SIAM J. CONTROL OPTIM.
Vol. 36, No. 6, pp. 2103–2127, November 1998

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MATRIX RATIONAL H_2 APPROXIMATION: A GRADIENT ALGORITHM BASED ON SCHUR ANALYSIS*

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Abstract. This paper deals with the rational approximation of specified order n to transfer functions which are assumed to be matrix-valued functions in the Hardy space for the complement of the closed unit disk endowed with the L_2-norm. An approach is developed leading to a new algorithm, the first one to our knowledge which concerns matrix-transfer functions in L_2-norm. This approach generalizes the ideas presented in [L. Baratchart, M. Cardelli, and M. Olivi, *Automatica*, 27(1991), pp. 413–418] in the scalar case but involves substantial new difficulties.

Using the Douglas–Shapiro–Shields factorization of transfer functions, the criterion for the rational approximation problem above is expressed in terms of inner matrix functions of McMillan degree n. These functions, which possess a manifold structure, are represented by means of local coordinate maps obtained in [D. Alpay, L. Baratchart, and A. Gombani, *Oper. Theory Adv. Appl.*, 73(1994), pp. 30–66] from a tangential Schur algorithm and for which the coordinates range over n copies of the unit ball. A gradient algorithm is then employed to solve the approximation problem using the coordinate maps to describe the manifold locally and changing from one coordinate map to another when required. However, while processing the gradient algorithm a boundary point can be reached. It is proved that such a point can be considered as an initial point for searching for a local minimum of lower degree while a local minimum of McMillan degree $k < n$ provides a starting point for searching for a local minimum at degree $k + 1$. The minimization process then pursues through different degrees. The convergence of this algorithm to a local minimum of appropriate degree is proved and demonstrated on a simple example.

Key words. rational approximation, identification, discrete time systems, inner matrices, gradient algorithm, Schur analysis

AMS subject classifications. Primary 41A20, 93B30; Secondary 47N70, 93B29

PII. S0363012995284230

1. Introduction.

The identification of linear time-invariant systems can be formalized as a rational approximation problem in which some criterion function is optimized over a set of systems. This approach has led to a wide variety in model structure, performance criteria, and actual methods of estimation (see [38] and the bibliography therein). Our interest is focused mainly on the particular class of discrete time, linear, time-invariant, and strictly causal systems and their strictly proper transfer functions. The *order* of such a system is defined to be its McMillan degree, that is, the dimension of the state space in its minimal realizations. The criterion which is chosen here is the L_2-norm, and our approximation problem states in the Hardy space $\bar{H}_2^{p\times m}$ of the complement of the unit disk: *given a transfer function $F \in \bar{H}_2^{p\times m}$, we are concerned in minimizing*

$$\|F - H\|_2^2 = \frac{1}{2\pi}\mathrm{Tr}\int_0^{2\pi}[F - H](e^{it})[F - H](e^{it})^* dt\,, \tag{1}$$

as H ranges over the set of rational stable (i.e., analytic for $|z| > 1$) functions of order at most n. Here, the symbol Tr stands for the trace and the superscript $*$ denotes transpose-conjugate. It should be noticed that in a stochastic framework, (1) is equal

*Received by the editors April 5, 1995; accepted for publication (in revised form) January 5, 1998; published electronically August 31, 1998.
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to the mean square error between the output of a given system and the output of a model of fixed order when both systems have the same white noise input (see [32]).

The above problem has received attention in [41], [40], [4], and more recently [37] and [30], either in the discrete-time form studied here or in the continuous-time equivalent. Many qualitative results have been proved in [6], such as the existence of a best approximant and the property usually called normality: if F is not itself rational of degree at most n, then a best local approximant H has degree exactly n. In [8], an algorithm to find local minima in the L_2 rational approximation problem is described for scalar systems. It is the purpose of this paper to present an algorithm which enables the results of this previous paper to be extended to the multivariable case.

Let us recall the main line of our approach in the scalar case (see [8] and [13]). In this case, the minimum in (1) must be performed over the set of irreducible fractions p/q, where q is a polynomial of degree n whose roots belong to the unit disk $\mathbb{U}$. Our optimization problem being linear with respect to the parameters of the numerator p, we are led to minimize a cost function Ψ^n defined on the set $\mathcal{P}_n^1$ of monic polynomials q of degree n whose roots belong to $\mathbb{U}$. This set can be described by the coefficients of q and is open and bounded in $\mathbb{R}^n$; the function Ψ^n is smooth, so that we can use a gradient algorithm, producing a sequence of improving estimates, which either converges to a local minimum or meets the boundary of the domain at some point having some roots of modulus one. However, roots on the unit circle cancel and the cost function Ψ^n extends to a neighborhood of the closure of $\mathcal{P}_n^1$. At the boundary the extension of Ψ^n can be interpreted as the cost function of a lower-order approximation problem. Thus the search for a local minimum can continue through different orders, until such a minimum (of order $k \leq n$) is actually reached. Conversely, multiplying by $z-1$ or $z+1$ a minimum of order k provides an initial point for the optimization problem at order $k+1$: at such a point, the opposite of the gradient points inside the domain. Finally, the procedure can continue until a local minimum of order n is actually found.

Transition to the multivariable case involves substantial new difficulties, mainly due to the fact that the domain of the cost function is no longer an open subset of a Euclidean space but it does possess a manifold structure. A manifold has a covering by countably many open coordinate neighborhoods, each of these coordinate neighborhoods corresponding to an open subset of some $\mathbb{R}^d$ by a local coordinate homeomorphism (d is then the dimension of the manifold). The methods developed for the Euclidean case then apply to each of the coordinate neighborhoods separately. Over a manifold, an optimization problem can be tackled by using a search algorithm through the manifold as a whole, using the coordinate maps to describe the manifold locally and changing from one coordinate map to another when required. Such a representation of the elements of the domain has the advantage to get rid of redundancy and ensure identifiability [22]. Using state space representations, it was first established by Hazewinkel and Kalman [26] that the set of stable transfer functions of fixed degree possesses a manifold structure. Several atlases of local coordinate maps (called sets of overlapping canonical forms in system theory) have been derived from this approach ([33], [25]). However, this manifold is never compact, and convergence of a gradient algorithm to points outside can occur. To avoid this problem, a transfer function will be represented by means of the inner-unstable or Douglas–Shapiro–Shields factorization (see [15]). The elimination of the parameters in which the system is linear (namely, those of the unstable factor) allows us to perform the search for an optimum on the manifold of *inner matrix functions* of degree n. We are then in a position to proceed to the generalization to the multivariable case of the above-mentioned procedure.

The paper is organized as follows: Section 2 states the problem within the framework of the Hardy spaces and introduces the cost function by means of the inner-unstable factorization. In section 3, we first recall some results of [1], in which the theory of reproducing kernel Hilbert spaces is used to construct local coordinates of the manifold of inner matrix functions of fixed McMillan degree: such functions are obtained by iterating a linear fractional transformation which changes an inner function into another one, the McMillan degree being increased by one. Then, a fractional representation of this transformation is given in which the numerator (a polynomial matrix) and the denominator (a polynomial) are polynomial functions in the local coordinates. This representation allows us, in section 4, to study the cost function on the boundary of the domain and to elaborate an algorithm which converges generically to a local minimum. The numerical aspects have been examined in section 5.

2. Minimizing over the set of inner matrices. The Hardy spaces H_2 and H_∞ of the unit disk are the closed subspaces of $L_2(\mathbb{T})$ and $L_\infty(\mathbb{T})$, respectively, consisting of functions whose Fourier coefficients (a_n) satisfy $a_n = 0$ when $n < 0$; while the Hardy space $\bar{H}_{2,0}$ consists of functions for which $a_n = 0$ when $n \geq 0$. Note the orthogonal decomposition

$$L_2(\mathbb{T}) = H_2 \oplus \bar{H}_{2,0}.$$

It is well known (see, e.g., [27]) that members of H_2 are the nontangential limits on $\mathbb{T}$ of analytic functions f in the unit disk for which the functions $f_r(t) = f(re^{it}),\ r < 1$, are bounded in L_2-norm as $r \to 1$. Members of H_∞ correspond to bounded holomorphic functions in this process. Similarly, members of $\bar{H}_{2,0}$ correspond to analytic functions f in the complement of the unit disk vanishing at infinity and satisfying an analogous growth condition for $r > 1$. Thus, f belongs to H_2 (resp., to $\bar{H}_{2,0}$) if and only if it can be written as

$$f(z) = \sum_{k\geq 0} a_k\ z^k \quad \left(\text{resp.,} \quad f(z) = \sum_{k>0} a_k\ z^{-k} \right), \quad \sum |a_k|^2 \ < \ \infty. \tag{2}$$

Note that (2) is the Taylor expansion at 0 (resp., at ∞) and at the same time the Fourier expansion if we substitute $z = e^{i\theta}$.

The space $L_2^{p\times m}(\mathbb{T})$ of $(p \times m)$-matrices whose entries belong to $L_2(\mathbb{T})$ becomes a Hilbert space when endowed with the scalar product

$$\langle F, G\rangle = \frac{1}{2\pi} \mathrm{Tr} \int_0^{2\pi} F(e^{it})G(e^{it})^*\, dt. \tag{3}$$

The corresponding norm will also be given, for $F = (f_{ij})$, by $\|F\|_2^2 = \sum_{i,j} \|f_{ij}\|_2^2$, and the orthogonal decomposition

$$L_2^{p\times m}(\mathbb{T}) = H_2^{p\times m} \oplus \bar{H}_{2,0}^{p\times m} \tag{4}$$

is still valid. Taking into account the fact that $\bar{z} = z^{-1}$ on $\mathbb{T}$, and using the notation

$$G^\sharp(z) = G(1/\bar{z})^*,$$

(3) may be converted into the line integral

$$\langle F, G\rangle = \frac{1}{2i\pi}\mathrm{Tr}\int_{\mathbb{T}} G^\sharp(z)F(z)\frac{dz}{z}.$$

The Banach space $L_\infty^{p\times m}(\mathbb{T})$ is endowed with the norm

$$\|F\|_\infty = \sup_\theta ||F(e^{i\theta})||,$$

where $||\ .\ ||$ denotes the operator norm $\mathbb{C}^m \to \mathbb{C}^p$. The prefix $\mathcal{R}$ in front of the name of some set ($\mathcal{R}\bar{H}_{2,0}^{p\times m}$, $\mathcal{R}H_2^{p\times m}$, etc.) will indicate that we consider the *real* subspace of functions whose Fourier coefficients are real. Such functions are relevant in most applications. However, the natural framework for our study is the complex case which plainly includes the real case by restriction. When necessary, the results will be stated for real transfer functions.

The normality result mentioned in the introduction allows us to state the rational approximation problem in degree n as follows: *Given $F \in \bar{H}_{2,0}^{p\times m}$, minimize* (1) *over the set $\Sigma_{p,m}^-(n)$ of rational stable functions of McMillan degree exactly n.* It is well known that $\Sigma_{p,m}^-(n)$ possesses the structure of a real analytic manifold of dimension $2n(m+p)$ (see, e.g., [24]). We shall now give a description of this set which suits our purpose by using the inner-unstable or Douglas–Shapiro–Shields factorization (see [15] and [11]).

Recall that a $\mathbb{C}^{p\times p}$-valued analytic function Q in the unit disk is called *inner* if it is analytic in $\mathbb{U}$ and takes unitary values on the unit circle $\mathbb{T}$:

$$Q(e^{it})\ Q(e^{it})^* \ = \ Q(e^{it})^*\ Q(e^{it}) \ = \ I_p, \tag{5}$$

where I_p is the identity matrix of size p. This equality implies that the inverse of a rational inner functions agrees with $Q^\sharp$ and thus is analytic outside the unit disk. Naturally associated with Q is the space $QH_2^p \subset H_2^p$ which is invariant by the shift operator (i.e., multiplication by z), and its orthogonal complement $\mathcal{H}(Q)$. Note that $\mathcal{H}(Q)$ consists of vectors $v \in H_2^p$ of the form Qu for some u in $\bar{H}_{2,0}^p$. These spaces and the inner-unstable factorization are closely related to the shift realization (see [19]). Observe that the McMillan degree of a rational matrix may be defined even if this matrix fails to be analytic at infinity, using, for instance, Smith–McMillan forms (see [28]). Furthermore, the McMillan degrees of Q and Q^{-1} agree.

Proposition 1 (inner-unstable factorization). *Any rational function H in $\bar{H}_{2,0}^{p\times m}$ can be written*

$$H \ = \ Q^{-1}\ C, \tag{6}$$

where Q is a $(p\times p)$-rational inner function and C a $(p\times m)$-rational matrix whose columns belong to $\mathcal{H}(Q)$. The matrices Q and C may be chosen left co-prime. With this condition, the factorization is unique up to a common left unitary factor and Q and H have same McMillan degree.

The matrix Q is called the *left inner factor* of H and the matrix Q^{-1} is usually named in system theory an *all-pass stable transfer function.* To ensure uniqueness in the inner-unstable factorization, we shall require that Q satisfies the condition

$$Q(1) = I_p. \tag{7}$$

The set of $\mathbb{C}^{p\times p}$-valued rational inner functions of degree n will be denoted by $\mathcal{I}_n^p$, and by $\mathcal{I}_n^p(1)$ we denote the subset of functions satisfying the extra condition (7). As previously mentioned, $\mathcal{RI}_n^p$ and $\mathcal{RI}_n^p(1)$ will denote the corresponding sets of *real* inner functions. It is proved in [1] that $\mathcal{I}_n^p$ and $\mathcal{I}_n^p(1)$ are smooth manifolds of dimension $2np+p^2$ and $2np$, respectively (embedded in $H_\infty^{p\times p}$), while $\mathcal{RI}_n^p$ and $\mathcal{RI}_n^p(1)$ have dimension $np+p(p-1)/2$ and np, respectively. Moreover, the set $\Sigma_{p,m}^-(n)$ is a vector bundle whose base space is $\mathcal{I}_n^p(1)$ and whose fiber above Q is the vector space $\mathcal{F}_Q$ of matrices C whose columns belong to $\mathcal{H}(Q)$ (see [12]).

Now, we can write our approximation problem as

$$\min_{Q,C} \|F - Q^{-1}C\|_2^2,$$

where $Q \in \mathcal{I}_n^p(1)$ and $C \in \mathcal{F}_Q$. Observe that for fixed Q, the minimum is obtained when C is the projection of QF onto $\mathcal{F}_Q$. Since $F \in \bar{H}_{2,0}^{p\times m}$, C is also the projection of QF onto $H_2^{p\times m}$ that we shall denote by $L(Q)$. Therefore, minimizing (1) is equivalent to minimizing the function

$$\begin{array}{rclcl} \Psi^n & : & \mathcal{I}_n^p(1) & \to & \mathbb{R}, \\ & & Q & \to & \|F - Q^{-1}L(Q)\|_2^2, \end{array} \tag{8}$$

which is going to be the main purpose of the remainder of this paper. The first step consists of studying the domain of this function and will be the content of the next section.

First of all, we give a fractional representation of an inner matrix which will be useful in the sequel. If q is a polynomial of degree n, we define its *reciprocal polynomial* as being

$$\widetilde{q}(z) = z^n\ q^\sharp(z), \tag{9}$$

and if D is a polynomial matrix whose degree does not exceed n, we also put

$$\widetilde{D}(z) \ = \ z^n\ D^\sharp(z). \tag{10}$$

Recall that the degree of a polynomial matrix is defined to be the degree of its highest degree entry. While both this degree and the McMillan degree are used in this work, there should be no confusion from the context which is used.

PROPOSITION 2. *An inner matrix $Q \in \mathcal{I}_n^p$ has a representation of the form $Q = D/\widetilde{q}$ by means of a polynomial matrix D whose degree does not exceed n and a polynomial q of exact degree n whose roots belong to the open unit disk, satisfying $D\widetilde{D} = q\widetilde{q}I_p$ and* $\det D = \epsilon q\widetilde{q}^{p-1}$, *$\epsilon$ being a complex number of modulus one. Conversely, these conditions are sufficient for the rational matrix $D/\widetilde{q}$ to belong to $\mathcal{I}_n^p$.*

Proof. Since Q^{-1} is analytic outside the unit disk, it has a representation of the form $\widetilde{D}/q$, where q is, up to a constant factor, its *polynomial of poles* (see [28]). Condition (5) yields an analogous representation for Q, i.e., $Q = D/\widetilde{q}$, so that $D\widetilde{D} = q\widetilde{q}I_p$. It also implies that $\det Q$ is an inner scalar rational function, that is to say a Blaschke product, and the number of zeros of $\det Q$ within $\mathbb{U}$ determines the McMillan degree of Q, by the Potapov decomposition [35]. ∎

3. Parametrization of inner matrices. We describe here a parametrization of the set of inner functions obtained in [1] from a matrix version of the classical Schur algorithm that we now explain; in a fundamental paper [36], Schur proved that every function $f \in \mathcal{I}_n^1$ can be uniquely parameterized by a sequence y_j, $j = n, \ldots, 1$, of complex numbers with $|y_j| < 1$. Moreover, Schur gave an algorithm for computing these parameters:

$$y_j = f_j(0),$$

where $f_n = f$ and

$$f_{j-1}(z) = \frac{f_j(z) - f_j(0)}{(1 - \overline{f_j(0)} f_j(z))\, z}, \quad j = n, \dots, 1. \tag{11}$$

Since f_j is an inner function it follows from the maximum modulus principle that $|y_j| < 1$, and f_j has degree j, since a zero is eliminated at each step. Since f has degree n, f_0 is equal to a constant of modulus one. Other sequences of inner functions of decreasing degree may be constructed from f in a similar way. The most general recursion formula is the following (see [21]):

$$\frac{f_{j-1}(z) + \mu_j}{1 + \bar{\mu}_j f_{j-1}(z)} = \frac{f_j(z) - y_j}{1 - \overline{y_j} f_j(z)} \; \frac{1 - \bar{w}_j z}{z - w_j}, \quad j = n, \dots, 1,$$

where the w_j's are the interpolation points, $y_j = f_j(w_j)$ and the μ_j's belong to $\mathbb{U}$. The w_j's and the μ_j's being given, the sequence of numbers y_j completely characterizes the function f, which can recover inductively by the linear fractional transformations:

$$f_j(z) = \frac{[(z - w_j) + \bar{\mu}_j y_j (1 - \bar{w}_j z)]\, f_{j-1}(z) + [\mu_j (z - w_j) + y_j (1 - \bar{w}_j z)]}{[\bar{y}_j (z - w_j) + \bar{\mu}_j (1 - \bar{w}_j z)]\, f_{j-1}(z) + [\mu_j \bar{y}_j (z - w_j) + (1 - \bar{w}_j z)]}. \tag{12}$$

The map $(y_1, \dots, y_n, f_0) \to f$ is a diffeomorphism from the product of n copies of the open unit disk and of a copy of the unit circle onto $\mathcal{I}_n^1$.

This Schur algorithm is related to the classical interpolation problems of Nevanlinna–Pick and Carathéodory–Fejér (see [39]), which have a remarkable diversity of applications in systems engineering (see [5], [29]). Several approaches allow the extension of these problems to matrix-valued analytic functions (see [42] and the bibliography therein); however, the operator-theoretic one, involving reproducing kernel Hilbert spaces, clarifies the connections between interpolation and realization theory and gives a unified presentation of these problems (see, e.g., [17], [16], [3]). Another fundamental treatment can be found in [18], which emphasizes the relevance of the commutant lifting theorem in these interpolations issues and also presents several applications to engineering problems.

3.1. Reproducing kernel Hilbert spaces. For the convenience of the reader, we shall recall some basic facts about reproducing kernel Hilbert spaces which may be found in [16]. A complex Hilbert space $\mathcal{H}$ of $\mathbb{C}^p$-valued functions defined on some Ω open in $\mathbb{C}$ is called a *reproducing kernel Hilbert space* (RKHS) if there exists a $\mathbb{C}^{p\times p}$-valued function $K(z, w)$ defined on $\Omega \times \Omega$ such that for every choice of $w \in \Omega$, $c \in \mathbb{C}^p$ and $f \in \mathcal{H}$:

(i) $K(., w)c \in \mathcal{H}$,

(ii) $\langle f, K(., w)c\rangle_{\mathcal{H}} = c^* f(w)$.

The function K is called the *reproducing kernel,* and the main facts are that it is unique and it is a *positive function* in the following sense:

$$\sum_{i,j=1}^{r} c_j^* K(w_j, w_i) c_i \geq 0 \tag{13}$$

for every choice of points $w_1, w_2, \ldots, w_r \in \Omega$, and vectors $c_1, c_2, \ldots, c_r \in \mathbb{C}^p$.

The Hardy space H_2^p is clearly a reproducing kernel Hilbert space whose kernel is

$$\frac{I_p}{1-\bar{w}z}, \quad w \in \mathbb{U}, \quad z \in \mathbb{U},$$

and property (ii) is just the Cauchy formula. Finite dimensional Hilbert spaces of $\mathbb{C}^p$-valued functions are also reproducing kernel Hilbert spaces. Let $(f_1, f_2, \ldots, f_N)$ be some base of a finite dimensional Hilbert space. Then its reproducing kernel is easily computed to be

$$K(z,w) = (f_1(z), f_2(z), \ldots, f_N(z))P^{-1}(f_1(w), f_2(w), \ldots, f_N(w))^*,$$

where $P = (P_{ij})$ is the Gram matrix with entries $P_{ij} = \langle f_j, f_i \rangle$. The space $\mathcal{H}(Q)$ introduced in the previous section as being the orthogonal complement of QH_2^p in H_2^p is a reproducing kernel Hilbert space with reproducing kernel

$$K_Q(z,w) = \frac{I_p - Q(z)Q(w)^*}{1-\bar{w}z}, \tag{14}$$

which is the projection onto $\mathcal{H}(Q)^p$ of the reproducing kernel of H_2^p. This is readily seen with the help of the evaluation

$$\pi^+\left(Q(z)^{-1}\frac{I_p c}{1-\bar{w}z}\right) = \frac{Q(w)^* c}{1-\bar{w}z}, \tag{15}$$

where π^+ denotes the orthogonal projection onto H_2^p.

More generally, a RKHS is attached to every J-inner function. The study of these spaces, which play a central role in the theory of realization and interpolation, originates with de Branges and Rovnyak (see [14]). Put

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}.$$

A $\mathbb{C}^{2p\times 2p}$-valued rational function Θ is called J-inner if at every point of analyticity of Θ in $\mathbb{U}$, $J - \Theta(z)J\Theta(z)^*$ is positive semidefinite:

$$\Theta(z)J\Theta(z)^* \leq J, \tag{16}$$

and equality holds for z point of analyticity on $\mathbb{T}$. Consider the space H_2^{2p} endowed with the sesquilinear Hermitian form, $\langle f, g\rangle_J = \langle f, Jg\rangle$. This form is not positive definite but it is nondegenerate. Hence, the space $\Theta\, H_2^{2p}$ has an orthogonal complement in H_2^{2p}, which we call $\mathcal{H}(\Theta)$. Restricted to $\mathcal{H}(\Theta)$, the form $\langle ., .\rangle_J$ is positive definite, so that $\mathcal{H}(\Theta)$ is a Hilbert space. Moreover, it is a reproducing kernel Hilbert space with reproducing kernel

$$K_\Theta(z,w) = \frac{J - \Theta(z)J\Theta(w)^*}{1-\bar{w}z} \tag{17}$$

and the dimension of $\mathcal{H}(\Theta)$ agrees with the McMillan degree of Θ. In the next section, we shall make an intensive use of one-dimensional $\mathcal{H}(\Theta)$ spaces; let f be the function defined by

$$f(z) = \frac{\begin{pmatrix} u \\ v \end{pmatrix}}{1-\bar{w}z},$$

where $w \in \mathbb{U}$, $u \in \mathbb{C}^p$ with $\|u\| = 1$, and $v \in \mathbb{C}^p$; let $\mathcal{M}$ be the linear span of f endowed with the form $\langle\ ,\ \rangle_J$. If $\|v\| \leq 1$, then the Gram matrix $P = \langle f, f\rangle_J$ is positive and $\mathcal{M}$ is of the form $\mathcal{H}(\Theta)$, where Θ is unique up to a J-unitary constant multiplier on the right. It can be specified by the formula

$$\Theta(z) \ = \ I_{2p} - (1-\bar{\xi}z)f(z)P^{-1}f(\xi)^*J$$

for any point $\xi \in \mathbb{T}$. In the sequel, we shall work with the J-inner function associated with $\mathcal{M}$ which satisfies the condition $\Theta(1) = I_{2p}$. It is given by

$$\Theta(w,u,v)(z) \ = \ I_{2p} - (1-z)\frac{1-|w|^2}{1-\|v\|^2}\frac{\begin{pmatrix} u \\ v \end{pmatrix}}{1-\bar{w}z}\frac{\begin{pmatrix} u^* & v^* \end{pmatrix}}{1-w}J. \tag{18}$$

3.2. The linear fractional transformation associated with a *J*-inner function. In this section we introduce the linear fractional transformation T_Θ, which generalizes (12) to the matrix case (for a precise comparison see the remark after Theorem 7). The statements and the proofs of this section and the following are adapted from [1].

LEMMA 3. *Let*

$$\Theta \ = \ \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \tag{19}$$

be a $(2p \times 2p)$-rational J-inner function analytic in $\mathbb{U}$ and let A be a $(p\times p)$-rational inner function. Then $(\Theta_{21}A + \Theta_{22})$ is invertible in $\mathbb{U}$ and

$$T_\Theta(A) \ = \ (\Theta_{11}A + \Theta_{12})\ (\Theta_{21}A + \Theta_{22})^{-1} \tag{20}$$

is inner. Note that if $\Theta(1) = I_{2p}$ and $A(1) = I_p$, then $[T_\Theta(A)](1) = I_p$, and if A and Θ have real coefficients, then $T_\Theta(A)$ also has real coefficients.

Proof. First, let us show that $(\Theta_{21}A + \Theta_{22})$ is invertible at every point of $\mathbb{U}$. Indeed, condition (16) implies

$$\Theta_{22}\Theta_{22}^* \ \geq \ I_p + \Theta_{21}\Theta_{21}^* \quad \text{in } \mathbb{U},$$

so that $\Theta_{22}\Theta_{22}^*$ is positive definite, and Θ_{22} is invertible at any point of $\mathbb{U}$. Now, we have

$$I_p \ \geq \ \Theta_{22}^{-1}(\Theta_{22}^{-1})^* + (\Theta_{22}^{-1}\Theta_{21})(\Theta_{22}^{-1}\Theta_{21})^* \quad \text{in } \mathbb{U},$$

and thus $\|\Theta_{22}(z)^{-1}\Theta_{21}(z)\| \ < \ 1$, $\forall z \in \mathbb{U}$. The matrix A, being inner, is contractive in $\mathbb{U}$: $\|A(z)\| \ \leq \ 1$, $\forall z \in \mathbb{U}$, so that $\|\Theta_{22}(z)^{-1}\Theta_{21}(z)A(z)\| \ < \ 1$, $\forall z \in \mathbb{U}$. Finally, $(\Theta_{21}A + \Theta_{22}) \ = \ \Theta_{22}\ (I_p + \Theta_{22}^{-1}\Theta_{21}A)$ is invertible at any point of $\mathbb{U}$, and thus $B = T_\Theta(A)$ is analytic in $\mathbb{U}$. Then, condition (5) for B can be written

$$B^*B - I_p \ = \ \begin{pmatrix} B^* & I_p \end{pmatrix}\ J \begin{pmatrix} B \\ I_p \end{pmatrix} \ = \ 0 \quad \text{on } \mathbb{T}.$$

Using the relation

$$\begin{pmatrix} B \\ I_p \end{pmatrix} \ = \ \Theta \begin{pmatrix} A \\ I_p \end{pmatrix} (\Theta_{21}A + \Theta_{22})^{-1}, \tag{21}$$

we obtain

$$B^*B - I_p \ = \ ((\Theta_{21}A + \Theta_{22})^{-1})^* \ \begin{pmatrix} A^* & I_p \end{pmatrix}\ \Theta^*J\Theta \begin{pmatrix} A \\ I_p \end{pmatrix} (\Theta_{21}A + \Theta_{22})^{-1},$$

and since condition (5) is satisfied for A, it will be satisfied for B as well. $\square$

LEMMA 4. *The matrix $B = T_{\Theta(w,u,v)}(A)$, where $\Theta(w,u,v)$ is given by* (18), *satisfies the interpolation condition*

$$B(w)^* u = v. \tag{22}$$

Proof. Indeed, it can be verified that $\Theta(w,u,v)$ satisfies the equation

$$\begin{pmatrix} u^* & -v^* \end{pmatrix} \Theta(w) = 0.$$

Thus

$$\begin{pmatrix} u^* & -v^* \end{pmatrix} \Theta(w) \begin{pmatrix} A(w) \\ I_p \end{pmatrix} = 0,$$

and together with (21) this implies our interpolation condition. ∎

Now, the question is the converse: let B be some rational inner matrix, and Θ J-inner analytic in $\mathbb{U}$. Can we write B in the form $B = T_\Theta(A)$ for some inner matrix A? First, note that if $B = T_\Theta(A)$, then A is the rational function given by

$$A = (\Theta_{11} - B\Theta_{21})^{-1}(B\Theta_{22} - \Theta_{12}), \tag{23}$$

unless $\det(\Theta_{11} - B\Theta_{21})$ vanishes identically. This may not happen since condition (16) for Θ implies $\Theta_{11}^*\Theta_{11} - \Theta_{21}^*\Theta_{21} = I_p$ on $\mathbb{T}$. So, $\Theta_{11} - B\Theta_{21}$ is invertible at any point of $\mathbb{T}$. However, it may fail to be invertible at some point of $\mathbb{U}$, so that A may not be analytic in $\mathbb{U}$. To ensure analyticity, we must make an additional assumption.

THEOREM 5. *Let B be a rational inner function, and let $\Theta(w,u,v)$ be the J-inner function* (18). *There exists an inner function A such that $B = T_\Theta(A)$ if and only if the interpolation condition $B(w)^*u = v$ is satisfied. We then have* $\deg B = \deg A + 1$.

Proof. This is a special case of a more general result proved in [1, Thm. 3.3] which is based on the links between $\mathcal{H}(\Theta)$ and $\mathcal{H}(Q)$ spaces. For more details on these problems we refer the reader to [2]. The content of the result of [1] is the following: let $B \in \mathcal{I}_n^p$ and Θ be a J-inner $(2p \times 2p)$-rational function; consider the map

$$\begin{array}{rccl} \tau : & \mathcal{H}(\Theta) & \to & H_2^p, \\ & \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} & \to & f_1 - Bf_2. \end{array}$$

Then there exists an inner function A such that $B = T_\Theta(A)$ if and only if τ is an isometry from $\mathcal{H}(\Theta)$ to $\mathcal{H}(B)$. Moreover, $\deg B = \deg\Theta + \deg A$. We shall admit this result.

If $\Theta(w,u,v)$ is given by (18), the conditions:

(i) τ is an isometry from $\mathcal{H}(\Theta)$ to $\mathcal{H}(B)$,

(ii) $B(w)^*u = v$

are equivalent. Indeed, τ sends the generator of $\mathcal{H}(\Theta)$ as follows:

$$\tau : \frac{\begin{pmatrix} u \\ v \end{pmatrix}}{1 - \bar{w}z} \to \frac{u - B(z)\,v}{1 - \bar{w}z}.$$

With the help of the evaluation (15), it is readily proved that $\frac{u - B(z)\,v}{1-\bar{w}z} \in \mathcal{H}(B)$ if and only if condition (ii) holds. In this case,

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle_J = \|f_1\|_2^2 - \|f_2\|_2^2 = \|f_1\|_2^2 - \|Bf_2\|_2^2 = \|f_1 - Bf_2\|_2^2,$$

and τ is an isometry from $\mathcal{H}(\Theta)$ (endowed with the scalar product $\langle\ ,\ \rangle_J$) to $\mathcal{H}(B)$. This proves the theorem. ∎

3.3. Description of the charts. If Θ is of the form (18), then $T_\Theta(A)$ and A have the same value at $z = 1$. We can construct from the identity matrix I_p, using n linear fractional transformations of this type, an inner matrix of degree n which belongs to $\mathcal{I}_n^p(1)$. Conversely, any matrix of $\mathcal{I}_n^p(1)$ can be obtained in this way. This is the content of the tangential Schur algorithm for which we need the following lemma.

LEMMA 6. *Let B be an inner function and $w \in \mathbb{U}$. Then, there exists $u \in \mathbb{C}^p$, $\|u\| = 1$, such that*

$$\|B(w)^* u\| < 1.$$

Proof. Suppose that for all unit vector $u \in \mathbb{C}^p$, $\|B(w)^* u\| = 1$. Then, since

$$\|K_B(.,w)u\|_2^2 \;=\; u^* K_B(w,w)u \;=\; \frac{1 - \|B(w)^* u\|}{1 - \bar{w}w},$$

for all $u \in \mathbb{C}^p$, $K_B(.,w)u \;=\; 0$. So, $K_B(.,w)$ is identically 0 and the matrix B must be constant. But this contradicts the fact that B has McMillan degree n. ∎

THEOREM 7 (tangential Schur algorithm). *Let $Q \in \mathcal{I}_n^p(1)$, and $w_k \in \mathbb{U}$, $k = n, \dots, 1$. Then, for $k = n, \dots, 1$ there exist unit vectors $u_k \in \mathbb{C}^p$ such that the vectors $y_k \in \mathbb{C}^p$ given by*

$$y_k = Q^{(k)}(w_k)^* u_k \tag{24}$$

satisfy $\|y_k\| < 1$, where $Q^{(n)} = Q$,

$$Q^{(k)} \;=\; T_{\Theta_k}(Q^{(k-1)}), \tag{25}$$

and $\Theta_k = \Theta(w_k, u_k, y_k)$ is the J-inner matrix given by (18). Then

$$Q \;=\; T_{\Theta_n}(T_{\Theta_{n-1}} \dots T_{\Theta_1}(I_p)) \dots \;=\; T_{\Theta_n \dots \Theta_1}(I_p).$$

Proof. This is an obvious consequence of Theorem 5 and Lemma 6. ∎

Let $\mathbf{w} = (w_1, w_2, \dots, w_n)$ and $\mathbf{u} = (u_1, u_2, \dots, u_n)$. Define the subset $\mathcal{V}_{(\mathbf{w},\mathbf{u})}$ of $\mathcal{I}_n^p(1)$ by

$$\mathcal{V}_{(\mathbf{w},\mathbf{u})} \;=\; \{Q \in \mathcal{I}_n^p(1) \;/\; \|Q^{(k)}(w_k)^* u_k\| < 1\},$$

and the function $\varphi_{(\mathbf{w},\mathbf{u})}$ by

$$\begin{array}{cccl} \varphi_{(\mathbf{w},\mathbf{u})} & : & \mathcal{V}_{(\mathbf{w},\mathbf{u})} & \to & \mathcal{B}_p^n, \\ & & Q & \to & (y_1, y_2, \dots, y_n), \end{array}$$

where the $Q^{(k)}$'s and the *Schur parameters* y_k's are defined recursively by (24) and (25), and $\mathcal{B}_p^n$ denotes the product of n copies of the unit ball of $\mathbb{C}^p$.

Remark. Note that in the scalar case and for $w_j = 0$, the transformation $Q^{(j)} = T_{\Theta_j}(Q^{(j-1)})$ is given by

$$Q^{(j)}(z) = \frac{(z - |y_j|^2)\, Q^{(j-1)}(z) + (1 - z)u_j \bar{y}_j}{(z - 1)\bar{u}_j y_j\, Q^{(j-1)}(z) + (1 - |y_j|^2 z)}.$$

This formula is exactly (12) in which y_j has been replaced by $u_j\bar{y}_j$, since the interpolation condition is now $Q^{(j)}(0) = u_j\bar{y}_j$, and μ_j is chosen to be $-u_j\bar{y}_j$. The general formula with an arbitrary μ_j can also be obtained by a T_Θ transformation where Θ is of the form (18) multiplied by an adequate constant J-inner function. In this case the normalization (7) is not conserved.

THEOREM 8. *The family $(\mathcal{V}, \varphi)$ defines a C^∞ atlas on $\mathcal{I}_n^p(1)$, which is compatible with its natural structure of embedded submanifold of $H_2^{p\times p}$.*

Proof. It follows from Lemma 6 that the collection of sets $\mathcal{V}_{(\mathbf{w},\mathbf{u})}$ covers $\mathcal{I}_n^p(1)$. It remains to prove that the map $\varphi_{(\mathbf{w},\mathbf{u})}$ is an homeomorphism and that the change of chart $\varphi_{(\mathbf{w},\mathbf{u})} \circ \varphi_{(\mathbf{w}',\mathbf{u}')}$ is smooth. The map $\varphi_{(\mathbf{w},\mathbf{u})}$ is one-to-one and onto by construction. The coefficients of $Q^{(k)}$ depend continuously on that of $Q^{(k-1)}$ and on $y_k, \ldots, y_1$, so that the coefficients of Q depend continuously on the Schur parameters. Since the matrix Q is inner and thus bounded in the unit disk, $\|Q(z)\| \le 1$, $\forall z \in \mathbb{U}$, Lebesgue's theorem finally implies that $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}$ is continuous. Conversely, note that the evaluation map $Q \to Q(w_n)^* u_n$ is continuous, so that $Q \to y_n$ is continuous. The coefficients of $Q^{(n-1)}$ depend continuously on that of Q and on y_n, and, if two normalized rational functions of bounded degree are closed in $H_2^{p\times p}$, then their coefficients must also be closed; then, the map $Q \to Q^{(n-1)}$ from $\mathcal{I}_n^p$ to $\mathcal{I}_{n-1}^p$, both endowed with the H_2-topology, is continuous. We thus prove inductively that $\varphi_{(\mathbf{w},\mathbf{u})}$ is continuous and consequently is an homeomorphism. Furthermore, the map $\varphi_{(\mathbf{w},\mathbf{u})} \circ \varphi_{(\mathbf{w}',\mathbf{u}')} \;:\; \mathcal{B}_p^n \to \mathcal{B}_p^n$ is C^∞, as a bounded rational function. ∎

In any chart of this atlas, *the local coordinates are the $2np$ real and imaginary parts of the components of the Schur parameters.* Note that an atlas for $\mathcal{RI}_n^p(1)$ can be obtained in a similar way, for which the w_i's lie in $(-1, 1)$, the u_i's and the y_i's have real components; indeed, in Lemma 6 u can be chosen real, and if A and Θ have real coefficients, $T_\Theta(A)$ also has real coefficients. The range of the charts is thus the product of n copies of the unit ball of $\mathbb{R}^p$.

3.4. Fractional representation in the local coordinates. In this section, we give a fractional representation of the form $D^{(k)}/\widetilde{q}^{(k)}$ for the matrix $Q^{(k)}$ (see Proposition 2). We introduce the map $\mathcal{S}_{(w,u)} \;:\; (A, y) \;\to\; T_{\Theta(w,u,y)}(A)$, so that the inner matrix $Q = \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(y)$ is computed by the iterative process:

$$I_p \to Q^{(1)} \to \cdots \to Q^{(k)} = \mathcal{S}_{(w_k,u_k)}(Q^{(k-1)}, y_k) \to \cdots \to Q^{(n)} = Q.$$

LEMMA 9. *For any $A \in \mathcal{I}_k^p$, $w \in \mathbb{U}$, $u \in \mathbb{C}^p$, $\|u\| = 1$, and $v \in \mathbb{C}^p$, $\|v\| < 1$, we have*

$$S_{(w,u)}(A, y) \;=\; A + \frac{1 - \beta_w}{1 - u^*Ay - \beta_w(y^*y - u^*Ay)}(u - Ay)(y^* - u^*A), \tag{26}$$

with $\beta_w = b_w/\widetilde{b}_w$, where $b_w(z) = (z - w)(1 - \bar{w})$.

Proof. Using (18) yields

$$T_{\Theta(w,u,y)}(A) \;=\; \left(A + (1 - \beta_w)\frac{u(y^* - u^*A)}{1 - y^*y}\right)\;\left(I_p + (1 - \beta_w)\frac{y(y^* - u^*A)}{1 - y^*y}\right)^{-1}.$$

A classical formula (see [28, Appendix A.20]) allows us to compute the inverse as follows

$$\left(I_p + (1 - \beta_w)\frac{y(y^* - u^*A)}{1 - y^*y}\right)^{-1} = I_p - (1 - \beta_w)\frac{y(y^* - u^*A)}{1 - u^*Ay - \beta_w(y^*y - u^*Ay)},$$

from which we deduce (26) by expanding the product. ∎

PROPOSITION 10. *A fractional representation $D^{(k)}/\widetilde{q}^{(k)}$ of the inner matrix $Q^{(k)} = T_{\Theta_k\cdots\Theta_1}(I_p)$ can be computed by the recursion formulas: $D^{(0)} = I_p$, $\widetilde{q}^{(0)} = 1$, and for $k = 1, \ldots, n$,*

$$\begin{aligned} D^{(k)} = (\widetilde{b}_{w_k} - y_k^* y_k\, b_{w_k}) D^{(k-1)} - (\widetilde{b}_{w_k} - \ b_{w_k}) \Bigg\{ & u_k u_k^* D^{(k-1)} + D^{(k-1)} y_k y_k^* \\ & - \widetilde{q}^{(k-1)}\ u_k y_k^* + \frac{u_k^* D^{(k-1)} y_k\ D^{(k-1)} - D^{(k-1)} y_k u_k^* D^{(k-1)}}{\widetilde{q}^{(k-1)}} \Bigg\}, \end{aligned} \tag{27}$$

$$\widetilde{q}^{(k)} = (\widetilde{b}_{w_k} - y_k^* y_k\, b_{w_k})\, \widetilde{q}^{(k-1)} - (\widetilde{b}_{w_k} - b_{w_k})\, u_k^* D^{(k-1)} y_k, \tag{28}$$

where $b_{w_k} = (1 - \bar{w}_k)(z - w_k)$. The stable polynomial $q^{(k)}$ has degree k, and the coefficients of the polynomials $\widetilde{q}^{(k)}$ and $d_{ij}^{(k)}$ (the entries of $D^{(k)}$) are polynomial functions in the local coordinates.

Proof. Assume that such a fractional representation has been obtained for $Q^{(k-1)}$. Replacing $Q^{(k-1)}$ by $D^{(k-1)}/\widetilde{q}^{(k-1)}$ in $S_{(w_k,u_k)}(Q^{(k-1)}, y_k)$ given by (26) yields a fractional representation for $Q^{(k)}$. Note that (27) actually defines a polynomial matrix, since $\widetilde{q}^{(k-1)}$ does indeed divide $u_k^* D^{(k-1)} y_k\ D^{(k-1)} - D^{(k-1)} y_k\ u_k^* D^{(k-1)}$. In order to prove this, put $u_k^* = (\bar{u}_1^k, \ldots, \bar{u}_p^k)$, $y_k^* = (\bar{y}_1^k, \ldots, \bar{y}_p^k)$, and $D^{(k-1)} = (d_{ij}^{(k-1)})$. A straightforward computation shows that

$$\begin{aligned} &\left(u_k^* D^{(k-1)} y_k\, D^{(k-1)} - D^{(k-1)} y_k\, u_k^* D^{(k-1)} \right)_{ij} \\ &\qquad = \sum_{l,m} \left(d_{lm}^{(k-1)} d_{ij}^{(k-1)} - d_{im}^{(k-1)} d_{lj}^{(k-1)} \right) \bar{u}_l^k y_m^k, \end{aligned}$$

where $d_{lm}^{(k-1)} d_{ij}^{(k-1)} - d_{im}^{(k-1)} d_{lj}^{(k-1)}$ is a minor of order 2 of $D^{(k-1)}$. But in the fractional representation $\widetilde{q}^{(k-1)}$ is, up to a constant factor, the polynomial of poles of $Q^{(k-1)}$, which is the least common denominator of all the minors of $Q^{(k-1)}$ (see [28]). Thus, it must divide all the minors of order 2 of $D^{(k-1)}$.

Now, let us prove by induction that, for $k = 1, \ldots, n$, the coefficients of $d_{ij}^{(k)}$ and $\widetilde{q}^{(k)}$ are polynomial functions in the local coordinates. This is true for $d_{ij}^{(1)}$ and $\widetilde{q}^{(1)}$ and we shall assume that this is also true for $d_{ij}^{(k-1)}$ and $\widetilde{q}^{(k-1)}$: for $l = 1, \ldots, n$, put

$$y_j^l = \xi_j^l + i\, \eta_j^l,$$

where y_j^l is the jth component of y_l; then the coefficients of $d_{ij}^{(k-1)}$ and $\widetilde{q}^{(k-1)}$ belong to the ring $\mathcal{P}_{k-1}$ of complex polynomials in the $2(k-1)p$ variables ξ_j^l and η_j^l, $l = 1, \ldots, k-1$, $j = 1, \ldots, p$. In order to prove our assumption at order k, we must verify that $\widetilde{q}^{(k-1)}$ divides all the minors of order 2 of $D^{(k-1)}$ in the ring $\mathcal{P}_{k-1}[z]$. Let

$$D^{(k-1)} \begin{pmatrix} i_1 & \cdots & i_l \\ j_1 & \cdots & j_l \end{pmatrix}$$

be the minor of $D^{(k-1)}$ computed from the lines $i_1, \ldots, i_l$ and the columns $j_1, \ldots, j_l$. Since the matrix $D/\widetilde{q}$ is the inverse of $\widetilde{D}/q$, the minors of order 2 of $D^{(k-1)}$ are related to those of order $p-2$ of $\widetilde{D}^{(k-1)}$ by the formula (see [20]):

$$\begin{aligned}&\{q^{(k-1)}\}^{p-3}\, D^{(k-1)}\begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}\\ &\quad = (-1)^{\{i_1+i_2+j_1+j_2\}}\, \widetilde{D}^{(k-1)}\begin{pmatrix} i'_1 & \cdots & i'_{p-2} \\ j'_1 & \cdots & j'_{p-2} \end{pmatrix} \widetilde{q}^{(k-1)},\end{aligned} \tag{29}$$

where $\{i_1, i_2, i'_1, \ldots, i'_{p-2}\}$ and $\{j_1, j_2, j'_1, \ldots, j'_{p-2}\}$ form complete sets of rows and columns. If $\widetilde{q}^{(k-1)}$ is irreducible we are done. We shall prove this still by induction. First, $\widetilde{q}^{(0)}$ is irreducible. Then, assume that $\widetilde{q}^{(l-1)}$ is irreducible while $\widetilde{q}^{(l)}$ can be factored as

$$\widetilde{q}^{(l)} = \alpha\beta, \quad \alpha \in \mathcal{P}_l[z], \quad \beta \in \mathcal{P}_l[z].$$

The polynomial $\widetilde{q}^{(l)}$ can be viewed as a polynomial in the $2p$ coordinates $\xi_1^l, \ldots, \xi_p^l$, $\eta_1^l, \ldots, \eta_p^l$, with coefficients in $\mathcal{P}_{l-1}[z]$:

$$\widetilde{q}^{(l)} = \left(\widetilde{b}_{w_l} - b_{w_l}\left\{\sum_{j=1}^{p} (\xi_j^l)^2 + (\eta_j^l)^2\right\}\right)\widetilde{q}^{(l-1)} - (\widetilde{b}_{w_l} - b_{w_l})\sum_{j=1}^{p}\left\{\sum_{i=1}^{p} \bar{u}_i^l d_{ij}^{(l-1)}\right\}(\xi_j^l + i\eta_j^l).$$

If α does not depend on ξ_1^l, for example, then α must divide $-b_{w_l}\widetilde{q}^{(l-1)}$ and since b_{w_l} does not divide $\widetilde{q}^{(l)}$, we must have $\alpha = \widetilde{q}^{(l-1)}$. Therefore, $\widetilde{q}^{(l-1)}$ must divide each component of $u_l^* D^{(l-1)}$ in $\mathcal{P}_{l-1}[z]$. But this is clearly impossible; indeed, since $Q^{(l-1)}(1) = I_p$ we should have $u_l^* D^{(l-1)} = \widetilde{q}^{(l-1)} u_l$ for every choice of local coordinates. Thus both α and β have degree one in each ξ_j^l and η_j^l. But then, writing $\alpha = \alpha_1\xi_1^l + \cdots + \alpha_p\xi_1^p + \alpha'_1\eta_1^l + \cdots + \alpha'_p\eta_1^p + \alpha_0$ and $\beta = \beta_1\xi_1^l + \cdots + \beta_p\xi_1^p + \beta'_1\eta_1^l + \cdots + \beta'_p\eta_1^p + \beta_0$, we can see that such a factorization is impossible, so that $\widetilde{q}^{(l)}$ is actually irreducible. ∎

Though the quotient in formula (27) is exact, we fail in searching for an explicit formula for it, and we do not know if such a formula exists.

4. A generic algorithm to find a local minimum. The closure of $\mathcal{I}_n^p(1)$ in $H_2^{p\times p}$ is a compact set, so that we can think of using a gradient algorithm to find a local minimum of the function Ψ^n defined by (8) in section 2. The elements of $\mathcal{I}_n^p(1)$ will be parameterized as explained in the previous section, the local coordinates being the real and imaginary parts of the components of the vectors $y_1, \ldots, y_n$. We shall work with the local representations of Ψ^n and denote by $\Psi^n_{(\mathbf{w},\mathbf{u})}$ the local representation associated with the chart defined by $(\mathbf{w}, \mathbf{u})$:

$$\begin{array}{rlll}\Psi^n_{(\mathbf{w},\mathbf{u})}: & \mathcal{B}_p^n & \to & \mathbb{R}, \\ & \mathbf{y} = (y_1, \ldots y_n) & \to & \Psi^n \circ \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y}).\end{array}$$

4.1. Limit points in the charts. The object of this section is to study what happens when, running a gradient algorithm, the norm of some Schur parameter tends to 1. In the scalar case, the structure of $\mathcal{I}_n^1(1)$ is particularly simple, since only one coordinate map is needed; as some $|y_k|$ tends to 1, the boundary of $\mathcal{I}_n^1(1)$ is reached. This boundary has been completely studied in the case of real functions; it has been proved in [7] that the set $\mathcal{RI}_n^1(1)$ can be identified with the set $\mathcal{P}_n^1$ of monic stable polynomials of degree n and established in [10] that its closure is a topological manifold with boundary, this boundary having corners. The smooth part of the boundary, which plays an important role in the algorithm, consists of those polynomials having exactly one irreducible factor over $\mathbb{R}$ whose roots are of modulus one. In the matrix case, as some $\|y_k\|$ tends to 1, either the chart is no more available and another one must be used, or some point of the boundary of $\mathcal{I}_n^p(1)$ is reached. Moreover, as we shall see later, the closure of $\mathcal{I}_n^p(1)$ is no more a topological manifold with boundary, and possesses some *singular* boundary points (see Proposition 13).

Proposition 11 below, describes *regular* boundary points. Observe that if $\|y\| = 1$, the J-inner function $\Theta(w, u, y)$ is no more defined; however, if u^*Ay is not identically equal to 1, the transformation $S_{(w,u)}$ keeps a sense and is given by

$$S_{(w,u)}(A, y) = A + \frac{(u - Ay)(y^*A - u^*)}{(1 - u^*Ay)}.$$

PROPOSITION 11. *Let* $\mathbf{y} \in \partial\mathcal{B}_p^n$*, the boundary of* $\mathcal{B}_p^n$*,* $\mathbf{w} \in \mathbb{U}^n$*, and* $\mathbf{u} \in \partial\mathcal{B}_p^n$*, and let* $(D^{(k)}, \widetilde{q}^{(k)})$ *be the sequence associated with* $\mathbf{w}, \mathbf{u}, \mathbf{y}$ *by the recursion formulas* (27) *and* (28)*. A sequence*

$$I_p \to Q^{(1)} \to \cdots \to Q^{(k)} = \mathcal{S}_{(w_k,u_k)}(Q^{(k-1)}, y_k) \to \cdots \to Q^{(n)}$$

of inner matrices can be computed, provided that $u_k^* Q^{(k-1)}(w_k) y_k$ *is not identically equal to* 1 *as* $\|y_k\| = 1$*, or equivalently,* $\widetilde{q}^{(k)}$ *does not vanish identically. In this case,* $\mathbf{y}$ *will be called a regular limit point in the chart defined by* $(\mathbf{w}, \mathbf{u})$*. Then* $Q^{(k)} = D^{(k)}/\widetilde{q}^{(k)}$*, and*

(a) $q^{(k)}$ *still has degree* k*,*

(b) *if* $\|y_k\| = 1$*, then* $\widetilde{q}^{(k)}$ *and* $D^{(k)}$ *have common roots on* $\mathbb{T}$ *and* $Q^{(k)}$ *has degree less than* k*.*

Moreover, there exists a neighborhood $\mathcal{W}$ *of* $\mathbf{y}$*, such that* $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}$ *extends smoothly to* $\mathcal{W}$*.*

Proof. Assume that these assertions have been proved until order $k-1$, and let us prove that they still hold at order k. If $\|y_k\| < 1$, there is nothing to prove. If $\|y_k\| = 1$, then

$$\widetilde{q}^{(k)} = (1 - |w_k|^2)(1 - z)(\widetilde{q}^{(k-1)} - u_k^* D^{(k-1)} y_k),$$

and since $Q^{(k-1)}$ is inner, by the maximum modulus principle, either $u_k^* Q^{(k-1)} y_k$ is identically equal to 1, and $\widetilde{q}^{(k)}$ vanishes identically, or

$$\widetilde{q}^{(k)}(0) = (1 - |w_k|^2)\widetilde{q}^{(k-1)}(0)(1 - u_k^* Q^{(k-1)}(0) y_k)$$

does not vanish and thus $q^{(k)}$ has degree k. In this case, $Q^{(k)} = \mathcal{S}_{(w_k,u_k)}(Q^{(k-1)}, y_k) = D^{(k)}/\widetilde{q}^{(k)}$ is well defined and still inner; 1, which is a root of $\widetilde{q}^{(k)}$, must also be a root of $D^{(k)}$, and the degree of $Q^{(k)}$ cannot exceed that of $Q^{(k-1)}$. More precisely,

$$\deg Q^{(k)} = \deg Q^{(k-1)} - \sharp\{\xi \in \mathbb{T}, y_k = Q^{(k-1)}(\xi)^* u_k\}.$$

By induction, the first part of the proposition is proved. Now, $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y}) = D^{(n)}/\widetilde{q}^{(n)}$ and since, by Proposition 10, the coefficients of the polynomials $\widetilde{q}^{(n)}$ and $d_{ij}^{(n)}$ are polynomial functions in the local coordinates, there exists a neighborhood $\mathcal{W}$ of $\mathbf{y}$, such that $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}$ extends smoothly to $\mathcal{W}$. ∎

In the next lemma, we shall prove that any inner matrix of degree strictly less than n can be viewed, up to a unitary matrix, as a boundary point of $\mathcal{I}_n^p(1)$ of this type.

LEMMA 12. *For each* $Q \in \mathcal{I}_d^p(1)$ *of degree* d *strictly less than* n*, there exist* $\mathbf{w}' \in \mathbb{U}^n$*,* $\mathbf{u}' \in \partial\mathcal{B}_p^n$*,* $\mathbf{y}' \in \partial\mathcal{B}_p^n$*, and a unitary matrix* $\mathcal{U}$*, such that* $\mathbf{y}'$ *is a regular limit point in the chart defined by* $(\mathbf{w}', \mathbf{u}')$ *and* $\mathcal{U}Q = \varphi_{(\mathbf{w}',\mathbf{u}')}^{-1}(\mathbf{y}')$.

Proof. Let $\mathbf{y} = (y_1, \ldots, y_d)$, $\mathbf{w} = (w_1, \ldots, w_d)$, and $\mathbf{u} = (u_1, \ldots, u_d)$ be such that $Q = \varphi_{(\mathbf{w},\mathbf{u})}^{-1}(\mathbf{y})$:

$$I_p \to Q^{(1)} = \mathcal{S}_{(w_1,u_1)}(I_p, y_1) \to Q^{(2)} \cdots \to Q = \mathcal{S}_{(w_d,u_d)}(Q^{(n-1)}, y_d).$$

The Schur transform $S_{(w,u)}$ applied to some unitary matrix $\mathcal{X}$ and some unit vector y such that $y \neq \mathcal{X}^* u$ will give another unitary matrix. Thus, we can construct a unitary matrix $\mathcal{U}$ in the following way:

$$\mathcal{U}_0 = I_p \to \mathcal{U}_1 = \mathcal{S}_{(w_1',u_1')}(I_p, y_1') \to \mathcal{U}_2 \cdots \to \mathcal{U} = \mathcal{S}_{(w_{n-d}',u_{n-d}')}(\mathcal{U}_{n-d-1}, y_{n-d}'),$$

where $w_1', \ldots, w_{n-d}'$, are chosen arbitrarily and $u_1', \ldots, u_{n-d}', y_1', \ldots, y_{n-d}'$, are unit vectors in $\mathbb{C}^p$, satisfying for $k = 1, \ldots, n-d$, ${u_k'}^* \,\mathcal{U}_{k-1}\, y_k' \neq 1$. Since we have

$$S_{(w,\,\mathcal{X}u)}(\mathcal{X}A, y) \;=\; \mathcal{X}S_{(w,\,u)}(A, y)$$

for any unitary matrix $\mathcal{X}$, $\mathcal{U}Q$ can be computed by the following iterative process:

$$I_p \to \mathcal{U}_1 = \mathcal{S}_{(w_1',u_1')}(I_p, y_1') \to \mathcal{U}_2 \cdots \to \mathcal{U} = \mathcal{S}_{(w_{n-d}',u_{n-d}')}(\mathcal{U}_{n-d-1}, y_{n-d}')$$

$$\to \mathcal{U}Q^{(1)} = \mathcal{S}_{(w_1,\mathcal{U}u_1)}(\mathcal{U}, y_1) \to \mathcal{U}Q^{(2)} \cdots \to \mathcal{U}Q = \mathcal{S}_{(w_d,\mathcal{U}u_d)}(\mathcal{U}Q^{(n-1)}, y_d).$$

Put

$$\begin{aligned}
\mathbf{w}' &= (w_1', \ldots, w_{n-d}', w_1, \ldots, w_d),\\
\mathbf{u}' &= (u_1', \ldots, u_{n-d}', \mathcal{U}u_1, \ldots, \mathcal{U}u_d),\\
\mathbf{y}' &= (y_1', \ldots, y_{n-d}', y_1, \ldots, y_d),
\end{aligned}$$

then $\mathbf{y}'$ is a regular limit point in the chart defined by $(\mathbf{w}', \mathbf{u}')$ and $\mathcal{U}Q = \varphi_{(\mathbf{w}',\mathbf{u}')}^{-1}(\mathbf{y}')$ as required. ∎

The next proposition is concerned with *singular* boundary points.

PROPOSITION 13. *Let* $\eta(t) : [0,1] \to \overline{\mathcal{B}_p^n}$ *be a smooth path whose terminal point* $\mathbf{y} = \eta(1)$ *belongs to* $\partial\mathcal{B}_p^n$ *and let* $D(t)/\widetilde{q}(t)$ *be the fractional representation of* $Q(t) = \varphi_{(\mathbf{w},\mathbf{u})}^{-1}(\eta(t))$ *obtained by the recursion formulas* (27) *and* (28)*. Assume that* $\widetilde{q}(t)$ *vanishes identically as* $t \to 1$*. Then,* $Q(t)$ *converges to some inner function* Q_η*, depending in general on the path and whose degree may be less than or equal to* n*; it is given by the number of roots of* $q(t)$ *which converges within the unit disk.*

Proof. Since $D(t)$ and $\widetilde{q}(t)$ are polynomial functions in the local coordinates, they do converge, respectively, to a polynomial matrix D and a polynomial $\widetilde{q}$. We deal with the case where $\widetilde{q}$ is the zero polynomial. However, the quotient $D(t)/\widetilde{q}(t)$ must converge to some inner function Q_η. Let $q(t)(z) = q_n(t)z^n + \cdots + q_1(t)z + q_0(t)$. As $t \to 1$, each coefficient tends to 0 while the quotients $q_k(t)/q_n(t)$ being the well-known elementary symmetric polynomials in the roots (of modulus at most 1) are bounded by the binomial coefficients $\binom{n}{k}$ and thus converges. The polynomial $q(t)/q_n(t)$ converges to some monic polynomial, which may have roots of modulus one, while the number of its roots within the unit circle gives the degree of Q_η.

Now, we are going to show that the limit Q_η actually depends on the path and may have degree as well less than or equal to n. Let us study the case where $n = 1$ ($\eta(t) = y_1(t)$). Formulas (27) and (28) yield

$$\widetilde{q}^{(1)}(t) = \widetilde{b}_{w_1} u_1^*(u_1 - y_1(t)) + b_{w_1}(u_1 - y_1(t))^* u_1 - b_{w_1}(u_1 - y_1(t))^*(u_1 - y_1(t)),$$

and

$$D^{(1)}(t) = \widetilde{q}^{(1)}(t) I_p - (\widetilde{b}_{w_1} - b_{w_1})(u_1 - y_1(t))(u_1 - y_1(t))^*,$$

so that

$$Q^{(1)}(t) = I_p - \frac{\widetilde{b}_{w_1} - b_{w_1}}{\widetilde{q}^{(1)}(t)}\ (u_1 - y_1(t))(u_1 - y_1(t))^*.$$

Now, as $t \to 1$, $\widetilde{q}^{(1)}(t)$ vanishes identically by assumption, and thus $y_1(t)$ must converge to u_1. Let

$$y_1(t) = u_1 - \sum_{k \geq l} (1-t)^k \xi_k, \quad \xi_l \neq 0, \quad \xi_k \in \mathbb{C}^p$$

be its expansion. Consequently, $\widetilde{q}^{(1)}(t) \sim (\widetilde{b}_{w_1} u_1^* \xi_l + b_{w_1} \xi_l^* u_1)(1-t)^l$ and $Q^{(1)}(t)$ converges to I_p, unless $u_1^* \xi_l = 0$. In this case, let s be the smallest index satisfying $s > l$ and $u_1^* \xi_s \neq 0$. Then, if $s < 2l$, $Q(t)$ still converges to I_p, while if $s \geq 2l$,

$$\widetilde{q}^{(1)}(t) \sim (\widetilde{b}_{w_1} u^* \xi_{2l} + b_{w_1} \xi_{2l}^* u_1 - \widetilde{b}_{w_1} \xi_l^* \xi_l)(1-t)^{2l},$$

and

$$Q^{(1)}(t) \to I_p - \frac{\widetilde{b}_{w_1} - b_{w_1}}{\widetilde{b}_{w_1} u_1^* \xi_{2l} + b_{w_1} \xi_{2l}^* u_1 - \widetilde{b}_{w_1} \xi_l^* \xi_l}\ \xi_l \xi_l^*,$$

which is an inner function of degree 1. In conclusion, as $y_1(t)$ converges to u_1, $Q^{(1)}(t)$ converges either to I_p or to some inner matrix of degree 1, in which case, we leave the domain of the chart while staying inside the manifold. The same situation arises at each order, though it may be more complicated if the norms of several Schur parameters go to 1. ∎

As an illustration, the closure of $\mathcal{RI}_1^2(1)$ can be viewed as a cone. The summit represents the identity matrix and is a singular boundary point while the base represents the circle of orthogonal matrices of determinant -1 and forms a regular boundary. Two charts are needed to describe this manifold. For example, the chart given by $w = 0$ and $u = (1, 0)^*$ parametrizes all the inner functions except for those of the form (a line of the cone) $\begin{pmatrix} 1 & 0 \\ 0 & \frac{z-a}{1-az} \end{pmatrix}$, $a \in (-1, 1)$, while the chart given by $w = 0$ and $u = (0, 1)^*$ parametrizes all the inner functions except for those of the form $\begin{pmatrix} \frac{z-a}{1-az} & 0 \\ 0 & 1 \end{pmatrix}$.

4.2. Properties of the local representations of the criterion. The object of this section is to study the behavior of the local representations of the criterion at the neighborhood of a boundary point of $\mathcal{I}_n^p(1)$. We have distinguished in the last section two kinds of limit points, the regular and the singular ones. In both cases, if $\eta(t)$ is a path whose terminal point $\mathbf{y}$ corresponds to a boundary point of degree $d < n$, say Q_η, then it is easily proved that

$$\lim_{t\to 1} \Psi^n_{(\mathbf{w},\mathbf{u})}(\eta(t)) = \Psi^d(Q_\eta).$$

However, regular limit points play a central role in our algorithm, mainly due to the fact that the local representations of the criterion extends smoothly at the neighborhood of such points. To prove this result, we shall need the following expression for Ψ^n.

PROPOSITION 14. *Let $G(z) = F^\sharp(z)/z$ and $Q = D/\widetilde{q}$ as in Proposition 2. Let R be the remainder in the Weierstrass division in $H_2^{p\times m}$ of $G\widetilde{D}$ by q:*

$$G\widetilde{D} \;=\; V\,q \;+\; R. \tag{30}$$

Then q divides RD and if P is the matrix quotient, of degree at most $n-1$, we have that

$$\Psi^n(Q) \;=\; \|F\|_2^2 - \left\langle F, \frac{\widetilde{P}}{q} \right\rangle. \tag{31}$$

Proof. Since $Q^{-1}L(Q)$ and $F - Q^{-1}L(Q)$ are orthogonal, the cost function can be rewritten:

$$\Psi^n(Q) = \|F\|_2^2 - \langle F, Q^{-1}L(Q)\rangle.$$

The orthogonal projection $L(Q)$ of QF onto $H_2^{p\times m}$ is easily computed from the division (30) as being given by $L(Q) = \widetilde{R}/\widetilde{q}$, where $\widetilde{R} = z^{n-1}\, R^\sharp(z)$. Now, multiplying (30) by D on the right shows that q divides RD, and $Q^{-1}L(Q) = \widetilde{D}/q\; \widetilde{R}/\widetilde{q} = \widetilde{P}/q$. ◻

PROPOSITION 15. *Assume that $G(z) = F^\sharp(z)/z$ is analytic in $D_r = \{z,\ \|z\| \le r\}$ for some $r > 1$. Let $\mathbf{y} \in \partial\mathcal{B}_p^n$ be a regular limit point in some chart defined by $(\mathbf{w},\mathbf{u})$ (see Proposition 11) and let $Q = \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y})$ belong to $\mathcal{I}_d^p$ for some $d < n$. Then, $\Psi^n_{(\mathbf{w},\mathbf{u})}$ extends in some open neighborhood of $\mathbf{y}$ to a smooth function still denoted by $\Psi^n_{(\mathbf{w},\mathbf{u})}$. Moreover, we have*

$$\Psi^n_{(\mathbf{w},\mathbf{u})}(\mathbf{y}) \;=\; \Psi^d\left(Q(1)^{-1}Q\right).$$

Proof. Let $\mathcal{W}$ be a neighborhood of $\mathbf{y}$ on which, by Proposition 11, $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}$ extends smoothly. We may assume that in $\mathcal{W}$, $|\widetilde{q}(0)| \ge \mu$, for some $\mu > 0$. In order to proceed to our extension, we shall use the expression (31) of Ψ^n, in which the polynomial matrices D, R, and P, and the polynomial q, depend on the local coordinates. A well-known integral representation for the remainder R (cf. [39]) is

$$R(z) \;=\; \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{G\widetilde{D}}{q}\,\frac{q(\xi)-q(z)}{\xi - z}\,d\xi.$$

We may also restrict $\mathcal{W}$ so that the roots of q lie in a disk $D_s = \{z, |z| < s\}$ for some s, $1 < s < r$. Then, we can extend R in $\mathcal{W}$ by putting

$$R(z) \;=\; \frac{1}{2i\pi}\int_{\Gamma} \frac{G\widetilde{D}}{q}\,\frac{q(\xi)-q(z)}{\xi - z}\,d\xi, \tag{32}$$

where Γ is any contour lying in the open annulus between D_s and D_r. Indeed, the coefficient of order k of R is given by

$$R_k \;=\; \frac{1}{2i\pi}\int_\Gamma \frac{G\widetilde{D}}{q}(\xi^{n-k-1}q_n+\cdots+q_{k+1})d\xi$$

and since $|q(\xi)| > \mu\ d(\Gamma, D_s)^n$, the integrand is bounded, and its derivatives are also bounded. Finally, Lebesgue's theorem says that the integral representation (32) defines a smooth function. The extension of R is still the remainder of the division (30). In $\mathcal{W}$, q keeps on dividing RD and the quotient extends smoothly P. As for R, $\Psi^n_{(\mathbf{w},\mathbf{u})}$ extends smoothly by the integral representation

$$\Psi^n_{(\mathbf{w},\mathbf{u})}(\mathbf{y}) \;=\; \|F\|_2^2 - \frac{1}{2i\pi}\mathrm{Tr}\overline{\int_\Gamma G(z)\frac{\widetilde{P}}{q}(z)dz}. \qquad \square$$

Let us give two important consequences of Proposition 15.

LEMMA 16. *Let $Q \in \mathcal{I}_k^p(1)$ for some $k < n$ and let $\mathbf{y} = (y_1,\dots,y_k)$ be its Schur parameters in some chart defined by $\mathbf{w} = (w_1,\dots,w_k)$ and $\mathbf{u} = (u_1,\dots,u_k)$. Let $w_0 \in \mathbb{U}$, u_0 and y_0 be two distinct unit vectors and put*

$$\mathcal{U} = I_p - \frac{(u_0-y_0)(u_0-y_0)^*}{1-u_0^*y_0},$$

$\mathbf{w}' = (w_0, w_1,\dots,w_k)$, $\mathbf{u}' = (u_0, \mathcal{U}u_1,\dots,\mathcal{U}u_k)$, and $\mathbf{y}' = (y_0, y_1,\dots,y_k)$. Then $\mathbf{y}'$ is a regular limit point in the chart defined by $(\mathbf{w}',\mathbf{u}')$ and $Q' = \mathcal{U}Q$ is given by $Q' = \varphi^{-1}_{(\mathbf{w}',\mathbf{u}')}(y')$. Moreover,

$$\psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}(\mathbf{y}') = \psi^k_{(\mathbf{w},\mathbf{u})}(\mathbf{y}). \tag{33}$$

COROLLARY 17. *Suppose that $\mathbf{y}$ is a local minimum of $\psi^k_{(\mathbf{w},\mathbf{u})}(\mathbf{y})$. Then, the gradient of $\psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}$ at $\mathbf{y}'$ is orthogonal to the surface $\mathcal{S} = \{(y_0,\dots,y_n),\ \|y_0\| = 1,\ \|y_j\| < 1,\ j = 1,\dots n\}$ and points outwards.*

Proof. From Proposition 15 we see that the projection of the gradient of $\psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}$ at $\mathbf{y}'$ on $\mathcal{S}$ is just the gradient of $\psi^k_{(\mathbf{w},\mathbf{u})}$ at $\mathbf{y}$, whence orthogonality holds. Moreover, it cannot point inwards because this would imply that Q' which is rational of order k is a local minimum at order $k+1$, and this is impossible except if F itself has degree $k+1$ (cf. [6]). $\square$

4.3. The algorithm. The algorithm searching for a local minimum at order n splits into four main operations.

A. *Choosing an initial point.* This choice involves the choice of $(\mathbf{w},\mathbf{u})$ indexing a chart. The initial point $Q_i = \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y}_i)$ may have degree less than or equal to the target order n.

B. *Minimizing at fixed order k.* A software is used which integrates the vector field $-\mathrm{grad}\,\Psi^k_{(\mathbf{w},\mathbf{u})}$ from an initial point $\mathbf{y}_i \in \mathcal{B}_p^k$. The cost function is computed by (31) where $q = q^{(k)}$ and $\widetilde{D} = \widetilde{D}^{(k)}$ are given by the following recursion formulas, immediately deduced from (27) and (28),

$$\widetilde{D}^{(l)} = (b_{w_l} - y_l^* y_l \widetilde{b}_{w_l})\widetilde{D}^{(l-1)} - (b_{w_l} - \widetilde{b}_{w_l}) \Bigg\{ \widetilde{D}^{(l-1)} u_l u_l^* + y_l y_l^* \widetilde{D}^{(l-1)} - q^{(l-1)}\ y_l u_l^* + \frac{y_l^* \widetilde{D}^{(l-1)} u_l\ \widetilde{D}^{(l-1)} - \widetilde{D}^{(l-1)} u_l y_l^* \widetilde{D}^{(l-1)}}{q^{(l-1)}} \Bigg\}, \tag{34}$$

$$q^{(l)} = (b_{w_l} - y_l^* y_l \widetilde{b}_{w_l})\, q^{(l-1)} - (b_{w_l} - \widetilde{b}_{w_l})\, y_l^* \widetilde{D}^{(l-1)} u_l, \tag{35}$$

and initialized by $q^{(0)} = 1$ and $\widetilde{D}^{(0)} = 1$. Then, one of the following possibilities occurs:

(i) *a local minimum is reached.* If $k = n$, we are done, while if $k < n$, this local minimum provides an initial point for searching for a minimum of order $k + 1$, as described in point D.

(ii) *the norm of some Schur parameter tends to* 1*.* This situation has been studied in section 4.1; either a change of chart is necessary, or a boundary point of the manifold is reached. More precisely, if the polynomial $\widetilde{q}^{(k)}$ nearly vanishes while its roots stay far from the unit circle, then the limit point belongs to $\mathcal{I}_k^p(1)$, and the first eventuality is true. In any other case, a boundary point is reached.

C. *Meeting a boundary point.* Such a boundary point, up to an unitary matrix, is an element Q_b of $\mathcal{I}_d^p(1)$ for some $d < k$, and the criterion at order k converges to $\Psi^d(Q_b)$. Then, a minimization process at order d can restart from Q_b. If only the first Schur parameter has norm 1, we can directly deduce from Lemma 16 some chart and Schur parameters for Q_b. Otherwise, the matrix Q_b must be computed from the recursion formulas (27) and (28), eliminating the roots of modulus one. Then, an adequate chart has to be provided.

D. *Choosing an adequate coordinate chart.* Given a normalized inner matrix Q, of order k, we must find a couple $(\mathbf{w}, \mathbf{u})$ such that Q belongs to the local neighborhood $\mathcal{V}_{(\mathbf{w},\mathbf{u})}$ defined in section 3.3, or equivalently such that a sequence $Q^{(k)} = Q, Q^{(k-1)}, \dots, Q^{(1)} = I_p$ of inner functions of decreasing degree can be constructed by the Schur algorithm. The fractional representation of $Q^{(l-1)}$ is computed from that of $Q^{(l)}$ by the recursion formulas

$$b_{w_l}\, \tilde{b}_{w_l}\, \tilde{D}^{(l-1)} = (\tilde{b}_{w_l} - y_l^* y_l\, b_{w_l})\tilde{D}^{(l)} - (\tilde{b}_{w_l} - b_{w_l}) \left(\tilde{D}^{(l)} u_l u_l^* + y_l y_l^* \tilde{D}^{(l)} - q^{(l)}\ y_l u_l^* + \frac{y_l^* \tilde{D}^{(l)} u_l\ \tilde{D}^{(l)} - \tilde{D}^{(l)} u_l y_l^* \tilde{D}^{(l)}}{q^{(l)}} \right),$$

$$b_{w_l}\, \tilde{b}_{w_l}\, q^{(l-1)} = (\tilde{b}_{w_l} - y_l^* y_l\, b_{w_l})\, q^{(l)} - (\tilde{b}_{w_l} - b_{w_l})\, y_l^* \tilde{D}^{(l)} u_l.$$

The polynomial $b_{w_l}\, \tilde{b}_{w_l}$ divides the right-hand sides, so that $\tilde{D}^{(l-1)}$ and $q^{(l-1)}$ actually are polynomial, and $Q^{(l-1)}$ has degree $l - 1$ as required.

E. *Increasing the degree.* When the minimization procedure leads to a local minimum of order $k < n$, say Q_m, then Lemma 16, for any choice of w_0, u_0, and $y_0 \neq u_0$, provides a boundary point Q'_m of $\mathcal{I}_{k+1}^p(1)$ together with a local parametrization $Q'_m = \varphi^{-1}_{(\mathbf{w}',\mathbf{u}')}(\mathbf{y}'_m)$, deduced from a local parametrization $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y}_m)$ of Q_m, satisfying (33). Since by Corollary 17, $-\mathrm{grad}\, \Psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}$ points inwards at $\mathbf{y}'$, this point can be used as an initial point for a minimization process at order $k + 1$.

The point is that the value of the criterion, where the criterion must be understood as being Ψ^k when working at order k, decreases continuously, being conserved while the order changes, so that the minimization process pursues through different orders.

To ensure the good behavior of the algorithm, we shall make two extra assumptions. First, we shall assume that grad Ψ^k *does not vanish on the boundary of* $\mathcal{I}_k^p(1)$*, for* $1 \leq k \leq n$. Second, we shall require *all the critical points of* Ψ^k *in* $\mathcal{I}_k^p(1)$ *to be nondegenerate,* i.e., to have a second derivative which is a nondegenerate quadratic form. These two properties hold generically, that is for almost every F in some sense, and we refer the reader to [8] for the first one, and [6] for the second one. They ensure in particular that critical points in $\mathcal{I}_k^p(1)$ are finite in number. Since the criterion decreases continuously, we never meet twice the same local minimum and *this ensures that the procedure eventually comes to an end.* Note that if the minimization process stops at a critical point which is not a minimum, since this point is nondegenerate, it will be unstable under small perturbations, thereby allowing us to continue the procedure.

The choice of an initial point is crucial for our purpose (see the example in the next section). In many problems, we hope that some more information or engineering judgment could help us to select an initial point which ensures rapid convergence of the procedure to the global minimum. However, it is well known that the L_2 approximation problem possesses many local minima. Since our final goal is to find the global minimum, we may think of initializing the algorithm at enough points to reach all local minima and compare between them. But we do not know what "enough" means and we do not have a bound for the number of initializing points. Consequently, more efficient strategies should be investigated. For instance, we can find all the local minima at order 1 and then, initialize our procedure at order 2, by replacing them on the boundary of $\mathcal{I}_2^p(1)$ as described in point D, choosing w_0, u_0, and y_0 in several ways, and so on, step by step, until the target order. This strategy gives rather good results.

The choice of a local chart at the neighborhood of a given point is an important and difficult task. The main purpose of using coordinates is to be able to perform calculations on a computer and as such it is desirable that the numerical conditioning of the chart is good. A criterion must be chosen to decide upon the quality of local coordinates around a point on a manifold. Moreover, a distortion occurs when mapping part of a manifold to Euclidean space, so that the sequence of improving estimates produced by an optimization algorithm is dependent on the choice of the chart, and it would be interesting to select the charts with the view to improve the convergence of the algorithm. But in this case, the selection strategy will depend upon the problem at hand and bring along a lot of "overhead costs." The present version of our algorithm uses a basic selection strategy, which minimizes the norm of the Schur parameters at each step of the Schur algorithm over a finite atlas. This point must be improved and is presently under study.

4.4. A numerical example. The sole purpose of the following example is to demonstrate the procedure of computing local minima. For more real-data simulations, we refer to the scalar case paper [8] or [9]. This example has been first considered in [31] to demonstrate the procedure of computing the minimal degree approximation in a Hankel-norm model reduction problem and refers to a fourth-order system:

$$F(z) = \begin{pmatrix} \dfrac{1+z}{z^2 - z + 1/4} & \dfrac{1}{z - 1/2} \\[2ex] \dfrac{-z^2 + z + 1}{z^3 + 1/2z^2 - 1/4z - 1/8} & \dfrac{z - 1/4}{z^2 + z + 1/4} \end{pmatrix},$$

or equivalently $F = N/d$, where

$$d(z) = z^4 - 1/2z^2 + 1/16$$

and

$$N(z) = \begin{pmatrix} z^3 + 2z^2 + 5/4z + 1/4 & z^3 + 1/2z^2 - 1/4z - 1/8 \\ -z^3 + 3/2z^2 + 1/2z - 1/2 & z^3 - 5/4z^2 + 1/2z - 1/16 \end{pmatrix}.$$

The system has four poles located at $1/2$, $1/2$, $-1/2$, $-1/2$. According to the theory, if we look for a minimum of (1) with $n = 4$, we must recover the function F itself, since from consistency, the criterion has no other critical points [12]. We shall use this fact to test the procedure.

The function to be approximated is represented in the program by a great number of Fourier coefficients (computed from frequency data in practice). Thus in this example, the input of the program is not actually the function F but the 200 first Fourier coefficients of its rational entries. The software package Scilab is used for the implementation. We have run a great number of tests changing the starting point and the initial chart. We present here a case in which every step of the algorithm must be visited before, according to the theory, we finally recover the function F.

Step 1. *We integrate at order* 4 *and reach the boundary.* The initial point has parameters $\mathbf{y} = ((0.5, 0.5)^*, ((0.5, 0.5)^*, (-0.5, -0.5)^*, (0.5, 0.5)^*)$ in the chart indexed by $\mathbf{w} = (0, 0, 0, 0)$ and $\mathbf{u} = ((1, 0)^*, (1, 0)^*, (0, 1)^*, (1, 0)^*)$, and corresponds to the inner matrix $Q_i = D_i/\widetilde{q}_i$, where

$$\widetilde{D}_i(z) =$$

$$\begin{pmatrix} -0.3 + 0.4z + 0.4z^2 - 0.8z^3 + 0.5z^4 & -0.5 + 0.8z - 0.4z^2 + 0.4z^3 - 0.3z^4 \\ 0.3 - 0.4z + 0.4z^2 - 0.8z^3 + 0.5z^4 & 0.5 - 0.8z + 0.4z^2 + 0.4z^3 - 0.3z^4 \end{pmatrix}, \tag{36}$$

$$q_i(z) = z^2(z^2 - 1.2z + 0.4). \tag{37}$$

Note that q_i is not exactly the stable polynomial $q^{(4)}$ computed from the recursion formulas (34) and (35) which has a leading coefficient equal to 0.3125. As we integrate the opposite of the gradient using the Scilab function "ode," the norm of the first parameter tends to 1, while $\widetilde{q}^{(1)}(0) = 0.49$ stays far from 0. Thus we have reached a *regular* boundary point Q_b of parameters

$$\mathbf{y}_b = ((0.509, 0.86)^*, (0.357, 0.55)^*, (-0.659, -0.405)^*, (0.556, 0.264)^*).$$

The criterion is equal to 3.786.

Step 2. *We integrate at order* 3 *and get a local minimum.* We put $u_0 = (1, 0)^*$ and $y_0 = (0.509, 0.86)^*$ and we compute the unitary matrix

$$\mathcal{U} = I_p - \frac{(u_0 - y_0)(u_0 - y_0)^*}{1 - u_0^* y_0}.$$

Lemma 16 implies that $Q_b = \mathcal{U}Q$, where Q is the normalized inner matrix of degree 3 of parameters $\mathbf{y} = ((0.357, 0.55)^*, (-0.66, -0.405)^*, (0.556, 0.265)^*))$ in the chart indexed by $\mathbf{w} = (0, 0, 0)$ and $\mathbf{u} = ((0.509, 0.86)^*, (0.86, -0.509)^*, (0.509, 0.86))$. According to the theory, the criterion at Q is still equal to 3.786. We restart the minimization procedure from this point and find a third degree minimum for

$$\mathbf{y}_m = ((-0.574, 0.652)^*, (0.0214, -0.433)^*, (0.205, 0.428)^*),$$

where the criterion is equal to 0.997 and the relative error to 0.05.

Step 3. *We increase the degree and get out of the domain of the chart.* This third order local minimum provides starting points for fourth order minimizations. For instance, applying Lemma 16 with $w_0 = 0$, $u_0 = (1, 0)^*$, and $y_0 = (0, 1)^*$, which are distinct unit vectors, yields to the initial point of parameters

$$\mathbf{y} = ((0, 1)^*, (-0.574, 0.652)^*, (0.021, -0.433)^*, (0.205, 0.428)^*)$$

in the chart indexed by $\mathbf{w} = (0, 0, 0, 0)$ and

$$\mathbf{u} = ((1, 0)^*, (0.86, 0.509)^*, (-0.509, 0.86)^*, (0.86, 0.509)^*).$$

The minimization process leads us to leave the domain of the chart. Indeed, it produces a sequence of inner functions whose denominators computed by formulas (27) and (28), have leading coefficients which tends to 0 but roots which stay far from the unit circle. We stop at

$$\mathbf{y} = ((0.88, 0.096)^*, (-0.688, 0.102)^*, (0.169, 0.232)^*, (0.264, -0.027)^*)$$

at which the value of $\widetilde{q}(0)$ is about 0.125 which can produce important errors in the computation.

Step 4. *We change the chart and recover the function F.* We choose to work with a finite subset of the atlas described in section 3.3; the family $(\mathcal{V}_{(\mathbf{w},\mathbf{u})}, \varphi_{(\mathbf{w},\mathbf{u})})$ where $\mathbf{w} = (0, 0, 0, 0)$, and $\mathbf{u}$ is composed of unit vectors either equal to $e_1 = (1, 0)^*$ or to $e_2 = (0, 1)^*$. This family is a covering of the manifold $\mathcal{I}_n^p(1)$. At each step of the Schur algorithm, we choose $u_k = e_j$, where e_j is the vector for which the norm of the Schur parameter $y_k = Q^{(k)}(0)^*(e_j)$ is the smallest. It may happen that this process provides Schur parameters of norm almost equal to 1. In this case we can try each chart of our finite atlas to find a better one. In our case this process gives a new chart indexed by $\mathbf{w} = (0, 0, 0, 0)$ and $\mathbf{u} = ((1, 0)^*, (1, 0)^*, (0, 1)^*, (1, 0)^*)$. The parameters of the point are given in this chart by $\mathbf{y} = ((0.632, -.278)^*, (-0.578, -.337)^*, (0.262, 0.192)^*, (0.157, -.142)^*)$. The minimization can continue and the minimum is reached for

$$\mathbf{y}_m = ((0.495, -0.32)^*, (-0.57, -0.328)^*, (0.266, 0.202)^*, (0.146, -0.129)^*).$$

The approximant computed from these parameters agrees with F with four significant digits.

If we start in the same initial chart $\mathbf{w} = (0, 0, 0, 0)$, and

$$\mathbf{u} = ((1, 0)^*, (1, 0)^*, (0, 1)^*, (1, 0)^*),$$

from the point

$$\mathbf{y} = ((0.5, -0.5)^*, (-0.5, -0.5)^*, (0.5, 0.5)^*, (0.5, -0.5)^*),$$

we immediately reach the minimum with a very good accuracy. This emphasizes the importance of the choice of the initial point. On the other hand, if we start from the same initial point Q_i given by (36) and (37), but in the chart indexed by $\mathbf{w} = (0, 0, 0, 0)$ and $\mathbf{u} = ((0, 1)^*, (1, 0)^*, (0, 1)^*, (0, 1)^*)$ (the Schur parameters are given by $\mathbf{y} = ((-.338, -.444)^*, (.0476, .506)^*, (.515, .515)^*, (-.3, -.3)^*))$, then we do not meet the boundary and we again reach the minimum with a very good accuracy. This illustrates the dependence on the chart of the iterative path produced by the gradient algorithm.

5. Conclusion. A rational approximation problem in L_2-norm has been studied. A new parametrization of stable all-pass transfer functions has been used, based on Schur analysis [1]. Such an overlapping parametrization (in differential geometry an atlas of charts) has allowed us to use classical optimization procedures within a local neighborhood, changing the neighborhood when necessary, in order to solve our minimization problem. Using the state space approach, other parametrizations of stable all-pass transfer functions are available as the one obtained in [25] in continuous-time, based on the work of Ober on balanced canonical forms [33]. A link between the two approaches is probable and a better understanding of the situation seems desirable. In this connection, a state space formulation of the Schur algorithm has been described in continuous-time in [23]. A balanced canonical form for discrete time stable all-pass systems has been obtained in the SISO case [34] by requiring the realization to be balanced and such that the reachability matrix is upper triangular with positive diagonal entries. This canonical form can be parametrized by the Schur parameters obtained in the classical algorithm (11). The generalization of these results to the multivariable case is under study.

Using this parametrization, a minimization algorithm has been described and its convergence to local minima has been proved. We have implemented this algorithm using the matrix-based scientific software Scilab and demonstrated the procedure of computing a local minimum in many simple examples. Later, using this work, a software package named Hyperion has been implemented by J. Grimm to solve a problem provided by the French CNES: identify from frequency data a 2×2 hyperfrequency filter of order 8. Very good results have been obtained on this problem [9]. However, the selection strategy algorithm used in this package is still basic and must be improved. This is going to be the object of forthcoming research.

Acknowledgments. The authors would like to thank D. Alpay who introduced them to Schur analysis and L. Baratchart for providing them with several helpful suggestions during this research. We would also like to thank E. Saff, A. Gombani, and N. Duddley Ward for their very careful reading of this paper and useful criticisms.

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