanced realizations associated with the (reversed) tangential Schur algorithm, expressed explicitly in terms of the parameters used. It also yields guidelines for the choices to be made in the tangential Schur algorithm in order to arrive at a recursive construction of balanced realizations exclusively in terms of multiplication of unitary matrices. An implementation of this state-space approach in the field of rational H_2-approximation (as previously studied in [13, 7] in the context of transfer functions) is currently under investigation.

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