Matrix Rational H^2 Approximation:

a State-Space Approach using Schur Parameters

J.P. Marmorat, M. Olivi, B. Hanzon, R. Peeters

Sophia–Antipolis, France

Introduction

We present a method to compute a stable rational L^2-approximation of specified order n to a given multivariable transfer function.

- it works for multivariable systems,
- it uses a nice parametrization of stable allpass systems, which
  - takes into account the stability constraint
  - ensures identifiability
  - is well-conditionned
- it uses a recursive search on the degree which improves the chances to reach the global minimum.

The L^2-criterion in state-space form

- $F(z) = \mathcal{C}(zI_N - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$, $m \times p$ given transfer function
- $H(z) = \gamma(zI_n - A)^{-1}B + \mathcal{D}$, approximant at order n

  (A, B) input-normal pair: $AA^* + BB^* = I$.

L^2-norm of the error $F - H$:

$$J(A, B) = \|F\|^2 - \mathrm{Tr}\left(\gamma\gamma^*\right),$$

where

$$\gamma = \mathcal{C}W,$$

and W solution to the Lyapunov equation:

$$\mathcal{A}WA^* + \mathcal{B}B^* = W.$$

Optimization set: stable-allpass systems

The two following sets (equivalence classes) are diffeomorphic:

- input-normal pairs (A, B)
- stable allpass functions $G(z) = D + C(zI_n - A)^{-1}B$
  up to a left constant unitary factor
  $$\begin{bmatrix} D & C \\ B & A \end{bmatrix}$$ unitary matrix.

Alpay, Baratchart, Gombani [1];

Parametrization issue

Desirable properties:

- ensures identifiability
- a small perturbation of the parameters preserves the stability and the order of the system?
- allows for the use of differential tools.

$\longrightarrow$ Differentiable manifold.

Atlas of charts or overlapping canonical forms :

a collection of local parametrizations with compatibility conditions

(changes of charts are smooth).

Atlases of charts

Two families can be found in the litterature

1. from a tangential Schur algorithm:

$$G_n(1/\bar{w})u = v, \|v\| < 1, \quad G_n \overset{LFT}{\Longrightarrow} G_{n-1}$$

$$G_n, \ldots, G_k \overset{(w_k, u_k, v_k)}{\Longrightarrow} G_{k-1}, \ldots, G_0.$$

   Alpay, Baratchart, Gombani [1]; Fulcheri, Olivi [2]

2. from state-space representations

   Hanzon, Ober [3]

A parametrization that combines the two approaches:

Peeters, Hanzon, Olivi [4]

Encoding stable-allpass systems

A $p \times p$ stable allpass systems of degree n is encoded by

- $w_1, w_2, \dots, w_n$ points of the unit circle,
- $u_1, u_2, \dots, u_n$ unit complex p-vectors,
- $v_1, v_2, \dots, v_n$ complex p-vectors, $\|v_i\| < 1$.

The w_i's and the u_i's define the chart while the v_i's are the Schur parameters of the system in the chart. In a given chart, a system is perfectly determined by its Schur parameters (identifiability).

Encoding stable-allpass systems (2)

The stable allpass system encoded in that way has unitary realization matrix

$$\begin{bmatrix} D_n & C_n \\ B_n & A_n \end{bmatrix}$$

computed by induction

$$\begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix} = \begin{bmatrix} V_k & 0 \\ 0 & I_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & D_{k-1} & C_{k-1} \\ 0 & B_{k-1} & A_{k-1} \end{bmatrix} \begin{bmatrix} U_k^* & 0 \\ 0 & I_{k-1} \end{bmatrix},$$

where A_k is $k \times k$, D_k is $p \times p$, and $D_0 = I_p$.

$\longrightarrow$ very nice numerical behavior

Encoding stable-allpass systems (3)

U_k and V_k are the $(p+1) \times (p+1)$ unitary matrices:

$$U_k = \begin{bmatrix} \xi_k u_k & I_p - (1 + \eta_k w_k) u_k u_k^* \\ \eta_k \overline{w_k} & \xi_k u_k^* \end{bmatrix}$$

$$V_k = \begin{bmatrix} \xi_k v_k & I_p - (1 - \eta_k) \frac{v_k v_k^*}{\|v_k\|^2} \\ \eta_k & -\xi_k v_k^* \end{bmatrix}$$

$$\xi_k = \frac{\sqrt{1 - |w_k|^2}}{\sqrt{1 - |w_k|^2 \|v_k\|^2}}, \quad \eta_k = \frac{\sqrt{1 - \|v_k\|^2}}{\sqrt{1 - |w_k|^2 \|v_k\|^2}}$$

Main steps of the algorithm

- finding an adapted chart:
  realization in Schur form (A lower triangular)

$$A = \begin{bmatrix} w_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & w_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{1-|w_n|^2}\, u_n^* \\ \vdots \end{bmatrix}$$

  $\rightarrow v_1 = v_2 = \ldots v_n = 0$ (Potapov factorization)

- optimization over the manifold

- recursive search on the degree (optional):
  minimum of degree $k \rightarrow$ initial point of degree $k+1$ (error preserved)

The RARL2 software

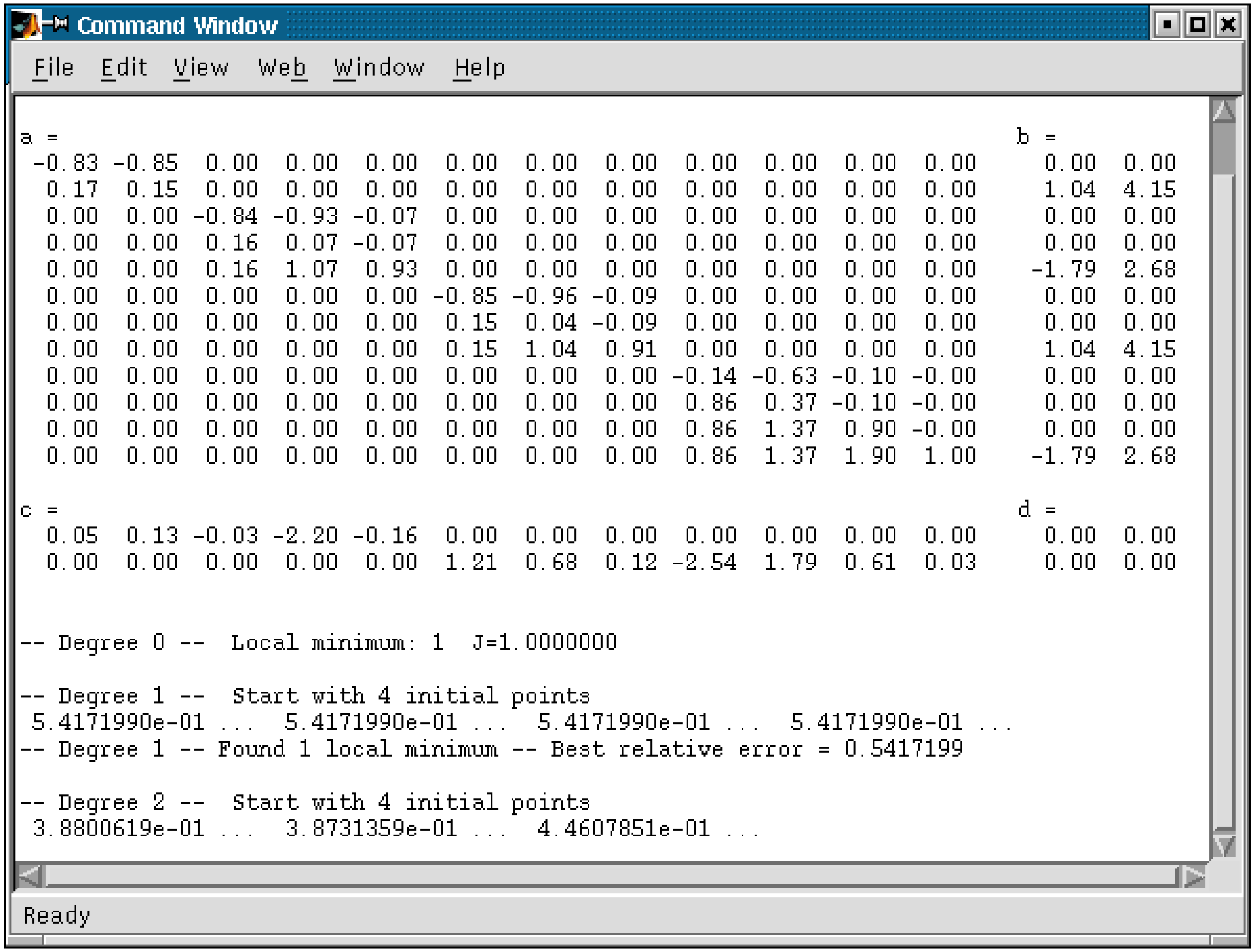
This software computes a stable rational L^2-approximation of specified order n to a multivariable transfer function given in one of the following forms:

- a realization
- a finite number of Fourier coefficients
- some pointwise values on the unit circle.

It has been implemented using standard MATLAB subroutines. The optimizer of the toolkit OPTIM is used to find a local minimum, given by a realization, of the nonlinear L^2-criterion.

`http://www-sop.inria.fr/miaou/Martine.Olivi/me.html`

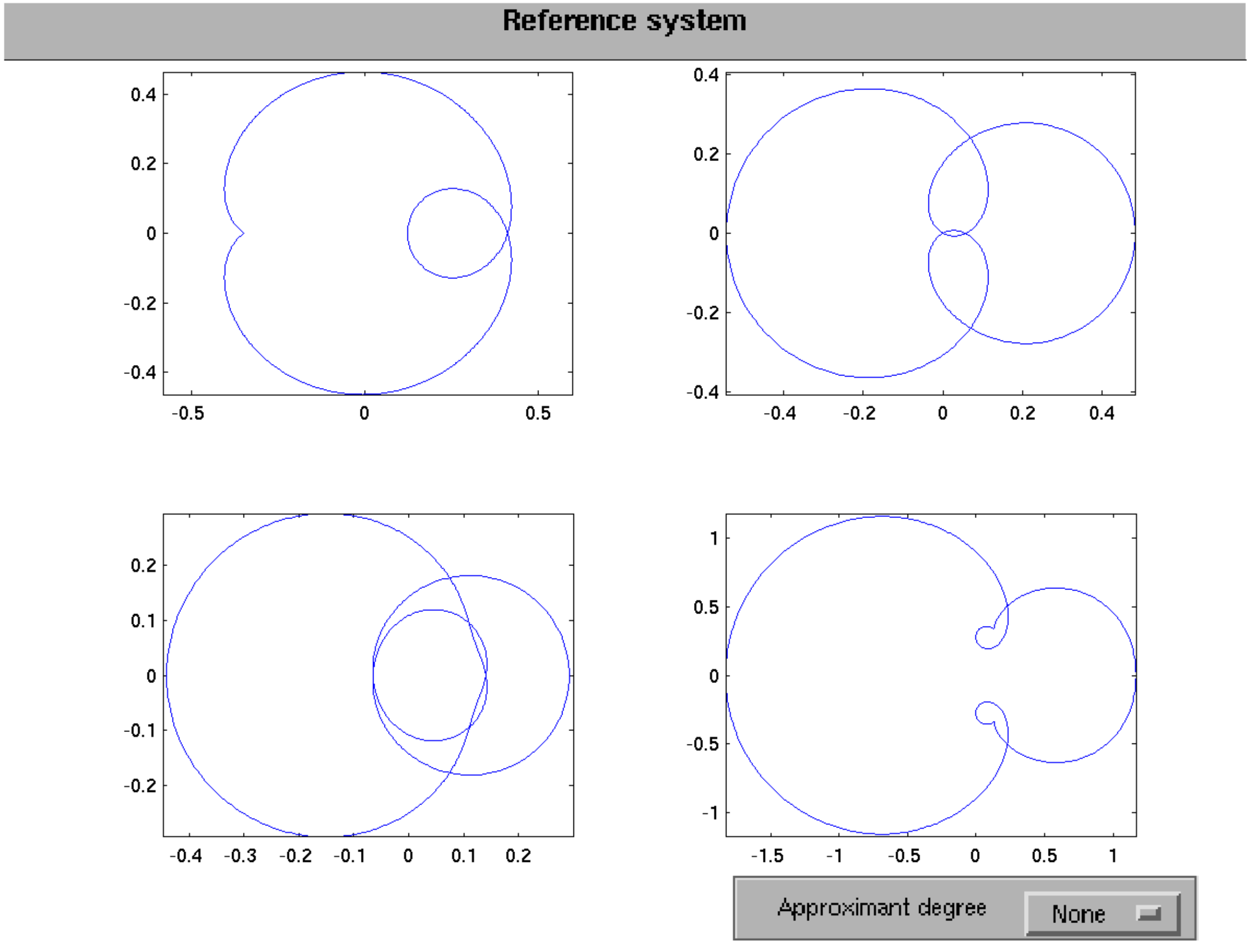
Automobile gas turbine



```
Command Window
File  Edit  View  Web  Window  Help

a =                                                                          b =
 -0.83 -0.85  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00       0.00  0.00
  0.17  0.15  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00       1.04  4.15
  0.00  0.00 -0.84 -0.93 -0.07  0.00  0.00  0.00  0.00  0.00  0.00  0.00       0.00  0.00
  0.00  0.00  0.16  0.07 -0.07  0.00  0.00  0.00  0.00  0.00  0.00  0.00       0.00  0.00
  0.00  0.00  0.16  1.07  0.93  0.00  0.00  0.00  0.00  0.00  0.00  0.00      -1.79  2.68
  0.00  0.00  0.00  0.00  0.00 -0.85 -0.96 -0.09  0.00  0.00  0.00  0.00       0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.15  0.04 -0.09  0.00  0.00  0.00  0.00       0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.15  1.04  0.91  0.00  0.00  0.00  0.00       1.04  4.15
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 -0.14 -0.63 -0.10 -0.00       0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.86  0.37 -0.10 -0.00       0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.86  1.37  0.90 -0.00       0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.86  1.37  1.90  1.00      -1.79  2.68

c =                                                                          d =
  0.05  0.13 -0.03 -2.20 -0.16  0.00  0.00  0.00  0.00  0.00  0.00  0.00       0.00  0.00
  0.00  0.00  0.00  0.00  0.00  1.21  0.68  0.12 -2.54  1.79  0.61  0.03       0.00  0.00


-- Degree 0 --  Local minimum: 1  J=1.0000000

-- Degree 1 --  Start with 4 initial points
 5.4171990e-01 ...  5.4171990e-01 ...  5.4171990e-01 ...  5.4171990e-01 ...
-- Degree 1 -- Found 1 local minimum -- Best relative error = 0.5417199

-- Degree 2 --  Start with 4 initial points
 3.8800619e-01 ...  3.8731359e-01 ...  4.4607851e-01 ...

Ready
```

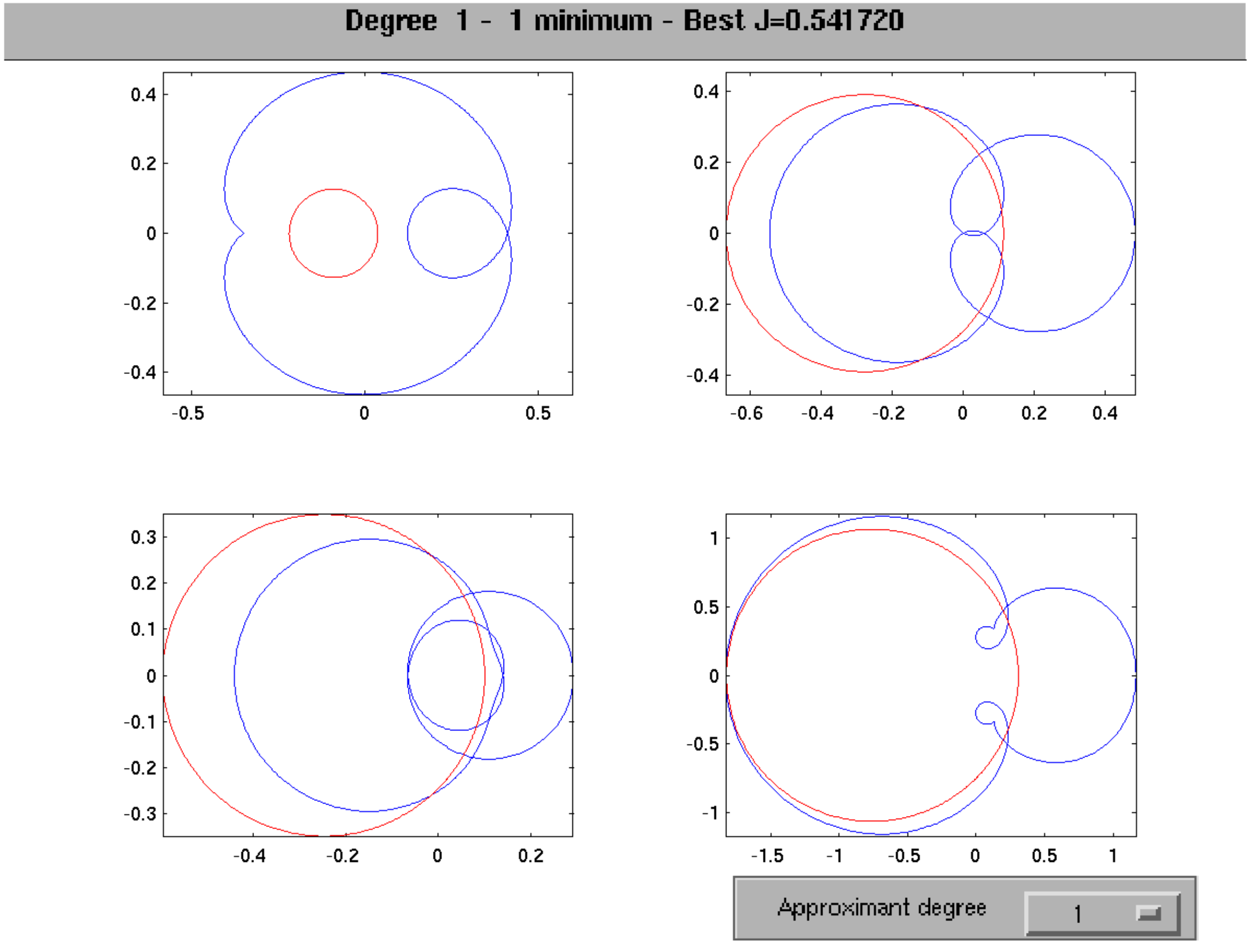
Hung, MacFarlane [6]; Glover [5]; Yan, Lam [7]

Nyquist diagrams

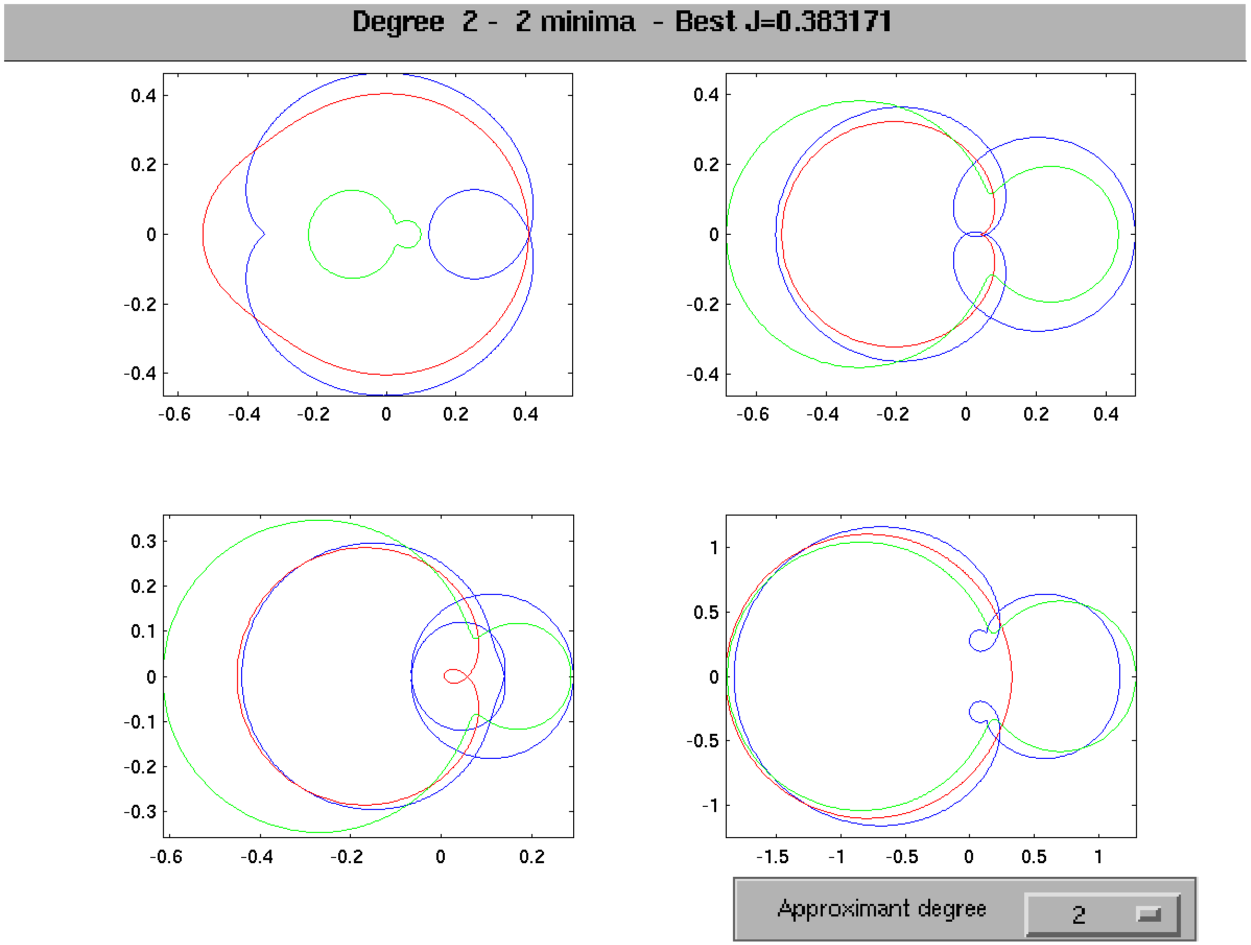


2×2; order 12.

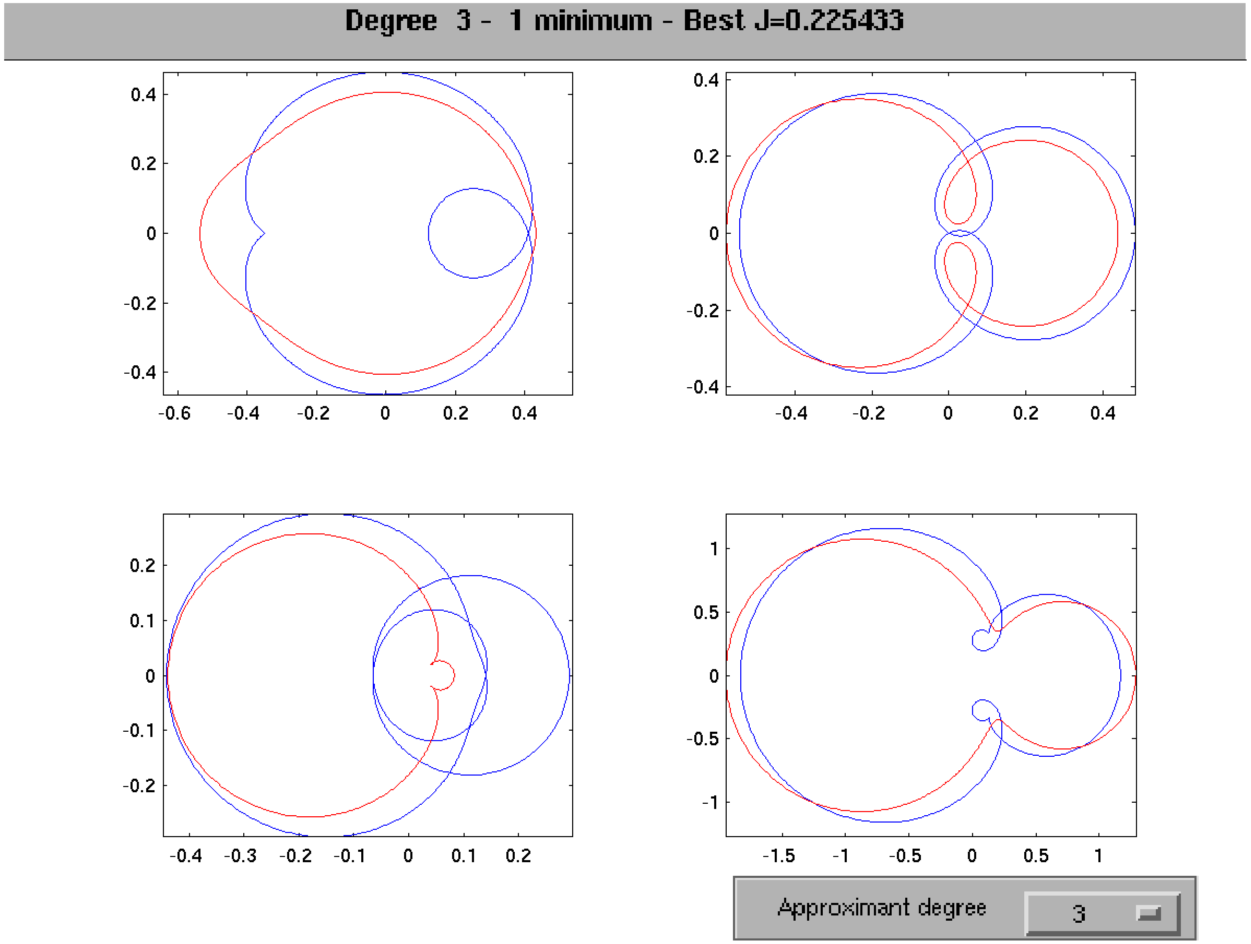
Approximants: order 1



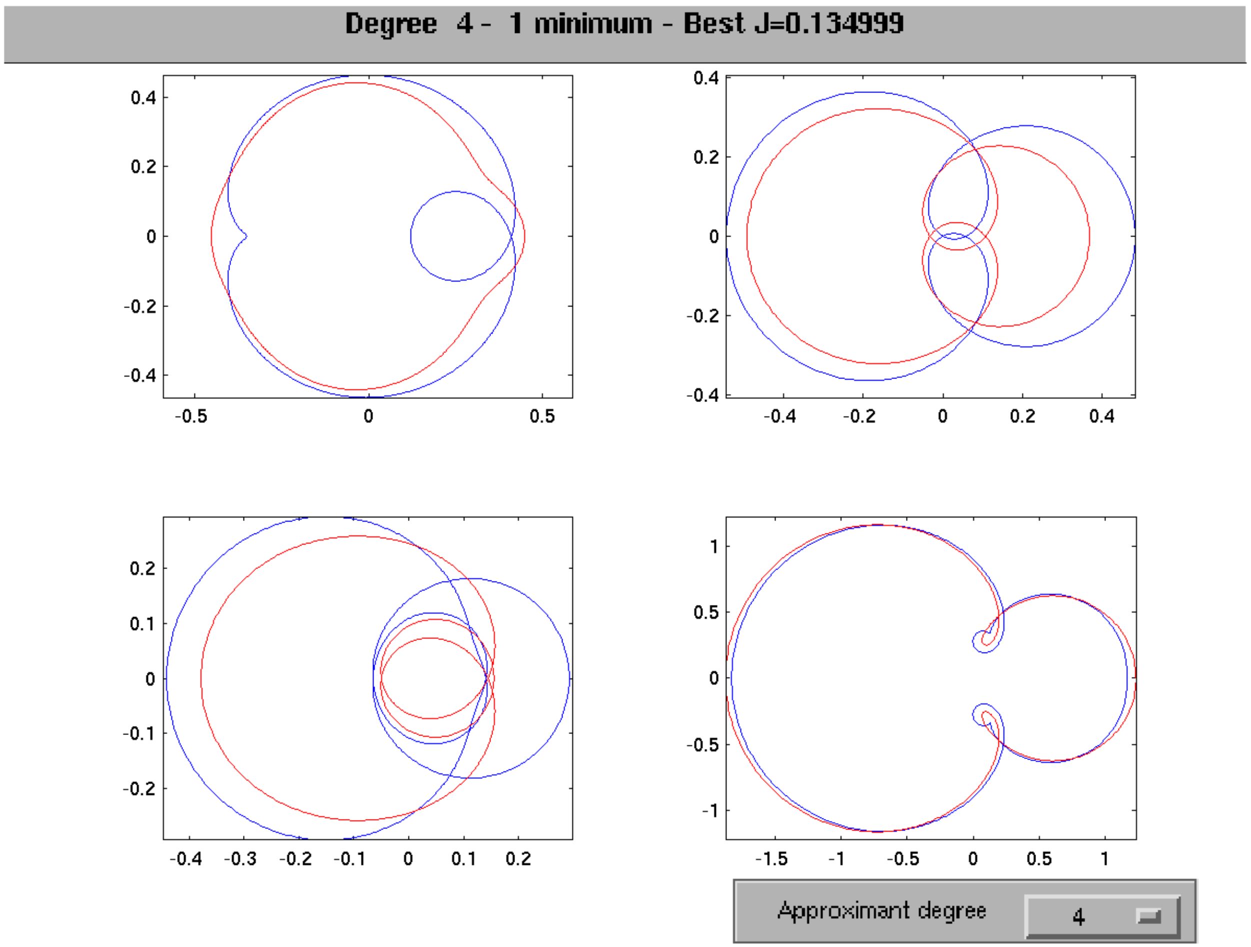
Approximants: order 2



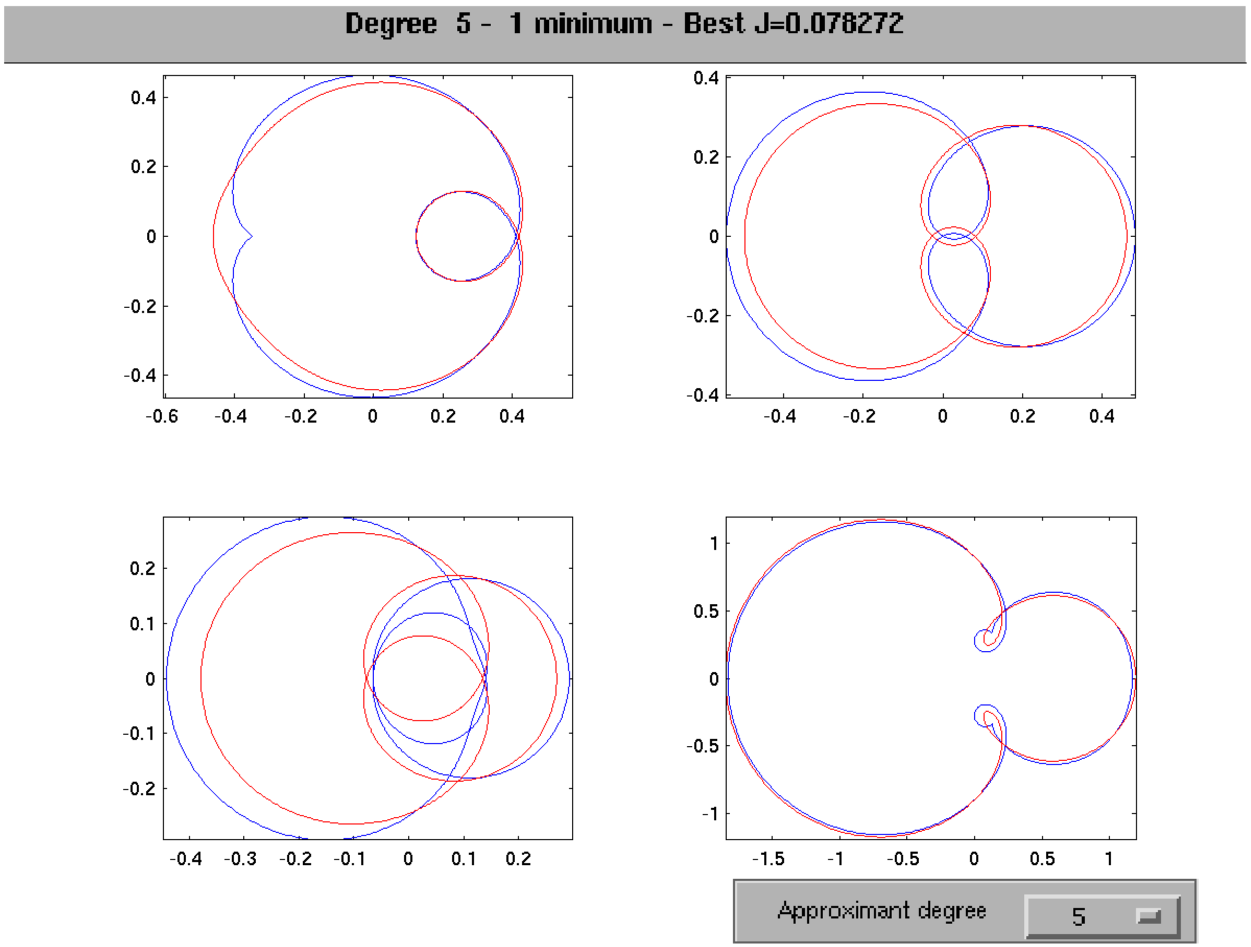
Approximants: order 3



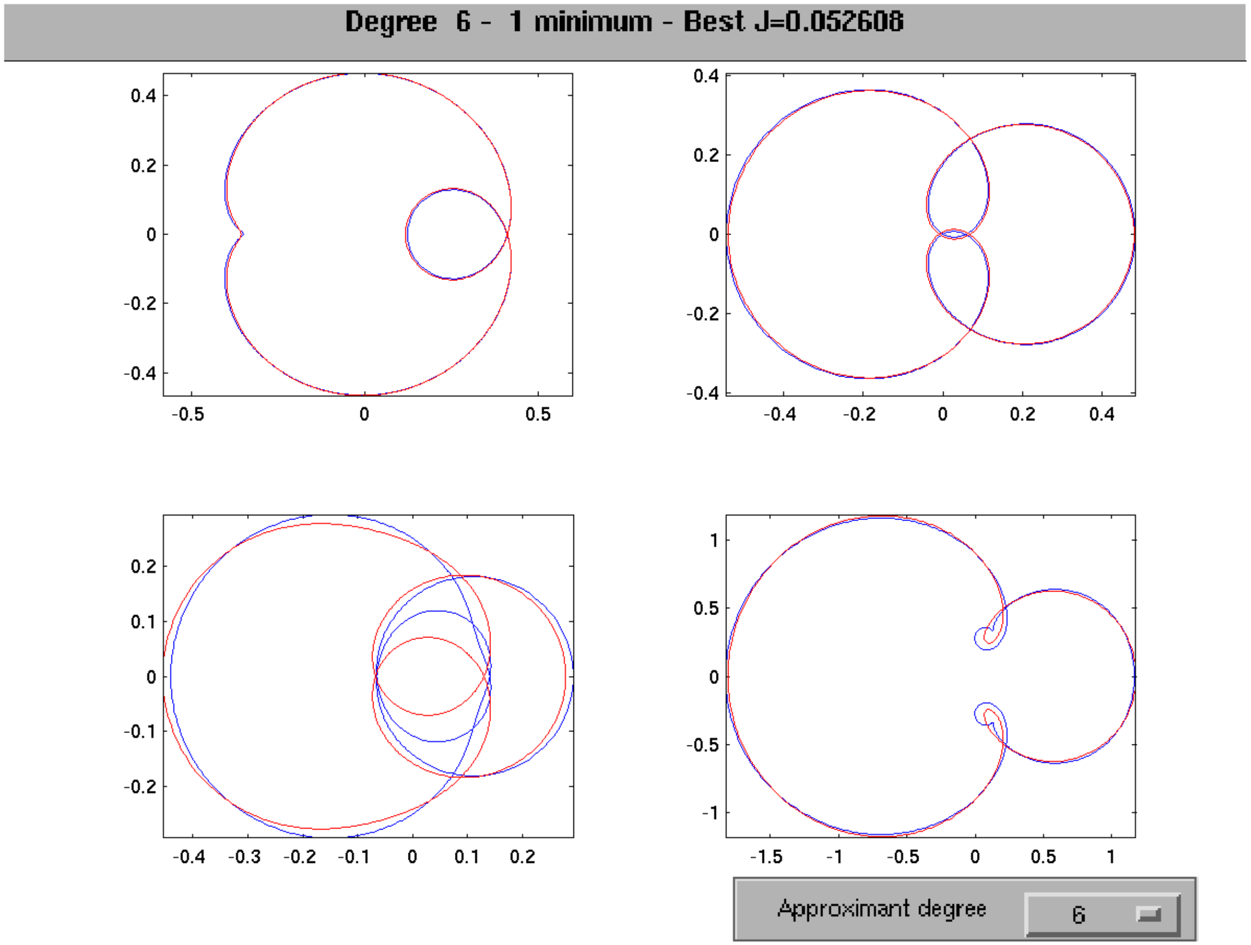
Approximants: order 4



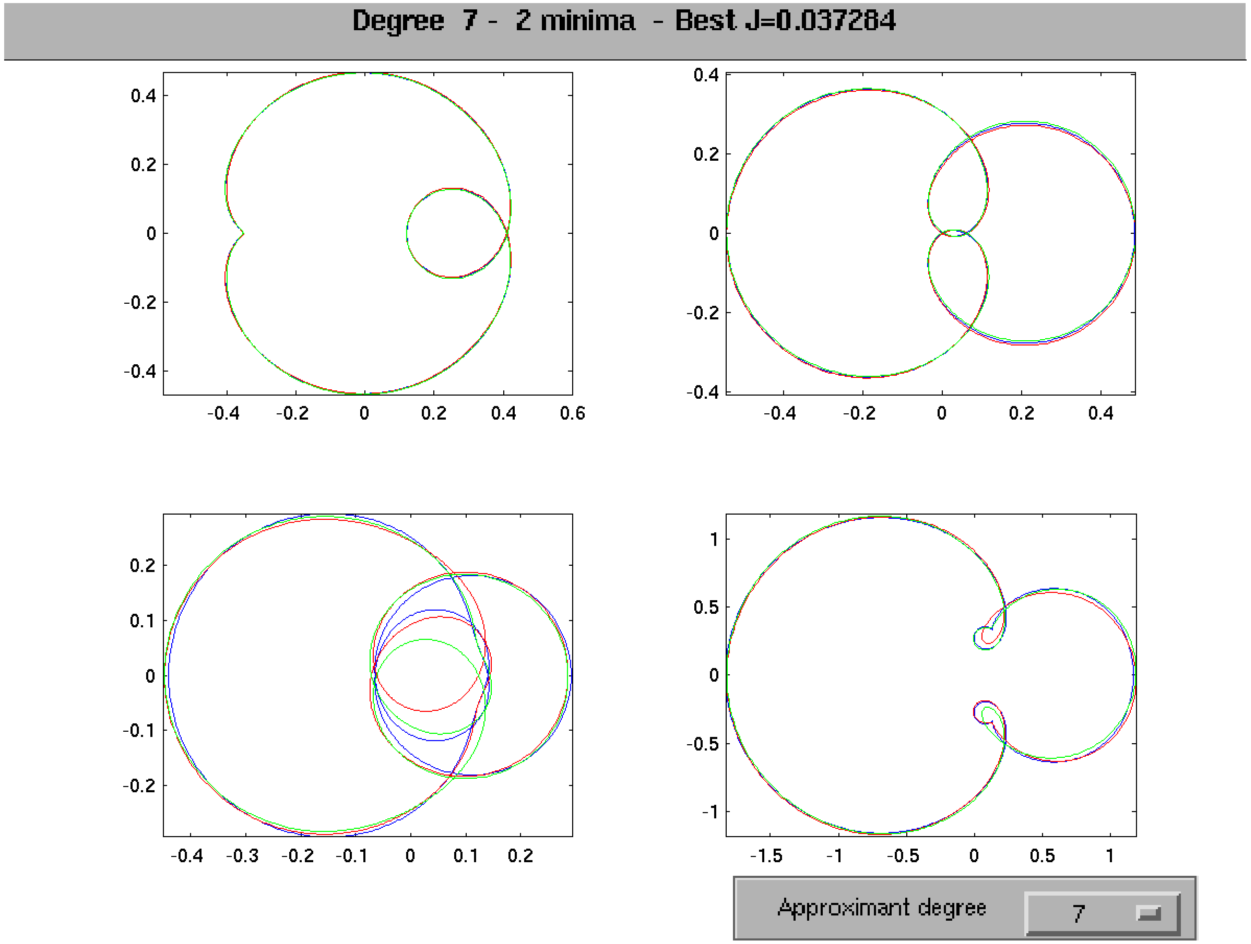
Approximants: order 5



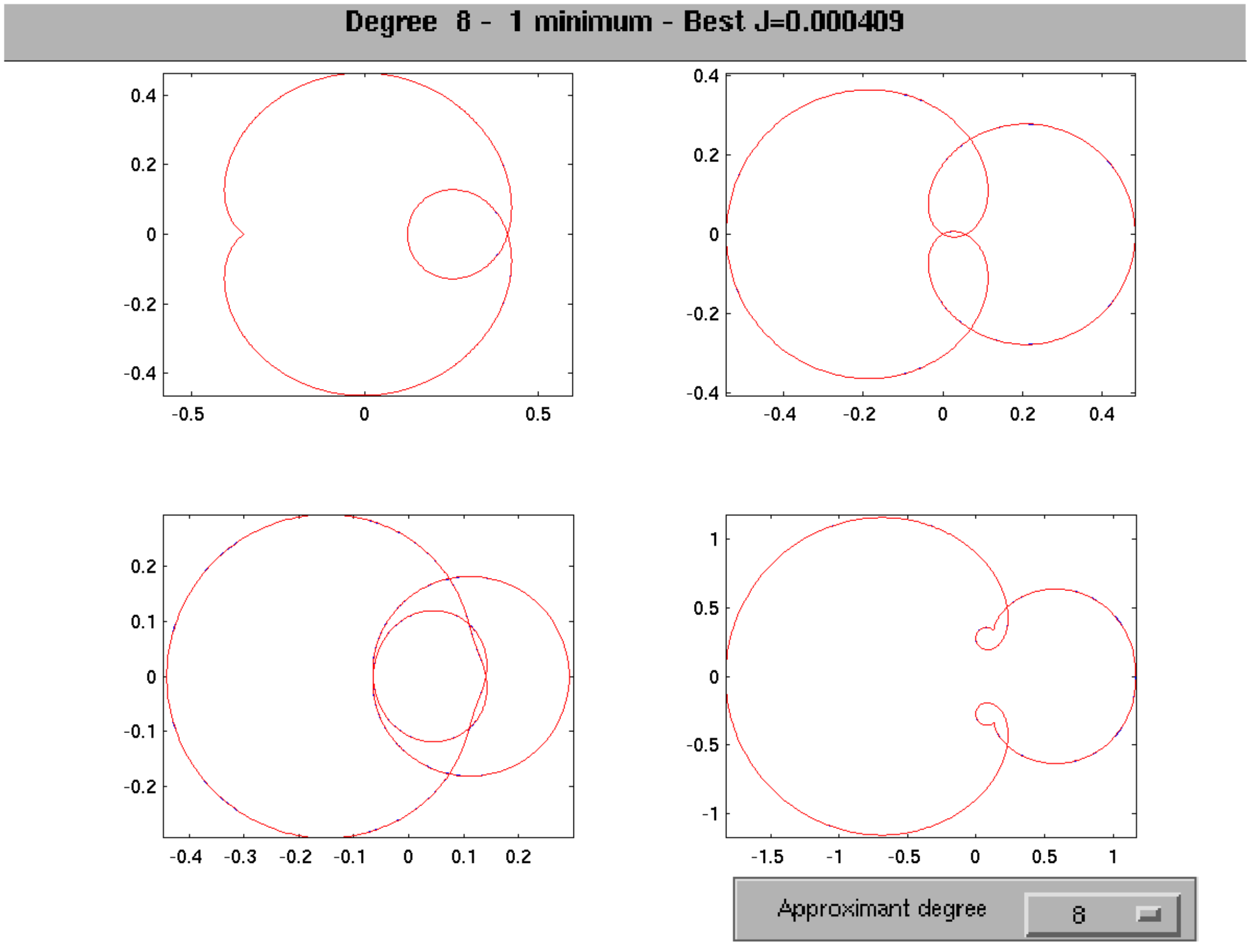
Approximants: order 6



Approximants: order 7



Approximants: order 8



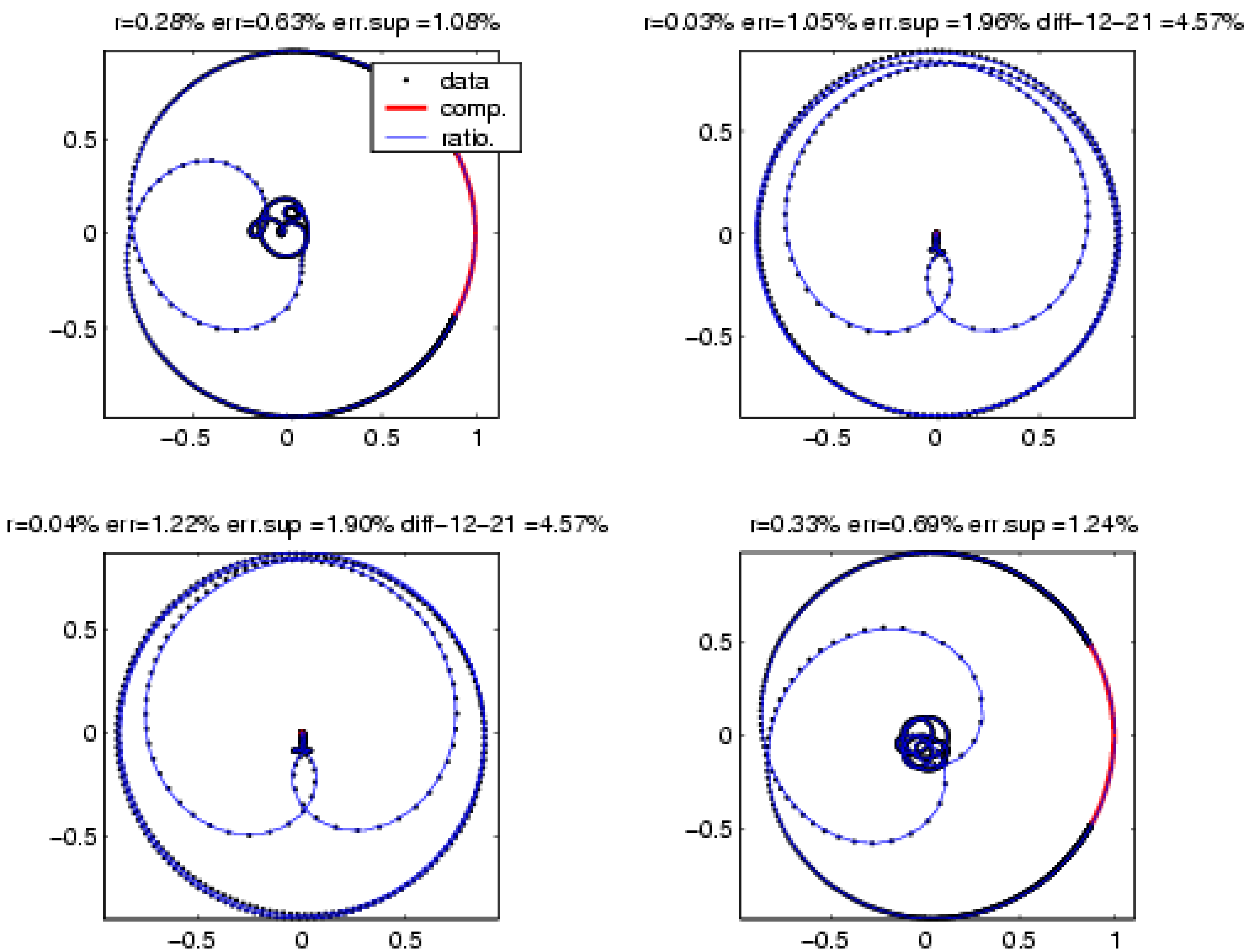
Hyperfrequency Filter

The problem: find a 8th order model of a MIMO (2×2) hyperfrequency filter, from experimental pointwise values in some range of frequencies provided by the CNES (French space agency).

First stage (interpolation/completion): compute a stable matrix transfer function of high order which approximates these data, given by a great number (800) of Fourier coefficients.

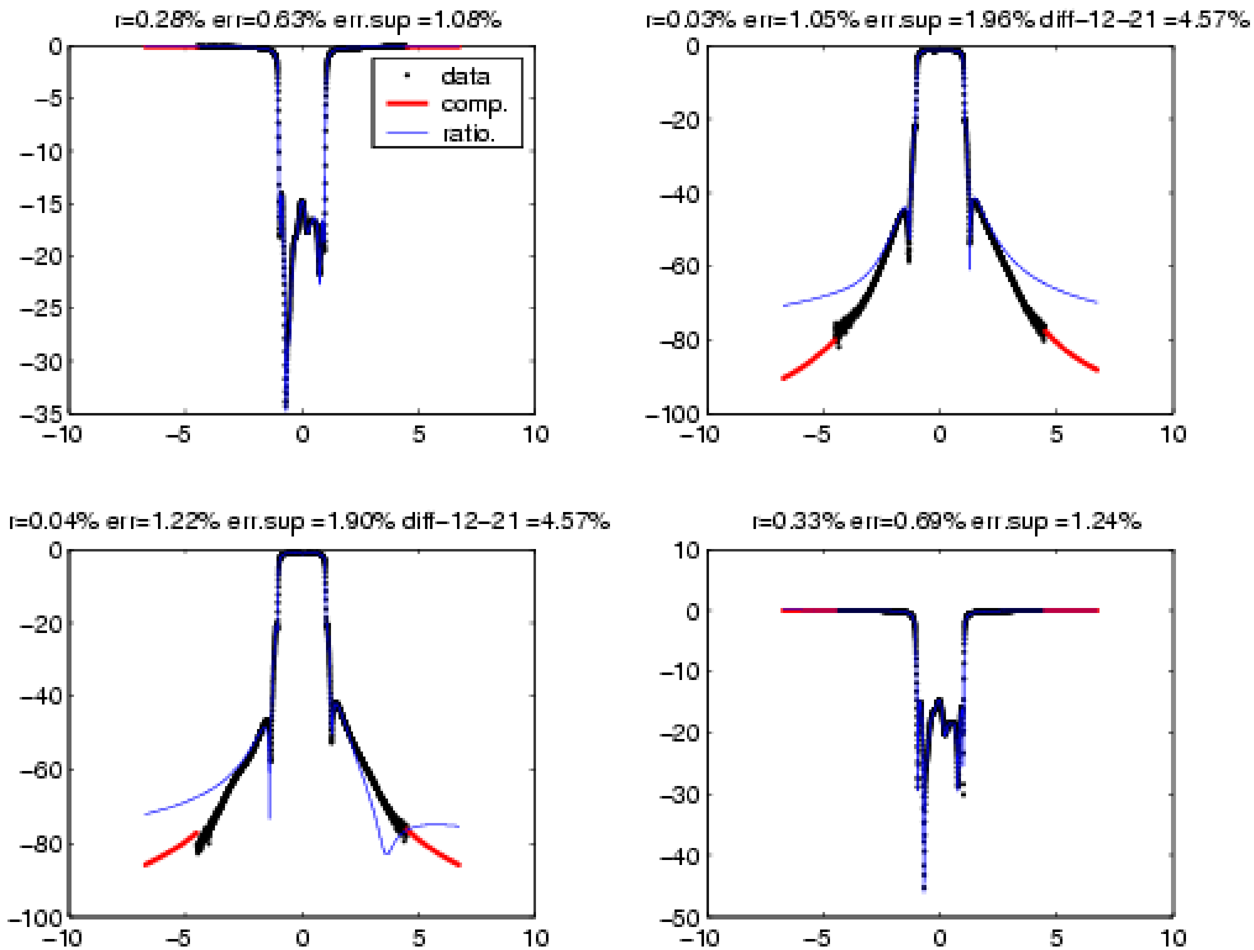
PRESTO-HF: software by F. Seyfert;
HYPERION: software by J. Grimm.

Data and approximant at order 8



Nyquist diagrams

Data and approximant at order 8



Bode diagrams

References

[1] D. ALPAY, L. BARATCHART, AND A. GOMBANI, On the differential structure of matrix-valued rational inner functions, Operator Theory : Advances and Applications, 73 (1994), pp. 30–66.

[2] P. FULCHERI AND M. OLIVI, Matrix Rational H^2-Approximation: a Gradient Algorithm Based on Schur Analysis, SIAM J. Contr. and Opt., 36 (1998), pp. 2103–2127.

[3] B. HANZON AND R.J. OBER, Overlapping block-balanced canonical forms for various classes of linear systems, Linear Algebra and its Applications, 281 (1998), pp. 171-225.

[4] R.L.M. PEETERS, B. HANZON AND M. OLIVI, Balanced realizations of discrete-time stable all-pass systems and the tangential Schur algorithm, in Proceedings of the ECC99, Karlsruhe, Germany, August 31-September 3.

[5] K. GLOVER, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, Int. J. Control 39 (1984), pp. 1115-1193.

[6] Y.S. HUNG AND A.G.J. MACFARLANE, Multivariable feedback: a quasi-classical approach, Lect. Notes in Control Sci. 40, Springer- Verlag, 1982.

[7] W.-Y. YAN AND J. LAM, An approximate approach to H^2 optimal model reduction, IEEE Trans. Autom. Control 44, No.7 (1999), pp 1341-1358.