Schur parametrizations and balanced realizations of real discrete-time stable all-pass systems.

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Abstract. We investigate the parametrization issue for real discrete-time stable all-pass multivariable systems by means of a real tangential Schur algorithm. A recursive construction of balanced realizations is associated with it, that possesses a very good numerical behavior. A chart selection strategy is given which makes use of the real Schur form of the dynamic matrix. Application to rational L^2-approximation is considered.

I. Introduction

Stable all-pass systems of fixed order have several applications in linear systems theory. Within the fields of system identification, approximation and model reduction, they have been used in connection with the Douglas-Shapiro-Shields factorization [2], to obtain effective algorithms for various purposes. In this respect, an essential issue is that of the parametrization of this class of systems. A "nice" parametrization is such that a small perturbation of the parameters preserves the stability and the order of the system and allows for the use of differential tools. In the multivariable case there is no global parametrization of this kind. However, the set of stable all-pass systems of fixed order possesses a manifold structure with its local parametrizations (charts) combined into an atlas which satisfy our requirements. Such parametrizations have been constructed in [1] by means of a tangential Schur algorithm. In each iteration step a linear fractional transformation is employed which is associated with a J-inner rational function of McMillan degree 1. In [7], a recursive construction of balanced realizations is associated with the Schur algorithm, that can be implemented as a product of unitary matrices and presents a very good numerical behavior. It combines the technical advantages of the Schur parametrizations to the practical ones of the state-space descriptions.

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The natural framework for these studies was that of complex functions. However, in most applications systems are real-valued and their transfer functions T are real, that is satisfy the relation $\overline{T(z)} = T(\bar{z})$. Even if the complex case includes the real case by restriction, a specific treatment is actually relevant. Indeed, the difficulty when dealing with atlases of charts, is to select a "good" chart or local parametrization for a given lossless function. In the complex case, the Potapov factorization of the function provides a chart in which all the Schur parameters are zero. The interpolation points of the Schur algorithm are then the poles of the lossless function. On the level of realizations, they are the eigenvalues of the dynamic matrix and the chart can be easily determined from a realization in Schur form. Of course, this very efficient strategy doesn't work in the real case. To solve this problem, we investigate here a *real Schur algorithm* which allows for iteration steps associated with J-inner rational function of McMillan degree 2, with complex conjugate poles. This algorithm also allows for a recursive construction of balanced realization with real entries.

II. Schur parametrizations of complex lossless functions.

Let

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}.$$

A function G is ($p \times p$)-lossless or stable all-pass, if and only if

$$G(z)G(z)^* \leq I_p, \quad |z| > 1,$$

with equality on the circle. We denote by $\mathcal{L}_n^p$ the set of ($p \times p$)-lossless functions of McMillan degree n and by $\mathcal{U}_p$ the set of constant unitary matrices.

For $G \in \mathcal{L}_n^p$, an interpolation condition is a relation of the form

$$G(1/\bar{w})u \;=\; v, \tag{1}$$

- $w \in \mathbb{C}, |w| < 1$,
- $u \in \mathbb{C}^p$, $\|u\| = 1$
- $v \in \mathbb{C}^p$, $\|v\| < 1$.

Remark. Note that $G(z)$ and w being given, some direction u can always be found, such that v given by (1) has norm *strictly less* than 1.

An interpolation condition being given, a $(p \times p)$ block-matrix function

$$\Theta(z) \;=\; \begin{pmatrix} \Theta_1(z) & \Theta_2(z) \\ \Theta_3(z) & \Theta_4(z) \end{pmatrix} \qquad (2)$$

can be defined, depending on w, u, v, such that the function $G(z)$ can be represented by the linear fractional transformation

$$G \;=\; (\Theta_4\, G_{n-1} + \Theta_3)\,(\Theta_2\, G_{n-1} + \Theta_1)^{-1}, \qquad (3)$$

for some *lossless function* $G_{n-1}(z)$ *of degree* $n-1$. The (J-unitary) matrix function $\Theta(z)$ is uniquely determined up to a constant (J-unitary) right factor H by

$$\Theta(z) = \left[I_{2p} + (z-1)\frac{1-|w|^2}{1-\|v\|^2} \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}^* J \right] H. \qquad (4)$$

The tangential Schur algorithm proceeds consist in repeating this process (see [1] or [4] for more details). A sequence of lossless functions $G_k(z)$ of degree k is constructed, G_k satisfying the interpolation condition

$$G_k(1/\bar{w}_k)u_k = v_k, \quad \|v_k\| < 1,$$

until G_0 which is a *constant unitary matrix*.

As in the scalar case, the interpolation values $v_n = v, v_{n-1}, \dots, v_1$, can be taken as parameters to describe the space $\mathcal{L}_n^p$. But in the matrix case, they only describe an open subset of the manifold. Associated with the sequences

$$\begin{aligned} \mathbf{w} &= (w_n = w, w_{n-1}, \dots, w_1), \\ \mathbf{u} &= (u_n = u, u_{n-1}, \dots, u_1), \end{aligned}$$

of interpolation points and interpolation directions, and with a chart $(\mathcal{W}, \psi)$ of $\mathcal{U}_p$, we define a chart $(\mathcal{V}, \varphi)$ by its domain

$$\mathcal{V} = \{G \in \mathcal{L}_n^p \;/\; \|G_k(1/\bar{w}_k)u_k\| < 1, G_0 \in \mathcal{W}\},$$

and its coordinate map :

$$\varphi \;:\; G \to (v_1, v_2, \dots, v_n, G_0).$$

The family $(\mathcal{V}, \varphi)$ defines a C^∞ atlas on $\mathcal{L}_n^p$.

In many application a parametrization of the quotient space $\mathcal{L}_n^p/\mathcal{U}_p$ is necessary. It is obtained by fixing in each chart the value of the last matrix G_0 obtained in the Schur algorithm.

III. Balanced realizations, Schur forms and adapted charts.

An important property of a lossless function is that it admits a balanced realization

$$G(z) = C(zI_n - A)^{-1}B + D,$$

such that the associated *realization matrix*

$$R = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \qquad (5)$$

is *unitary* (see [7] and the bibliography therein).

In [7] it is proved that for a particular choice of the factor H in $\Theta(z)$ (see (4)), a very simple transformation on realizations corresponds to the linear fractional transformation (10). Let R_{n-1} be a unitary realization matrix of $G_{n-1}(z)$, then a unitary realization matrix R_n of $G(z)$ is given by

$$R_n = \begin{bmatrix} V & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{n-1} \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I_{n-1} \end{bmatrix} \qquad (6)$$

where U and V are *unitary* $(p+1)\times(p+1)$ complex matrices depending on u, v, w as follows

$$\begin{aligned} U &= \begin{bmatrix} \xi u & I_p - (1+w\eta)uu^* \\ \overline{w}\eta & \xi u^* \end{bmatrix}, \\ V &= \begin{bmatrix} \xi v & I_p - (1-\eta)\frac{vv^*}{\|v\|^2} \\ \eta & -\xi v^* \end{bmatrix}, \end{aligned}$$

with

$$\xi = \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}, \quad \eta = \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}}.$$

In a chart associated with this particular Schur algorithm, a *unitary realization matrix* of the current lossless function G can be computed by iterating formula (6), which presents a very nice numerical behavior since it only involves *multiplications by unitary matrices*.

Observe that if the Schur parameter v is equal to zero, recursion (6) applied to a unitary realization matrix (5) gives

$$\left[\begin{array}{c|cc} \hat{D} & \sqrt{1-|w|^2}Du & C \\ \hline \sqrt{1-|w|^2}u^* & w & 0 \\ B(I_p - (1+\bar{w})uu^*) & \sqrt{1-|w|^2}Bu & A \end{array}\right] \qquad (7)$$

where $\hat{D} = D(I_p - (1+\bar{w})uu^*)$. This suggests the following strategy which leads to a chart in which all the Schur parameters are equal to the null vector and which corresponds to a Potapov factorization [8]. We start with a balanced realization (A_n, B_n, C_n, D_n) of the lossless function $G(z) \in \mathcal{L}_n^p$, in which A_n is lower triangular (Schur form). Let

$$A_n = \begin{bmatrix} w_n & 0\cdots 0 \\ \beta_n & A_{n-1} \end{bmatrix}, \quad B_n = \begin{bmatrix} b_n \\ \vdots \end{bmatrix},$$
$$C_n = \begin{bmatrix} c_n & C_{n-1} \end{bmatrix},$$

where w_n is a complex number, b_n a row vector of size p, and c_n a column vector of size p. Comparing with (7), we choose w_n as first interpolation point, $u_n = b_n^*/\|b_n\|$ and $v_n = 0$. The corresponding function G_{n-1}

has realization $(A_{n-1}, B_{n-1}, C_{n-1}, D_{n-1})$ which is still in Schur form. Repeating this process, we get a sequence of interpolation points $(w_n, \ldots, w_1)$, the eigenvalues of A_n, and a sequence of unit vectors $(u_n, \ldots, u_1)$ that index a chart in which G has Schur parameters $v_n = \ldots = v_1 = 0$.

IV. A Schur algorithm for real functions.

To deal with the case of real functions, we may specialize the Schur algorithm to real interpolations points and directions and allow for the Schur parameters only real values. This defines an atlas, but then the Schur form does not provide an adapted chart. To avoid this problem, it is necessary to consider a more general Schur algorithm which allows for steps of order two, in which the degree of the lossless function decreases by two.

We shall denote by $\mathcal{RL}_n^p$ the set of real $(p \times p)$-lossless of McMillan degree n and by $\mathcal{O}_p$ the set of $p \times p$ orthogonal matrices. Let $G \in \mathcal{RL}_n^p$ and consider a couple of interpolation conditions

$$\left\{ \begin{array}{lcl} G(1/\bar{w})u & = & v \\ G(1/w)\bar{u} & = & \bar{v}, \end{array} \right. \tag{8}$$

- $w \in \mathbb{C}, w \notin \mathbb{R}, |w| < 1$,
- $u \in \mathbb{C}^p, \ \|u\| = 1$,
- $v \in \mathbb{C}^p, \ \|v\| < 1$.

Note that G being real, the two interpolation conditions are equivalent.

Following [1], we associate to these interpolation conditions the J-inner function of McMillan degree two

$$\Theta(z) = \\ \left[I_{2p} + (z-1)\mathcal{C}(I_2 - z\mathcal{A})^{-1}\mathcal{P}^{-1}(I_2 - \mathcal{A})^{-*}\mathcal{C}^* J\right] H,$$

where

$$\mathcal{A} = \left[\begin{array}{cc} \bar{w} & 0 \\ 0 & w \end{array} \right], \quad w = \lambda + i\,\mu,$$

$$\mathcal{C} = \left[\begin{array}{cc} u & \bar{u} \\ v & \bar{v} \end{array} \right], \quad \|u\| = 1, \quad \|v\| < 1,$$

and $\mathcal{P}$ satisfies the equation

$$\mathcal{P} - \mathcal{A}^*\mathcal{P}\mathcal{A} = \mathcal{C}^* J \mathcal{C}.$$

The function $\Theta(z)$ is associated with the $\mathcal{H}(\Theta)$ space spanned by

$$f(z) = \frac{\left[\begin{array}{c} u \\ v \end{array} \right]}{(1-\bar{w}z)}, \quad \bar{f}(z) = \frac{\left[\begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right]}{(1-wz)},$$

which is a subspace of the Hardy space H^2 endowed with the J-inner product $< \ , J \ >_{H^2}$. It is well-known that $\mathcal{P}$ is the Gram matrix associated with this basis with respect to the J-inner product. It is thus given by

$$\mathcal{P} = \left[\begin{array}{cc} r & s \\ \bar{s} & r \end{array} \right],$$

$$r = \frac{1 - \|v\|^2}{1 - |w|^2}, \quad s = \frac{u^*\bar{u} - v^*\bar{v}}{1 - \bar{w}^2} = e^{i\psi}|s|, \tag{9}$$

and it is easily proved that $\mathcal{P}$ is strictly positive. This is the necessary condition to run a Schur step (see [1, prop.3.2]): the function $G(z)$ can be represented by the linear fractional transformation

$$G \ = \ (\Theta_4\, G_{n-2} + \Theta_3)\ (\Theta_2\, G_{n-2} + \Theta_1)^{-1}, \tag{10}$$

for some *lossless function $G_{n-2}(z)$ of degree $n-2$.*

Moreover, if we assume H is real, then Θ is real as can be seen from the following form

$$\Theta(z) = \\ \left[I_{2p} + (z-1)\mathcal{C}_r(I_2 - z\mathcal{A}_r)^{-1}\mathcal{P}_r^{-1}(I_2 - \mathcal{A}_r)^{-*}\mathcal{C}_r\right] H, \tag{11}$$

obtained by the transformation

$$\begin{array}{lcl} \mathcal{A}_r & = & S\mathcal{A}S^*, \\ \mathcal{C}_r & = & \mathcal{C}S^*, \\ \mathcal{P}_r & = & S\mathcal{P}S^*, \end{array}$$

where S is the unitary transformation

$$S = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} 1 & 1 \\ -i & i \end{array} \right] \left[\begin{array}{cc} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{array} \right],$$

where ψ is the argument of s: $s = e^{i\psi}|s|$. Then

$$\mathcal{A}_r = \left[\begin{array}{cc} \lambda & \mu \\ -\mu & \lambda \end{array} \right],$$

and

$$\mathcal{P}_r = \left[\begin{array}{cc} r + |s| & 0 \\ 0 & r - |s| \end{array} \right].$$

The *real Schur algorithm* consists in the construction of a sequence of real lossless functions of decreasing degree using either a Schur step of degree 1 from a real interpolation condition, or a Schur step of degree 2 from two complex conjugate interpolation conditions.

An atlas can be describe as follows. Each chart will be associated with a chart $(\mathcal{W}, \psi)$ of $\mathcal{O}_p$, a sequence of interpolation points and a sequence of associated directions :

$$\begin{array}{lcl} \mathbf{w} & = & (w = w_{m+l}, w_{m+l-1}, \ldots, w_1), \\ \mathbf{u} & = & (u = u_{m+l}, u_{m+l-1}, \ldots, u_1), \end{array}$$

where m is the number of $w_k \in \mathbb{R}$, l the number of $w_k \notin \mathbb{R}$ and $m + 2l = n$. A chart $(\mathcal{V}, \varphi)$ is defined by its domain

$$\mathcal{V} = \{G \in \mathcal{RL}_n^p \ / \ \|G_k(1/\bar{w}_k)u_k\| < 1, G_0 \in \mathcal{W}\},$$

and its coordinate map :

$$\varphi \ : \ G \rightarrow (v_1, v_2, \ldots, v_{m+l}, G_0).$$

Then, the family $(\mathcal{V}, \varphi)$ defines a C^∞ atlas on $\mathcal{RL}_n^p$. As previously, the quotient space is described by fixing in each chart the last lossless function G_0.

Remark. Note that the Schur parameters v_k's are not all of the same nature : if $w_k \in \mathbb{R}$, then $v_k \in \mathbb{R}^p$, while if $w_k \notin \mathbb{R}$, then $v_k \in \mathbb{C}^p$ and contains $2p$ real parameters. This gives $mp + 2lp = np$ real parameters which is precisely the dimension of the manifold $\mathcal{RL}_n^p/\mathcal{O}_p$.

V. Second order recursions on balanced realizations.

Now we would like to be able to choose H in (11) so that a Schur step of degree 2 in the real Schur algorithm corresponds to a recursion on unitary matrix realizations of the form

$$R_n = \begin{bmatrix} V & 0 \\ 0 & I_{n-2} \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & R_{n-2} \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I_{n-2} \end{bmatrix}, \tag{12}$$

where U and V are *unitary* $(p+2)\times(p+2)$ complex matrices depending on u, v, w.

Recursion (12) does not depend on the choice of a minimal realization and defines a mapping

$$\tilde{G} = \mathcal{F}_{U,V}(G)$$

on proper rational matrix functions as follows (see [7])

$$\widetilde{G}(z) = F_1(z) + F_2(z)(zI_k - F_4(z))^{-1}F_3(z), \tag{13}$$

with $F_1(z)$ of size $p\times p$, $F_2(z)$ of size $p\times 2$, $F_3(z)$ of size $2\times p$ and $F_4(z)$ of size 2×2 defined by:

$$F(z) = \begin{bmatrix} F_1(z) & F_2(z) \\ F_3(z) & F_4(z) \end{bmatrix} = V \begin{bmatrix} I_2 & 0 \\ 0 & G(z) \end{bmatrix} U^*. \tag{14}$$

Let the unitary $(p+2)\times(p+2)$ matrices U and V be partitioned as

$$U = \begin{bmatrix} \alpha_U & M_U \\ k_U & \beta_U^* \end{bmatrix}, \quad V = \begin{bmatrix} \alpha_V & M_V \\ k_V & \beta_V^* \end{bmatrix},$$

with k_U and k_V are (2×2) matrices. The mapping $\mathcal{F}_{U,V}$ can be written as a linear fractional transformation

$$\mathcal{F}_{U,V}(G) = \mathcal{T}_\Phi(G),$$

for Φ given by

$$\Phi(z) = \left[I_{2p} + (z-1)\,\alpha(k_V - z\,k_U)^{-1}(k_V - k_U)^{-*}\alpha^* J\right] K$$

with

$$K = M + \alpha(k_V - k_U)^{-1}\beta^* J, \tag{15}$$

where

$$M = \begin{bmatrix} M_U & 0 \\ 0 & M_V \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_U \\ \alpha_V \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_U \\ \beta_V \end{bmatrix}.$$

Now, the question is : can we choose the blocks in U and V and the left J-unitary matrix H in (11) so that Θ and Φ coincides? This is only possible if (up to an orthogonal transformation)

$$\left\{ \begin{array}{rcccl} k_U k_V^{-1} & = & \mathcal{P}_r^{1/2}\mathcal{A}_r\mathcal{P}_r^{-1/2} & = & W^* \\ \alpha\, k_V^{-1} & = & \mathcal{C}_r\mathcal{P}_r^{-1/2} & = & \begin{bmatrix} X \\ Y \end{bmatrix}. \end{array} \right. \tag{16}$$

The matrix V being unitary, we have

$$Y^*Y + I_2 = k_V^{-*}k_V^{-1}.$$

The matrix $Y^*Y + I_2$ being positive definite, it admits a unique positive square root (see [3]) $(I_2 + Y^*Y)^{1/2}$ which is symmetric, and k_V is given (up to an orthogonal matrix on the right) by

$$k_V = (I_2 + Y^*Y)^{-1/2}.$$

Thus,

$$\alpha = \begin{bmatrix} X \\ Y \end{bmatrix} (I_2 + Y^*Y)^{-1/2}$$

Furthermore,

$$M_V M_V^* = I_p - Y(I_2 + Y^*Y)^{-1}Y^*,$$

and it is easily seen that

$$I_p - Y(I_2 + Y^*Y)^{-1}Y^* = (I_p + YY^*)^{-1}$$

and is positive definite. Thus we can take

$$M_V = (I_p + YY^*)^{-1/2},$$

and finally,

$$\beta_V^* = -Y^*(I_p + YY^*)^{-1/2},$$

so that

$$V = \begin{bmatrix} Y(I_2 + Y^*Y)^{-1/2} & (I_p + YY^*)^{-1/2} \\ (I_2 + Y^*Y)^{-1/2} & -Y^*(I_p + YY^*)^{-1/2} \end{bmatrix}.$$

Now, by (16)

$$\begin{array}{rcl} \alpha_U & = & X((I_2 + Y^*Y)^{-1/2} \\ k_U & = & W^*((I_2 + Y^*Y)^{-1/2} \end{array}$$

Exploiting the unitarity of U, we obtain

$$U = \begin{bmatrix} X(I_2 + Y^*Y)^{-1/2} & M_U \\ W^*(I_2 + Y^*Y)^{-1/2} & -W^{-1}X^*M_U \end{bmatrix},$$

with

$$M_U = (I_p - X(I_p + Y^*Y)^{-1}X^*)^{1/2}.$$

The matrices X, Y, and W can be computed from u, v, and w by the formulas

$$X = \begin{bmatrix} \frac{e^{i\psi/2}u + e^{-i\psi/2}\bar{u}}{\sqrt{2}\sqrt{r+|s|}} & \frac{i(e^{i\psi/2}u - e^{-i\psi/2}\bar{u})}{\sqrt{2}\sqrt{r-|s|}} \end{bmatrix} \tag{17}$$

$$Y = \begin{bmatrix} \frac{e^{i\psi/2}v + e^{-i\psi/2}\bar{v}}{\sqrt{2}\sqrt{r+|s|}} & \frac{i(e^{i\psi/2}v - e^{-i\psi/2}\bar{v})}{\sqrt{2}\sqrt{r-|s|}} \end{bmatrix} \tag{18}$$

$$W = \left[\begin{array}{cc} \lambda & -\frac{\sqrt{r-|s|}}{\sqrt{r+|s|}}\mu \\ \frac{\sqrt{r+|s|}}{\sqrt{r-|s|}}\mu & \lambda \end{array} \right], \qquad (19)$$

where r, s and ψ are given by (9).
Now, if we choose $H = K$, where K is given by (15), then Θ and Φ coincides as required. A recursive construction of balanced realization can be associated with the real Schur algorithm.

VI. REAL SCHUR FORM AND ADAPTED CHARTS.

If the vector v in the couple of interpolation conditions (8) is zero, then Y is equal to zero too and recursion (12) applied to an unitary realization matrix of the form (5) gives

$$\left[\begin{array}{c|cc} DM_U & D\beta_U & C \\ \hline X^* & W & 0 \\ BM_U^* & B\beta_U & A \end{array} \right].$$

To find an adapted chart for a given lossless function G, we shall proceed as follows: we start from a balanced realization (A_n, B_n, C_n, D_n) of the lossless function $G(z) \in \mathcal{RL}_n^p$ in which A_n is in real Schur form

$$A_n = \left[\begin{array}{cccc} W_1 & 0 & \cdots & 0 \\ \star & W_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & W_k \end{array} \right],$$

where W_i is either a real number or a (2×2) block with complex conjugate eigenvalues. If W_i is a (2×2) block, we shall impose its form

$$\left[\begin{array}{cc} \lambda & \mu_1 \\ \mu_2 & \lambda \end{array} \right], \quad \mu_1 > 0, \quad |\mu_1| < |\mu_2|.$$

This ensures uniqueness of the real Schur form up to a permutation of the diagonal blocks.
If W_1 is real, we proceed as in section III. If it is a (2×2) block, the interpolation point w and direction u of the corresponding Schur step of degree 2 can be determined by solving

$$\left\{ \begin{array}{lcl} W & = & W_1 \\ X & = & X_1 \end{array} \right.$$

where X_1^* is the matrix consisting in the two first rows of B_n, and W and X are given by (19) and (17) in which v is zero. Then, running the second order Schur step provides a new realization matrix of order $n-2$ still in real Schur form. Then we can continue and determine the sequences $\mathbf{w}$ and $\mathbf{u}$ of a chart in which G as all its Schur parameters equal to the null vector.

VII. APPLICATION TO RATIONAL APPROXIMATION.

In [6], the parametrization described in section II and III was used to compute a best *stable* rational L^2 approximation of specified order to a given *multivariable* transfer function. The fact that this parametrization takes into account the stability constraint and possesses a good numerical behavior makes possible the use of constrained optimization techniques to find local minima. Moreover it provides a model in state-space form, which is very useful in practice. This approach was demonstrated on several numerical examples coming from system identification or model reduction.
However, in most applications, systems are real-valued, and even if they can be handled by using complex rational approximation, this is not satisfactory. The number of parameters is unnecessarily doubled, introducing extra-errors. Moreover, a ”real” system may have a complex global minimum. For example, the function

$$f(z) = \frac{1 - z^2}{z^3},$$

admits tree minima: a real and two complex ones, which realize the best relative error.
For all these reasons, a specific treatment for real systems is desirable and is provided by the parametrization of real stable all-pass systems presented in this paper. Some numerical simulations will be presented
- a MIMO model reduction problem : the automobile gas turbine model with 2 inputs, 2 outputs and 12 states, given by a realization in [5, p.168], and already considered in [6].
- a scalar function known to be hardly approximated, the scalar function $f(z) = (1 - z)^{1/2}$. It can be shown that the poles of its rational approximants tends to the unit circle which leads to some numerical difficulties.

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