Matrix rational H_2 approximation and Schur parameters.

Pascale Fulcheri, Martine Olivi[1]

Abstract. This paper deals with the rational approximation of specified order n to transfer functions which are assumed to be matrix valued functions in the Hardy space for the complement of the closed unit disk endowed with the L_2-norm. An approach is developed leading to a new algorithm, the first one to our knowledge which concerns matrix transfer functions in L_2-norm. This algorithm proceeds inductively on the order n of the approximant using a gradient algorithm to find local minima through the manifold of inner functions of McMillan degree n. These functions are represented by means of local coordinate maps that come from a matricial version of the Schur algorithm.

1 Introduction

The identification of linear time-invariant systems can be formalized as a rational approximation problem in which some criterion function is optimized over a set of systems. This approach has led to a wide variety in model structure, performance criteria and actual methods of estimation (see [16] and the bibliography therein). Our interest is focused mainly on the particular class of discrete-time, linear, time-invariant and strictly causal systems and their strictly proper transfer functions. The criterion which is chosen here is the L_2-norm, so that our approximation problem states in $\bar{H}_{2,0}^{p\times m}$, where $\bar{H}_{2,0}$ denotes the orthogonal complement in $L^2(\mathbb{T})$ of the Hardy space H_2 of the unit disk $\mathbb{U}$ (see [10]): *given a transfer function $F \in \bar{H}_{2,0}^{p\times m}$, we are concerned in minimizing*

$$\|F-H\|_2^2 = \frac{1}{2\pi}\mathrm{Tr}\int_0^{2\pi}[F-H](e^{it})[F-H](e^{it})^*dt\,,$$

as H ranges over the set of rational stable (i.e. analytic for $|z|>1$) strictly proper functions of McMillan degree n.

[1] INRIA, BP 93, 06902 Sophia-Antipolis Cedex, FRANCE, {olivi}@sophia.inria.fr, phone: 33 4 92 38 78 77, fax: 33 4 92 38 78 58

In [2], an algorithm to find local minima in this problem is described for scalar systems. It proceeds inductively on the McMillan degree, using at each step a gradient algorithm to find one or several local minima which provide initial points for the next step. It is the purpose of this paper to present an algorithm which enables the results of this previous paper to be extended to the multivariable case. Such a generalization involves substantial new difficulties, mainly due to the fact that the domain of the cost function is no longer an open subset of a Euclidean space but it does possess a manifold structure. A manifold has a covering by countably many open coordinate neighborhoods, each of these coordinate neighborhoods corresponding to an open subset of some $\mathbb{R}^d$ by a local coordinate homeomorphism (d is then the dimension of the manifold). The methods developed for the Euclidean case then apply to each of the coordinate neighborhoods separately. Over a manifold, an optimization problem can be tackled by using a search algorithm through the manifold as a whole, using the coordinate maps to describe the manifold locally and changing from one coordinate map to another when required. Such a representation of the set of approximants has the advantage to get rid of redundancy and ensure identifiability [9]. A transfer function will be represented by means of the inner-unstable or Douglas–Shapiro–Shields factorization (see. [4]):

$$H \;=\; Q^{-1}\,C, \tag{1}$$

where Q is a $(p\times p)$-rational inner function and C a $(p\times m)$-rational matrix whose columns belong to $\mathcal{H}(Q)$ the orthogonal complement of QH_2^p in H_2^p. The factorization is unique up to a common left unitary factor and Q and H have same McMillan degree. To ensure uniqueness in the inner-unstable factorization, we shall require that Q satisfies the normalization condition

$$Q(1) = I_p. \tag{2}$$

The set of $\mathbb{C}^{p\times p}$-valued rational inner functions of degree n will be denoted by $\mathcal{I}_n^p$, and by $\mathcal{I}_n^p(1)$ we denote the subset of functions satisfying the extra condition (2). It is proved in [1] that $\mathcal{I}_n^p$ and $\mathcal{I}_n^p(1)$ are smooth manifolds of dimension $2np + p^2$ and $2np$ respectively.

The elimination of the parameters in which the system is linear (namely those of the unstable factor) leads to minimize the function

$$\begin{array}{cccc} \Psi^n & : & \mathcal{I}_n^p(1) & \rightarrow & \mathbb{R} \\ & & Q & \rightarrow & \|F - Q^{-1}L(Q)\|_2^2, \end{array} \qquad (3)$$

where $L(Q)$ denotes the projection of QF onto $H_2^{p\times m}$. To describe $\mathcal{I}_n^p(1)$ we shall use the tangential Schur algorithm which gives local coordinates.

The natural framework for our study is the complex case, that is the case of functions whose Fourier coefficients can be complex. It plainly includes the real case, which is relevant in most applications, by restriction. Due to the space limitations, we give no proofs here and refer the reader to [8].

2 Parametrization of inner matrices.

We give here a parametrization of the set of inner functions obtained in [1] from a matricial version of the classical Schur algorithm, involving reproducing kernel Hilbert spaces [6]. Other extensions of the Schur algorithm to matrix-valued analytic functions are available for example in [5], [7], etc.. The one presented here, rather general, has the advantage to give rise in a simple way to an atlas of charts for the manifold of inner functions of fixed degree. The statements of this section are adapted from [1].

Let

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix},$$

and let

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \qquad (4)$$

be a $(2p \times 2p)$-rational J-inner function analytic in $\mathbb{U}$. For any $(p \times p)$-rational inner function A, $(\Theta_{21}A + \Theta_{22})$ is invertible in $\mathbb{U}$ and we define

$$T_\Theta(A) = (\Theta_{11}A + \Theta_{12})\,(\Theta_{21}A + \Theta_{22})^{-1}. \qquad (5)$$

It is easily proved that $T_\Theta(A)$ is inner. Moreover, if $\Theta(1) = I_{2p}$ the normalization condition (2) is preserved.

The following function will play a special role in the sequel. Fix $w \in \mathbb{U}$, $u \in \mathbb{C}^p$ with $\|u\| = 1$, and $v \in \mathbb{C}^p$ and define

$$\Theta_{w,u,v}(z) = I_{2p} - (1-z)\frac{1-|w|^2}{1-\|v\|^2}\frac{\begin{pmatrix} u \\ v \end{pmatrix}}{1-\bar{w}z}\frac{\begin{pmatrix} u \\ v \end{pmatrix}^*}{1-w}J. \qquad (6)$$

The function $\Theta_{w,u,v}$ is J-inner of degree one and any J-inner function of degree one can be written in this form up to a right J-inner constant function. We then have:

Theorem 1 *Let B be a rational inner function, and let $\Theta_{w,u,v}$ be the J-inner function (6). There exists an inner function A such that*

$$B = T_{\Theta_{w,u,v}}(A)$$

if and only if the interpolation condition

$$B(w)^* u = v$$

is satisfied. Then $\deg B = \deg A + 1$.

The tangential Schur algorithm then proceeds as follows:

Tangential Schur algorithm: let $Q \in \mathcal{I}_n^p(1)$, and $w_k \in \mathbb{U}$, $k = n, \ldots, 1$. Then, for $k = n, \ldots, 1$, there exist unit vectors $u_k \in \mathbb{C}^p$, such that the vectors $y_k \in \mathbb{C}^p$, given by

$$y_k = Q^{(k)}(w_k)^* u_k, \qquad (7)$$

satisfy $\|y_k\| < 1$, where $Q^{(n)} = Q$,

$$Q^{(k)} = T_{\Theta_k}(Q^{(k-1)}), \qquad (8)$$

and $\Theta_k = \Theta_{w_k,u_k,y_k}$ is given by (6). Then

$$Q = T_{\Theta_n}(T_{\Theta_{n-1}} \ldots T_{\Theta_1}(I_p)) \ldots = T_{\Theta_n \ldots \Theta_1}(I_p).$$

Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and define

$$\mathcal{V}_{(\mathbf{w},\mathbf{u})} = \{Q \in \mathcal{I}_n^p(1) \;/\; \|Q^{(k)}(w_k)^* u_k\| < 1\},$$

$$\begin{array}{cccc} \varphi_{(\mathbf{w},\mathbf{u})} & : & \mathcal{V}_{(\mathbf{w},\mathbf{u})} & \rightarrow & \mathcal{B}_p^n \\ & & Q & \rightarrow & \mathbf{y} = (y_1, y_2, \ldots, y_n), \end{array}$$

where $\mathcal{B}_p^n$ denotes the product of n copies of the unit ball of $\mathbb{C}^p$. Then, the family $(\mathcal{V}, \varphi)$ defines a C^∞ atlas on $\mathcal{I}_n^p(1)$ which is compatible with its natural structure of embedded sub-manifold of $H_2^{p\times p}$.

3 Local representations of the cost function.

The elements of $\mathcal{I}_n^p(1)$ will be parameterized as explained in the previous section. In any chart of this atlas, *the local coordinates are the* $2np$ *real and imaginary parts of the components of the Schur parameters* $y_1, \dots, y_n$. We shall work with the local representations of Ψ^n and denote by $\Psi^n_{(\mathbf{w},\mathbf{u})}$ the local representation associated with the chart defined by $(\mathbf{w}, \mathbf{u})$:

$$\begin{array}{llcl} \Psi^n_{(\mathbf{w},\mathbf{u})}: & \mathcal{B}_p^n & \rightarrow & \mathbb{R} \\ & \mathbf{y} = (y_1, \dots y_n) & \rightarrow & \Psi^n \circ \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y}). \end{array}$$

For any matrix-valued function $A(z)$, we define

$$A^\sharp(z) = A(1/\bar{z})^*;$$

if q is a polynomial of formal degree n, we define its *reciprocal polynomial* as being

$$\widetilde{q}(z) = z^n \; q^\sharp(z), \tag{9}$$

and if D is a polynomial matrix whose degree does not exceed n, we also put

$$\widetilde{D}(z) \;=\; z^n \; D^\sharp(z). \tag{10}$$

An inner matrix $Q \in \mathcal{I}_n^p$ has a representation of the form

$$Q = D/\widetilde{q} \tag{11}$$

by means of a polynomial matrix D whose degree does not exceed n and a polynomial q of exact degree n whose roots belong to the open unit disk, satisfying $D\widetilde{D} = q\widetilde{q}I_p$ and $\det D = \epsilon q \widetilde{q}^{p-1}$, ϵ being a complex number of modulus one. Conversely, these conditions are sufficient for the rational matrix $D/\widetilde{q}$ to belong to $\mathcal{I}_n^p$.

We introduce the map

$$\mathcal{S}_{(w,u)} \;:\; (A, y) \;\rightarrow\; T_{\Theta_{w,u,y}}(A),$$

so that the inner matrix $Q = \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(y)$ is computed by the iterative process:

$$I_p \rightarrow Q^{(1)} \rightarrow Q^{(2)} \rightarrow \dots \rightarrow Q = Q^{(n)},$$

where $Q^k = \mathcal{S}_{(w_k,u_k)}(Q^{(k-1)}, y_k)$. A classical formula (see [13, Appendix A.20]) allows us to compute the inverse in (5) and we have

$$S_{(w,u)}(A, y) \;=\; A + \frac{(1-\beta_w)\,(u - Ay)(y^* - u^*A)}{1 - u^*Ay - \beta_w(y^*y - u^*Ay)},$$

with $\beta_w = b_w/\widetilde{b}_w$, and $b_w(z) = (z-w)(1-\bar{w})$.

Proposition 1 *A fractional representation $D_k/\widetilde{q}_k$ of the inner matrix $Q^{(k)} = T_{\Theta_k \dots \Theta_1}(I_p)$ can be computed by the recursion formulas:*
$D_0 = I_p$, $\widetilde{q}_0 = 1$, and for $k = 1, \dots, n$,

$$\begin{aligned} D_k = {} & (\widetilde{b}_{w_k} - y_k^* y_k \, b_{w_k}) D_{k-1} - (\widetilde{b}_{w_k} - \; b_{w_k}) \\ & \Big[u_k \; u_k^* D^{(k-1)} + D^{(k-1)} y_k y_k^* - \widetilde{q}^{(k-1)} u_k y_k^* \\ & + \frac{u_k^* D_{k-1} y_k \; D_{k-1} - D_{k-1} y_k u_k^* D_{k-1}}{\widetilde{q}_{k-1}} \Big] \end{aligned}$$

$$\widetilde{q}_k = (\widetilde{b}_{w_k} - y_k^* y_k \, b_{w_k}) \widetilde{q}_{k-1} - (\widetilde{b}_{w_k} - b_{w_k}) \, u_k^* D_{k-1} y_k,$$

where $b_{w_k} = (1-\bar{w}_k)(z - w_k)$.

The stable polynomial q_k has degree k, and the coefficients of the polynomials $\widetilde{q}_k$ and of the entries of D_k are polynomial functions in the local coordinates.

Though the quotient in the expression of D_k is exact, we fail in searching for an explicit formula for it, and we do not know if such a formula exists.

Now, the cost function can be computed from the fractional representation $D/\widetilde{q}$ of Q given by Proposition 1 as follows: let R be the remainder in the Weierstrass division in $H_2^{p\times m}$ of $G\widetilde{D}$ by q,

$$G\widetilde{D} \;=\; V \; q \;+\; R, \tag{12}$$

then q divides RD and if P is the matrix quotient, of degree at most $n-1$, we have that

$$\Psi^n(Q) \;=\; \|F\|_2^2 - < F, \frac{\widetilde{P}}{q} > . \tag{13}$$

4 A generic algorithm to find local minima.

The closure of $\mathcal{I}_n^p(1)$ in $H_2^{p\times p}$ is a compact set, so that we can think of using a gradient algorithm to find a local minimum of the function Ψ^n. We then have to study what happen when, running a gradient algorithm, the norm of some Schur parameter tends to 1. In the scalar case, the structure of $\mathcal{I}_n^1(1)$ is particularly simple, since only one coordinate map is needed: as some $\|y_k\|$ tends to 1, the boundary of $\mathcal{I}_n^1(1)$ is reached. In the matrix case, as some $\|y_k\|$ tends to 1, either the chart is no more available and another one must be used, or some point of the boundary of $\mathcal{I}_n^p(1)$ is reached.

Observe that if $\|y\| = 1$, the J-inner function $\Theta_{w,u,y}$ is no more defined, however, if u^*Ay is not identically equal to 1, the transformation $S_{(w,u)}$ keeps a sense and is given by

$$S_{(w,u)}(A,y) = A + \frac{(u - Ay)(y^*A - u^*)}{(1 - u^*Ay)}.$$

Regular limit points in the chart given by $(\mathbf{w},\mathbf{u})$: a point $\mathbf{y}$ of the boundary of $\mathcal{B}_p^n$, is a regular limit point in this chart if a sequence of inner matrices

$$I_p \to Q^{(1)} \to Q^{(2)} \to \ldots \to Q^{(n)},$$

where $Q^{(k)} = \mathcal{S}_{(w_k,u_k)}(Q^{(k-1)}, y_k)$ can be computed, that is if $u_k^* Q^{(k-1)}(w_k) y_k$ is not identically equal to 1 as $\|y_k\| = 1$, or equivalently if $\widetilde{q}_k$ (computed as in Proposition 1) does not vanish identically.

In this case, if $Q^{(k)} = D_k/\widetilde{q}_k$ is the fractional representation computed as in Proposition 1, then
(a) q_k still has degree k,
(b) if $\|y_k\| = 1$, then $\widetilde{q}_k$ and D_k have common roots on $\mathbb{T}$ and $Q^{(k)}$ has degree less than k.
Moreover, there exists a neighborhood $\mathcal{W}$ of $\mathbf{y}$, such that $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}$ extends smoothly to $\mathcal{W}$.

Any inner matrix of degree strictly less than n can be viewed, up to a unitary factor on the left, as a boundary point of $\mathcal{I}_n^p(1)$ of this type. Regular limit points play a central role in our algorithm, mainly due to the fact that the local representations of the criterion extends smoothly at the neighborhood of such points.

Proposition 2 *Assume that $G(z) = F^\sharp(z)/z$ is analytic in $D_r = \{z, \ \|z\| \le r\}$ for some $r > 1$. Let $\mathbf{y}$ be a regular limit point in some chart defined by $(\mathbf{w},\mathbf{u})$ and let $Q = \varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y})$ belong to $\mathcal{I}_d^p$ for some $d < n$. Then, $\Psi^n_{(\mathbf{w},\mathbf{u})}$ extends in some open neighborhood of $\mathbf{y}$ to a smooth function still denoted by $\Psi^n_{(\mathbf{w},\mathbf{u})}$. Moreover, we have*

$$\Psi^n_{(\mathbf{w},\mathbf{u})}(\mathbf{y}) = \Psi^d\left(Q(1)^{-1}Q\right).$$

Let us give an important consequence of Proposition 2.

Lemma 1 *Let $Q \in \mathcal{I}_k^p(1)$ for some $k < n$ and let $\mathbf{y} = (y_1,\ldots,y_k)$ be its Schur parameters in some chart defined by $\mathbf{w} = (w_1,\ldots,w_k)$ and $\mathbf{u} = (u_1,\ldots,u_k)$. Let $w_0 \in \mathbb{U}$, u_0 and y_0 be two distinct unit vectors and put*

$$\mathcal{U} = I_p - \frac{(u_0 - y_0)(u_0 - y_0)^*}{1 - u_0^* y_0},$$

*$\mathbf{w}' = (w_0, w_1,\ldots,w_k)$, $\mathbf{u}' = (u_0, \mathcal{U}u_1,\ldots,\mathcal{U}u_k)$ and $\mathbf{y}' = (y_0, y_1,\ldots,y_k)$.
Then
(1) $\mathbf{y}'$ is a regular limit point in the chart defined by $(\mathbf{w}',\mathbf{u}')$ and $Q' = \mathcal{U}Q$ is given by*

$$Q' = \varphi^{-1}_{(\mathbf{w}',\mathbf{u}')}(\mathbf{y}').$$

Moreover,

$$\psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}(\mathbf{y}') = \psi^k_{(\mathbf{w},\mathbf{u})}(\mathbf{y}). \tag{14}$$

(2) if $\mathbf{y}$ is a local minimum of $\psi^k_{(\mathbf{w},\mathbf{u})}(\mathbf{y})$, the gradient of $\psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}$ at $\mathbf{y}'$ is orthogonal to the surface $\{(y_0,\ldots,y_n), \ \|y_0\| = 1, \ \|y_j\| < 1, \ j = 1,\ldots,n\}$ and points outwards.

4.1 The algorithm.

The algorithm searching for a local minimum at order n splits into five main operations:

A. *Choosing an initial point.*
This choice involves the choice of $(\mathbf{w},\mathbf{u})$ indexing a chart. The initial point may have degree less than or equal to the target order n. The choice of an initial point is crucial for our purpose. In many problems, we hope that some more information or engineering judgment could help us to select an initial point which ensures rapid convergence of the procedure to the global minimum. If this is not the case, since the L_2 approximation problem possesses many local minima and our final goal is to find the global one, more efficient strategies must be investigated. For instance, we can find all the local minima at order 1 and then, initialize our procedure at order 2, by replacing them on the boundary of $\mathcal{I}_2^p(1)$ as described in point E, choosing w_0, u_0 and y_0 in several ways, and so on, step by step, until the target order. This strategy gives rather good results [3].

B. *Minimizing at fixed order k.*
A software is used which integrates the vector field $-\mathrm{grad}\,\Psi^k_{(\mathbf{w},\mathbf{u})}$ from an initial point $\mathbf{y}_i \in \mathcal{B}_p^k$. One of

the following possibilities occurs:
(i) *a local minimum is reached.*
If $k = n$, we are done, while if $k < n$, this local minimum provides an initial point for searching for a minimum of order $k+1$, as described in point E.
(ii) *the norm of some Schur parameter tends to* 1.
Either a change of chart is necessary, or a boundary point of the manifold is reached. *More precisely, if the polynomial $\widetilde{q}_k$ nearly vanishes while its roots stay far from the unit circle, then the limit point belongs to $\mathcal{I}_k^p(1)$, and the first eventuality is true. In any other case, a boundary point is reached.*

C. *Meeting a boundary point.*
Such a boundary point, up to a unitary matrix, is an element Q_b of $\mathcal{I}_d^p(1)$ for some $d < k$, and the criterion at order k converges to $\Psi^d(Q_b)$. Then, a minimization process at order d can restart from Q_b. If only the first Schur parameter has norm 1, we can directly deduce from Lemma 1 some chart and Schur parameters for Q_b. Otherwise, the matrix Q_b must be computed and an adequate chart has to be provided.

D. *Choosing an adequate coordinate chart.*
The choice of a local chart at the neighborhood of a given point is an important and difficult task. The main purpose of using coordinates is to be able to perform calculations on a computer and as such it is desirable that the numerical conditioning of the chart is good. A criterion must be chosen to decide upon the quality of local coordinates around a point on a manifold. Moreover, a distortion occurs when mapping part of a manifold to Euclidean space, so that the sequence of improving estimates produced by an optimization algorithm is dependent on the choice of the chart, and it would be interesting to select the charts with the view to improve the convergence of the algorithm. But in this case, the selection strategy will depend upon the problem at hand and bring along a lot of 'overhead costs'. The present version of our algorithm uses a basic selection strategy, which minimizes the norm of the Schur parameters at each step of the Schur algorithm over a finite atlas. This point must be improved and is presently under study.

E. *Increasing the degree.*
When the minimization procedure leads to a local minimum of order $k < n$, say Q_m, then Lemma 1, for any choice of w_0, u_0 and $y_0 \neq u_0$, provides a boundary point Q'_m of $\mathcal{I}_{k+1}^p(1)$ together with a local parametrization $Q'_m = \varphi^{-1}_{(\mathbf{w}',\mathbf{u}')}(\mathbf{y}'_m)$, deduced from a local parametrization $\varphi^{-1}_{(\mathbf{w},\mathbf{u})}(\mathbf{y}_m)$ of Q_m and satisfying (14). Since $-\mathrm{grad}\,\Psi^{k+1}_{(\mathbf{w}',\mathbf{u}')}$ points inwards at $\mathbf{y}'$, this point can be used as an initial point for a minimization process at order $k+1$. $\square$

The point is that the value of the criterion, where the criterion must be understood as being Ψ^k when working at order k, decreases continuously, being conserved while the order changes, so that the minimization process pursues through different orders. Under some generic assumptions, it can be proved that critical points in $\mathcal{I}_k^p(1)$ are finite in number. Since the criterion decreases continuously, we never meet twice the same local minimum and *this ensures that the procedure eventually comes to an end.*

5 Conclusion

A rational approximation problem in L_2-norm has been studied. A new parametrization of the stable all-pass transfer functions has been used, based on the tangential Schur algorithm [1], allowing us to use classical optimization procedures. Using the state space approach, other parametrizations of the stable all-pass transfer functions are now available as the one obtained in [12], based on the work of Ober on balanced canonical forms [14], involving nice selections and Kronecker indices. It must be noticed that a link between the two approaches has been established in [15] generalizing the work of Peeters and Hanzon in [11]. It gives rise to balanced realizations and overlapping canonical forms directly in terms of the parameters used in the tangential Schur algorithm. This can be very useful and help us in the choice of a good chart.

The algorithm described here has been implemented by J. Grimm[1] to solve a problem provided by the French CNES: identify from frequency data a 2×2 hyperfrequency filter of order 8. Very good results have been obtained on this problem [3]. Another domain of application for this study could be the surface acoustic wave filters. As in geophysics (see [7]), in this technology Schur parameters appear as reflection coefficients and the synthesis of SAW filters could be approach by rational approximation.

[1]INRIA, BP 93, 06902 Sophia-Antipolis Cedex, FRANCE (Jose.Grimm@sophia.inria.fr)

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