Parameter determination for surface acoustic wave filters[1]

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Abstract. Since many years now Surface Acoustic Wave filters have been used in electronic devices; nevertheless, some physical constraints make the optimal tuning an interesting mathematical problem. We investigate some aspects of this problem and its relation to the well-known Schur parameters which naturally arise due to the presence of internal reflectors.

1 Introduction

The filter we are interested in (see fig. 1) is constituted of two transducers Σ_1 and Σ_2 with inputs:

- incoming waves $E = \begin{bmatrix} E_g \\ E_d \end{bmatrix}$
- voltages $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$,

and outputs:

- outgoing waves $S = \begin{bmatrix} S_d \\ S_g \end{bmatrix}$
- currents $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$. The physical model used

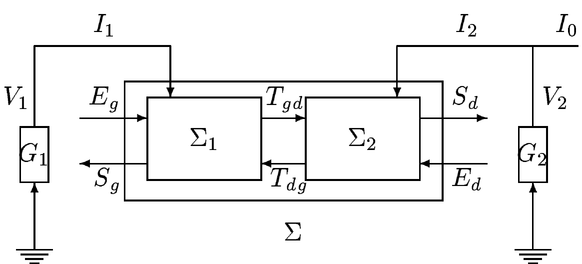

Figure 1: The filter.

in this work is described in [4]. Each transducer is made of a number of cells containing a reflection center with reflection coefficient r_i and a electroacoustic center with coefficient g_i. Each cell (see fig. 2) possess the same delay τ and the position of the

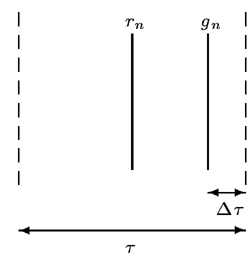

Figure 2: A cell of the left transducer.

electroacoustic center is determine so that, near some given frequency, say f_0, Σ_1 is unidirectional to the right while Σ_2 is unidirectional to the left. It happens when the delay between the electroacoustic center and the boundary of the cell is precisely

$$\Delta\tau = \frac{1}{8f_0}.$$

Note that the two transducers then present a symmetric structure (see fig 3 and 4). The cells are numbered from the left to the right, taking into account the distance between the transducers which is a multiple $T\tau$ of the delay τ associated with a cell.

The transfer function of the filter is the so-called "mixed matrix" given by

$$\begin{bmatrix} S \\ I \end{bmatrix} = \begin{bmatrix} M & \alpha \\ \beta & Y \end{bmatrix} \begin{bmatrix} E \\ V \end{bmatrix}.$$

The matrix M is the diffraction or scattering matrix, α is the electroacoustic matrix and Y the admittance (they are two by two matrices). The physical laws of reciprocity and energy conservation imply the following relations: M and Y are symmetric, $\beta = -\alpha^T$, where the superscript T means

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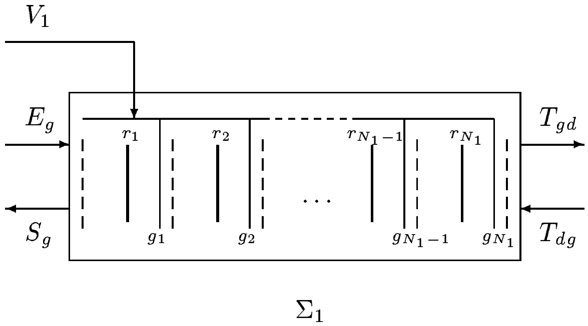

Figure 3: The left transducer.

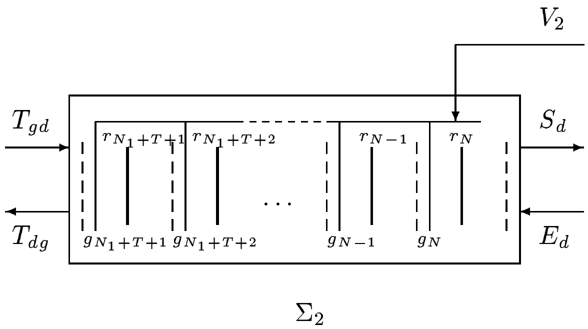

Figure 4: The right transducer.

transpose, and

$$\begin{aligned} MM^* &= Id, & (1)\\ \alpha^*\alpha &= Y + Y^* & (2)\\ \alpha &= M\beta^*, & (3) \end{aligned}$$

where the superscript $*$ denotes transpose-conjugate.

As we shall see, the entries of M, α, β, and Y are analytic functions of the complex variable

$$z = e^{-j2\pi f\tau},$$

where f is the frequency and τ the delay for one cell. Thus, (1) means that M *is inner*, (3) is the *Douglas-Shapiro-Shields factorization* of α (see [2]), while (2) means that α is the *spectral factor* and Y the *real positive function* of some density Φ.

The problem is to find the electroacoustic and reflection parameters of both transducers in order to produce a bandpass filter for some specified frequency in terms of power transmission. The power transmission is represented by the electrical transfer function

$$E = 2\sqrt{G_1 G_2}\frac{V_1}{I_0},$$

where G_1 and G_2 are the load impedances (see fig. 1), and is in fact equal to the entry 12 of the function

$$S = (Y+G)^{-1}(Y-G), \qquad G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}.$$

In the sequel, we propose a mathematical description of the transfer functions M, β, Y and S.

2 Chain and Diffraction matrices

We first focus on the acoustic waves. From this point of view, the filter can be considered as a single transducer, composed with

$$N = N_1 + N_2 + T$$

cells, where N_1 is the number of cells of Σ_1, N_2 the number of cells of Σ_2 and $T\tau$ is the delay between the transducers, with reflection coefficients

$$r_1, r_2, \ldots, r_{N_1}, 0, \ldots, 0, r_{N_1+T+1}, \ldots, r_N,$$

which explain our notations (see fig. 3 and 4).

Let us consider a set of $n - m + 1$ cells containing each a reflector. The diffraction or scattering ma-

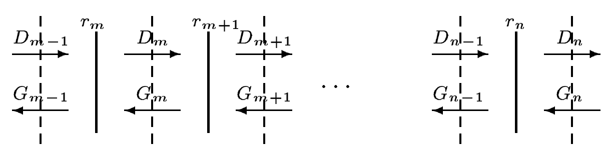

Figure 5: A set of cells

trix associated to this set relates incoming waves to outgoing waves,

$$\begin{bmatrix} G_{m-1} \\ D_n \end{bmatrix} = M_{m,n} \begin{bmatrix} D_{m-1} \\ G_n \end{bmatrix},$$

while the chain matrix is defined by

$$\begin{bmatrix} D_n \\ G_n \end{bmatrix} = C_{m,n} \begin{bmatrix} D_{m-1} \\ G_{m-1} \end{bmatrix}.$$

These two matrices are connected by the following linear fractional transformation:

$$M_{m,n} = \left[\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} C_{m,n} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right] \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} C_{m,n} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right]^{-1}. \quad (4)$$

The diffraction matrix of a single cell is known to be

$$\begin{bmatrix} G_{n-1} \\ D_n \end{bmatrix} = \begin{bmatrix} -j\, r_n\, z & t_n\, z \\ t_n\, z & -j\, r_n\, z \end{bmatrix} \begin{bmatrix} D_{n-1} \\ G_n \end{bmatrix},$$

$$t_n = \sqrt{1 - r_n^2},$$

from which we deduce the chain matrix of a single cell:

$$\begin{bmatrix} D_n \\ G_n \end{bmatrix} = \frac{1}{t_n} \begin{bmatrix} z & -j\, r_n \\ j\, r_n & \frac{1}{z} \end{bmatrix} \begin{bmatrix} D_{n-1} \\ G_{n-1} \end{bmatrix}.$$

It is then easily established that the chain matrix $C_{1,n}$ has the form

$$C_{1,n} = \frac{1}{P_n z^n} \begin{bmatrix} \phi_n(z^2) & -jz\widetilde{\psi}_n(z^2) \\ jz^{-1}\psi_n(z^2) & \widetilde{\phi}_n(z^2) \end{bmatrix},$$

$$P_n = t_1 t_2 \dots t_n,$$

where $\phi_n(\zeta)$ and $\psi_n(\zeta)$ are the Schur polynomials of degree n satisfying the Levinson recursions (see [1])

$$\begin{bmatrix} \phi_{n+1}(\zeta) & \widetilde{\psi}_{n+1}(\zeta) \\ \psi_{n+1}(\zeta) & \widetilde{\phi}_{n+1}(\zeta) \end{bmatrix} = \begin{bmatrix} \zeta & r_{n+1} \\ r_{n+1}\zeta & 1 \end{bmatrix} \begin{bmatrix} \phi_n(\zeta) & \widetilde{\psi}_n(\zeta) \\ \psi_n(\zeta) & \widetilde{\phi}_n(\zeta) \end{bmatrix}, \quad (5)$$

$$\phi_0 = 1, \quad \psi_0 = 0,$$

and where

$$\widetilde{\phi}_n(\zeta) = \zeta^n \phi_n(1/\zeta), \quad \widetilde{\psi}_n(\zeta) = \zeta^n \psi_n(1/\zeta),$$

are the reciprocal polynomials. Using the linear fractional transformation (4), we have

Lemma 1 *The inner matrix $M_{1,n}$ has McMillan degree $2n$ and can be written as*

$$M_{1,n} = \frac{\begin{bmatrix} -j\, z^{-1}\, \psi_n(z^2) & P_n\, z^n \\ P_n\, z^n & -j\, z\, \widetilde{\psi}_n(z^2) \end{bmatrix}}{\widetilde{\phi}_n(z^2)}. \quad (6)$$

Suppose that we reverse the transducer, so that from the left to the right the successsive cells are numbered $N, N-1, \dots, 2, 1$. We shall denote with a superscript R all the objects which refer to the reverse transducer. We have that

$$C_{n+1,N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [C^R_{1,N-n}]^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where

$$C^R_{1,N-n} = \frac{P_n z^n}{P_N z^N} \begin{bmatrix} \phi^R_{N-n}(z^2) & -jz\widetilde{\psi}^R_{N-n}(z^2) \\ jz^{-1}\psi^R_{N-n}(z^2) & \widetilde{\phi}^R_{N-n}(z^2) \end{bmatrix}.$$

Since $C_{1,N} = C_{n+1,N} C_{1,n}$, we obtain the following relation between the two kind of polynomials:

$$\begin{bmatrix} \phi_N(z^2) & -jz\widetilde{\psi}_N(z^2) \\ jz^{-1}\psi_N(z^2) & \widetilde{\phi}_N(z^2) \end{bmatrix} = \begin{bmatrix} \phi^R_{N-n}(z^2) & -jz^{-1}\psi^R_{N-n}(z^2) \\ jz\widetilde{\psi}^R_{N-n}(z^2) & \widetilde{\phi}^R_{N-n}(z^2) \end{bmatrix} \begin{bmatrix} \widetilde{\phi}_n(z^2) & -jz\widetilde{\psi}_n(z^2) \\ jz^{-1}\psi_n(z^2) & \phi_n(z^2) \end{bmatrix}.$$

3 The structure of β

In the sequel, we assume that

$$\delta = e^{j2\pi f \Delta\tau} \approx e^{j2\pi f_0 \Delta\tau}$$

is constant in the bandwidth, which is actually the case in most examples. For any matrix-valued function $A(z)$ we define

$$A^\sharp(z) = A(1/\bar{z})^*.$$

Recall that β is given by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \beta \begin{bmatrix} E_g = D_0 \\ E_d = G_N \end{bmatrix}.$$

We put for $n = 1, \dots, N$,

$$V_n = C_{1,n} \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} M\right]. \quad (7)$$

Now $I_1 = \sum_{n=1}^{N_1} I_n$, where the current I_n in the nth cell of the transducer Σ_1 can be computed as follows:

$$\begin{aligned} I_n &= jg_n\left(\delta D_n + \bar{\delta} G_n\right), \\ &= jg_n \begin{bmatrix} \delta & \bar{\delta} \end{bmatrix} C_{1,n} \begin{bmatrix} D_0 \\ G_0 \end{bmatrix}, \\ &= jg_n \begin{bmatrix} \delta & \bar{\delta} \end{bmatrix} V_n \begin{bmatrix} D_0 \\ G_N \end{bmatrix}. \end{aligned}$$

In the same way, $I_2 = \sum_{n=N_1+T+1}^{N} I_n$, where the current I_n in the nth cell of the transducer Σ_2 can be computed as follows:

$$\begin{aligned} I_n &= jg_n\left(\bar\delta D_{n-1} + \delta G_{n-1}\right), \\ &= jg_n \begin{bmatrix} \bar\delta & \delta \end{bmatrix} C_{1,n-1} \begin{bmatrix} D_0 \\ G_0 \end{bmatrix}, \\ &= jg_n \begin{bmatrix} \bar\delta & \delta \end{bmatrix} V_{n-1} \begin{bmatrix} D_0 \\ G_N \end{bmatrix}. \end{aligned}$$

Theorem 1 *The function β has representation*

$$\beta = j \begin{bmatrix} \sum_{n=1}^{N_1} & g_n & \begin{bmatrix} \delta & \bar\delta \end{bmatrix} & V_n \\ \\ \sum_{n=N_1+T}^{N-1} & g_{n+1} & \begin{bmatrix} \bar\delta & \delta \end{bmatrix} & V_n \end{bmatrix}. \qquad (8)$$

Also put

$$V_0 = \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} M\right]. \qquad (9)$$

Proposition 1 *The columns of $\alpha = M\beta^\sharp$ belongs to the orthogonal complement $H(M)$ of MH^2. For $n = 0, \ldots, N$, let v_n and w_n be the column vectors of*

$$V_n^T = [v_n \; w_n].$$

Then $\nu = (v_1, \ldots, v_N, w_0, \ldots, w_{N-1})$ is an orthogonal basis of $H(M)$.

Proof. It is easily verified that for $n = 0, \ldots, N$,

$$\begin{aligned} v_n &= \begin{bmatrix} P_n z^n & 0 \\ 0 & \frac{P_N z^N}{P_n z^n} \end{bmatrix} \begin{bmatrix} \frac{\widetilde{\phi}^R_{N-n}(z^2)}{\widetilde{\phi}_N(z^2)} \\ -jz\frac{\widetilde{\psi}_n(z^2)}{\widetilde{\phi}_N(z^2)} \end{bmatrix}, \\ w_n &= \begin{bmatrix} P_n z^n & 0 \\ 0 & \frac{P_N z^N}{P_n z^n} \end{bmatrix} \begin{bmatrix} -jz\frac{\widetilde{\psi}^R_{N-n}(z^2)}{\widetilde{\phi}_N(z^2)} \\ \frac{\widetilde{\phi}_n(z^2)}{\widetilde{\phi}_N(z^2)} \end{bmatrix}, \end{aligned}$$

and that

$$\begin{aligned} v_n &= M\,\bar w_n(1/z), \\ w_n &= M\,\bar v_n(1/z). \end{aligned}$$

Except for $v_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $w_N = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, the vectors v_n and w_n clearly belongs to $H(M)$. □

4 State-space realizations

From (8) we deduce an expression of α in the basis ν:

$$\alpha = -j \begin{bmatrix} \sum_{n=1}^{N_1} g_n\left(\delta v_n + \bar\delta w_n\right) & \sum_{n=1}^{N_1} g_{n+1}\left(\bar\delta v_n + \delta w_n\right) \end{bmatrix}.$$

With the help of the recurrence relation

$$V_n = C_{nn} V_{n-1},$$

we obtain

Theorem 2 *The strictly proper function $\alpha^\sharp$ has McMillan degree $2(N-1)$ and realization*

$$\alpha^\sharp(z) = C(zI_{2(N-1)} - A)^{-1}B$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 0 & \ldots & & & 0 \\ t_2 & 0 & \ldots & & \vdots \\ 0 & t_3 & 0 & \ldots & \\ \vdots & 0 & \ddots & 0 & \\ 0 & & \ddots & t_{N-1} & 0 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} jr_1 & 0 & \ldots & & 0 \\ 0 & jr_2 & 0 & & \vdots \\ \vdots & 0 & \ddots & & \\ & & & \ddots & 0 \\ 0 & \ldots & & 0 & jr_{N-1} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} jr_2 & 0 & \ldots & & 0 \\ 0 & jr_3 & 0 & & \\ \vdots & 0 & \ddots & & \\ & & & \ddots & 0 \\ 0 & \ldots & & & jr_N \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 0 & t_2 & 0 & \ldots & 0 \\ \vdots & 0 & t_3 & 0 & \\ & & \ddots & \ddots & 0 \\ & & & 0 & t_{N-1} \\ 0 & \ldots & & & 0 \end{bmatrix}$$

$$C = j \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} \bar\delta I_{N-1} & \delta I_{N-1} \\ \delta I_{N-1} & \bar\delta I_{N-1} \end{bmatrix} \qquad (10)$$

where C_1 *and* C_2 *are given by:*

$$\begin{aligned} C_1 &= [g_1\ g_2\ \dots\ g_{N_1}\ 0 \dots\ 0] \\ C_2 &= [0\ \dots\ 0\ g_{N_1+T+1}\ \dots\ g_N] \end{aligned}$$

and

$$B^T = \begin{bmatrix} t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & t_N \end{bmatrix} \tag{11}$$

The controllability gramian of (B, A) *is the identity.*

Corollary 1 *Let* $\alpha^\sharp$ *have realization*

$$\alpha^\sharp = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$$

as in Theorem 2, and suppose $Y + Y^\sharp = \alpha^\sharp \alpha$ *and* $S = (Y+I)^{-1}(Y-I)$. *Then* $Y^\sharp$ *has realization:*

$$Y^\sharp = \left(\begin{array}{c|c} A & AC^* \\ \hline C & \frac{1}{2}CC^* \end{array}\right) \tag{12}$$

and $S^\sharp$ *has realization:*

$$S^\sharp = \left(\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right),$$

where

$$\left\{ \begin{aligned} A_S &= A\left(I - C^*(\tfrac{1}{2}CC^* + I)^{-1}C\right) \\ B_S &= \sqrt{2}AC^*(\tfrac{1}{2}CC^* + I)^{-1} \\ C_S &= \sqrt{2}(\tfrac{1}{2}CC^* + I)^{-1}C \\ D_S &= (\tfrac{1}{2}CC^* - I)(\tfrac{1}{2}CC^* + I)^{-1} \end{aligned} \right. \tag{13}$$

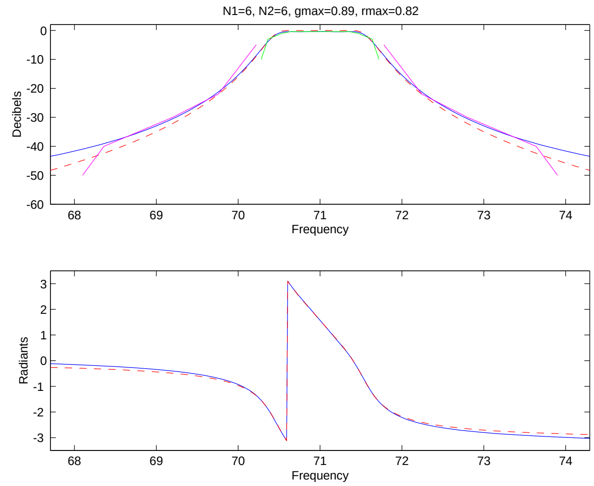

Figure 6: Amplitude and phase of S_{12}.

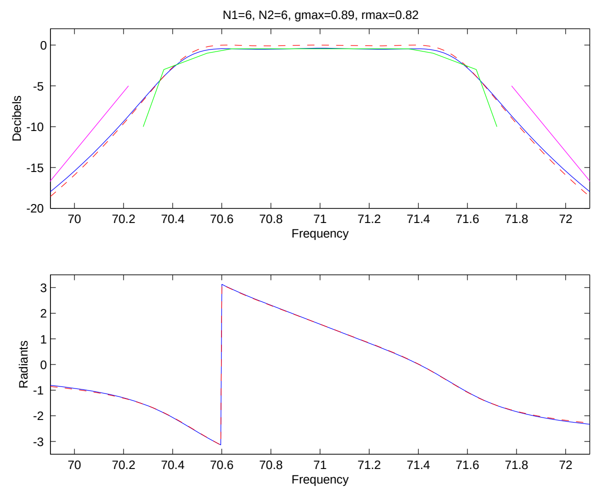

Figure 7: Amplitude and phase of S_{12} in the bandwidth.

5 Optimization

We tackle our parameters determination problem has an approximation problem in L^2 norm: given a reference filter satisfying the specifications, we minimize the distance from it to the set of transfer functions of the form S_{12}. The criterion is expressed in terms of state space realizations. We present some results on figure 7. The reference filter is a Chebyschev filter of degree 3 (dotted line) and the model we obtain contains $N_1 = 6$ cells in the left transducer and $N_2 = 6$ cells in the right transducer. Thought this model matches quite well the specifications, the values of the parameters are not realistic from a physical point of view and at this time we are not able to improve this result. Being confident in this approach, we think that we lack for a good reference filter taking into account the particular form of the functions S_{12}. Observe that S_{12} is a rational function of degree $2(N-1)$ described by only $2(N_1+N_2)$ parameters r and g, so that it cannot be any rational function. The caracterization of these functions or at least their asymptotic behavior is under study and would enable us to obtain a good reference filter.

References

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