Parameter determination for surface acoustic wave filters\textsuperscript{1}

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Abstract. Since many years now Surface Acoustic Wave filters have been used in electronic devices; nevertheless, some physical constraints make the optimal tuning an interesting mathematical problem. We investigate some aspects of this problem and its relation to the well-known Schur parameters which naturally arise due to the presence of internal reflectors.

1 Introduction

The filter we are interested in (see fig. 1) is constituted of two transducers $\Sigma_1$ and $\Sigma_2$ with inputs:

- incoming waves $E = \begin{bmatrix} E_g \\ E_d \end{bmatrix}$
- voltages $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$,

and outputs:

- outgoing waves $S = \begin{bmatrix} S_d \\ S_g \end{bmatrix}$
- currents $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$. The physical model used center with reflection coefficient $r$, and a electroacoustic center with coefficient $g$. Each cell (see fig. 2) possess the same delay $\tau$ and the position of the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A cell of the left transducer.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The filter.}
\end{figure}

Note that the two transducers then present a symmetric structure (see fig 3 and 4). The cells are numbered from the left to the right, taking into account the distance between the transducers which is a multiple $T\tau$ of the delay $\tau$ associated with a cell.

The transfer function of the filter is the so-called "mixed matrix" given by

$$
\begin{bmatrix} S \\ I \end{bmatrix} = \begin{bmatrix} M & \alpha \\ \beta & Y \end{bmatrix} \begin{bmatrix} E \\ V \end{bmatrix}.
$$

The matrix $M$ is the diffraction or scattering matrix, $\alpha$ is the electroacoustic matrix and $Y$ the admittance (they are two by two matrices). The physical laws of reciprocity and energy conservation imply the following relations: $M$ and $Y$ are symmetric, $\beta = -\alpha^T$, where the superscript $T$ means
transmission is represented by the electrical transfer function

\[ E = 2\sqrt{G_1 G_2} \frac{V_1}{I_0}, \]

where \( G_1 \) and \( G_2 \) are the load impedances (see fig. 1), and is in fact equal to the entry 12 of the function

\[ S = (Y + G)^{-1}(Y - G), \quad G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}. \]

In the sequel, we propose a mathematical description of the transfer functions \( M, \beta, Y \) and \( S \).

2 Chain and Diffraction matrices

We first focus on the acoustic waves. From this point of view, the filter can be considered as a single transducer, composed with

\[ N = N_1 + N_2 + T \]

cells, where \( N_1 \) is the number of cells of \( \Sigma_1 \), \( N_2 \) the number of cells of \( \Sigma_2 \) and \( T \) is the delay between the transducers, with reflection coefficients

\[ r_1, r_2, \ldots, r_{N_1}, 0, \ldots, 0, r_{N_1+T+1}, \ldots, r_N, \]

which explain our notations (see fig. 3 and 4).

Let us consider a set of \( n - m + 1 \) cells containing each a reflector. The diffraction or scattering matrix associated to this set relates incoming waves to outgoing waves,

\[ \begin{bmatrix} G_{m-1} \\ D_n \end{bmatrix} = M_{m,n} \begin{bmatrix} D_{m-1} \\ G_n \end{bmatrix}, \]

while the chain matrix is defined by

\[ \begin{bmatrix} D_n \\ G_n \end{bmatrix} = C_{m,n} \begin{bmatrix} D_{m-1} \\ G_{m-1} \end{bmatrix}. \]

\[ z = e^{-j2\pi f\tau}, \]

where \( f \) is the frequency and \( \tau \) the delay for one cell. Thus, (1) means that \( M \) is inner, (3) is the Douglas-Shapiro-Shields factorization of \( \alpha \) (see [2]), while (2) means that \( \alpha \) is the spectral factor and \( Y \) the real positive function of some density \( \Phi \).

The problem is to find the electroacoustic and reflection parameters of both transducers in order to produce a bandpass filter for some specified frequency in terms of power transmission. The power
These two matrices are connected by the following linear fractional transformation:

$$M_{m,n} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} C_{m,n} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} G_{m,n} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} C_{m,n} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} G_{m,n} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]^{-1} \cdot (4)$$

The diffraction matrix of a single cell is known to be

$$G_{n-1} = \begin{bmatrix} -j r_n z & t_n z \\ t_n z & -j r_n z \end{bmatrix} \begin{bmatrix} D_{n-1} \\ G_{n-1} \end{bmatrix}, \quad t_n = \sqrt{1 - r_n^2},$$

from which we deduce the chain matrix of a single cell:

$$D_n = \frac{1}{t_n} \begin{bmatrix} z & -j r_n \\ j r_n & \frac{1}{z} \end{bmatrix} \begin{bmatrix} D_{n-1} \\ G_{n-1} \end{bmatrix}.$$ 

It is then easily established that the chain matrix $C_{1,n}$ has the form

$$C_{1,n} = \frac{1}{P_n z^n} \begin{bmatrix} \phi_n(z^2) & -j \tilde{\psi}_n(z^2) \\ j \tilde{\psi}_n(z^2) & \phi_n(z^2) \end{bmatrix}, \quad P_n = t_1 t_2 \ldots t_n,$$

where $\phi_n(\zeta)$ and $\psi_n(\zeta)$ are the Schur polynomials of degree $n$ satisfying the Levinson recursions (see [1])

$$\begin{bmatrix} \phi_{n+1}(\zeta) & \tilde{\psi}_{n+1}(\zeta) \\ \tilde{\psi}_{n+1}(\zeta) & \tilde{\phi}_{n+1}(\zeta) \end{bmatrix} = \begin{bmatrix} \zeta & r_{n+1} \\ r_{n+1} & 1 \end{bmatrix} \begin{bmatrix} \phi_n(\zeta) & \tilde{\psi}_n(\zeta) \\ \psi_n(\zeta) & \phi_n(\zeta) \end{bmatrix}, \quad (5)$$

$$\phi_0 = 1, \quad \psi_0 = 0,$$

and where

$$\tilde{\phi}_n(\zeta) = \zeta^n \phi_n(1/\zeta), \quad \tilde{\psi}_n(\zeta) = \zeta^n \psi_n(1/\zeta),$$

are the reciprocal polynomials. Using the linear fractional transformation (4), we have

**Lemma 1** The inner matrix $M_{1,n}$ has McMillan degree $2n$ and can be written as

$$M_{1,n} = \begin{bmatrix} -j \tilde{\psi}_n(z^2) & P_n z^n \\ P_n z^n & -j \tilde{\phi}_n(z^2) \end{bmatrix}. \quad (6)$$

Suppose that we reverse the transducer, so that from the left to the right the successive cells are numbered $N, N-1, \ldots, 2, 1$. We shall denote with a superscript $R$ all the objects which refer to the reverse transducer. We have that

$$C_{n+1,N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left[ C_{1,n}^R \right]^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where

$$C_{1,n}^R = \frac{P_n z^n}{P_N z^N} \begin{bmatrix} \phi_{N-n}^R(z^2) & -j \tilde{\psi}_{N-n}^R(z^2) \\ j \tilde{\psi}_{N-n}^R(z^2) & \phi_{N-n}^R(z^2) \end{bmatrix}.$$ 

Since $C_{1,n} = C_{n+1,N} C_{1,n}$, we obtain the following relation between the two kind of polynomials:

$$\begin{bmatrix} \phi_n(z^2) & -j \tilde{\psi}_n(z^2) \\ j \tilde{\psi}_n(z^2) & \phi_n(z^2) \end{bmatrix} = \begin{bmatrix} \phi_n(z^2) & -j \tilde{\psi}_n(z^2) \\ j \tilde{\psi}_n(z^2) & \phi_n(z^2) \end{bmatrix},$$

3 The structure of $\beta$

In the sequel, we assume that

$$\delta = e^{i2\pi / \Delta \tau} \approx e^{i2\pi f_0 \Delta \tau}$$

is constant in the bandwidth, which is actually the case in most examples. For any matrix-valued function $A(z)$ we define

$$A^\delta(z) = A(1/z)^\delta.$$

Recall that $\beta$ is given by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \beta \begin{bmatrix} E_g = D_0 \\ E_d = G_N \end{bmatrix}.$$ 

We put for $n = 1, \ldots, N,$

$$V_n = C_{1,n} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} M. \quad (7)$$

Now $I_n = \sum_{n=1}^{N-1} I_n$, where the current $I_n$ in the $n$th cell of the transducer $\Sigma_1$ can be computed as follows:

$$I_n = j g_n \left( \delta D_n + \delta G_n \right),$$

$$= j g_n \begin{bmatrix} \delta & \delta \end{bmatrix} C_{1,n} \begin{bmatrix} D_0 \\ G_0 \end{bmatrix},$$

$$= j g_n \begin{bmatrix} \delta & \delta \end{bmatrix} V_n \begin{bmatrix} D_0 \\ G_N \end{bmatrix}.$$
In the same way, \( I_n = \sum_{n=N_1+T+1}^{N_1} I_n \), where the current \( I_n \) in the \( n \)th cell of the transducer \( \Sigma_2 \) can be computed as follows:

\[
I_n = j g_n (\delta D_{n-1} + \delta G_{n-1}),
\]

\[
= j g_n \left[ \begin{array}{cc} \delta & \delta \end{array} \right] C_{1,n-1} \left[ \begin{array}{c} D_0 \\ G_0 \end{array} \right],
\]

\[
= j g_n \left[ \begin{array}{cc} \delta & \delta \end{array} \right] V_{n-1} \left[ \begin{array}{c} D_0 \\ G_N \end{array} \right].
\]

**Theorem 1** The function \( \beta \) has representation

\[
\beta = j \sum_{n=1}^{N_1} g_n \left[ \begin{array}{cc} \delta & \delta \end{array} \right] V_n.
\]

Also put

\[
V_0 = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right] + \left[ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right] M.
\]

**Proposition 1** The columns of \( \alpha = M \beta \) belong to the orthogonal complement \( H(M) \) of \( MH^2 \). For \( n = 0, \ldots, N \), let \( v_n \) and \( w_n \) be the column vectors of

\[
V_n^T = [v_n \ w_n].
\]

Then \( \nu = (v_1, \ldots, v_N, w_0, \ldots, w_{N-1}) \) is an orthogonal basis of \( H(M) \).

**Proof.** It is easily verified that for \( n = 0, \ldots, N \),

\[
v_n = \left[ \begin{array}{c} P_n z^n \\ 0 \\ \frac{P_n z^n}{P_n z^n} \end{array} \right] \left[ \begin{array}{c} \tilde{\phi}_n(z^2) \\ \phi_n(z^2) \\ -j z \phi_n(z^2) \end{array} \right],
\]

\[
w_n = \left[ \begin{array}{c} P_n z^n \\ 0 \\ \frac{P_n z^n}{P_n z^n} \end{array} \right] \left[ \begin{array}{c} -j z \tilde{\phi}_n(z^2) \\ \phi_n(z^2) \\ \phi_n(z^2) \end{array} \right],
\]

and that

\[
v_n = M \bar{w}_n(1/z),
\]

\[
w_n = M \sigma_n(1/z).
\]

Except for \( v_0 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \) and \( w_N = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \), the vectors \( v_n \) and \( w_n \) clearly belongs to \( H(M) \).

4 State-space realizations

From (8) we deduce an expression of \( \alpha \) in the basis \( \nu \):

\[
\alpha = -j \left[ \sum_{n=1}^{N_1} g_n (\delta v_n + \delta w_n) \sum_{n=1}^{N_1} g_{n+1} (\delta v_n + \delta w_n) \right].
\]

With the help of the recurrence relation

\[
V_n = C_{nn} V_{n-1},
\]

we obtain

**Theorem 2** The strictly proper function \( \alpha(z) \) has McMillan degree \( 2(N-1) \) and realization

\[
\alpha(z) = C(z I_{2(N-1)} - A)^{-1} B
\]

where

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]

\[
A_{11} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \cdots & 0 \\ 0 & \cdots & 0 & t_{N-1} \end{bmatrix}
\]

\[
A_{12} = \begin{bmatrix} j r_1 & 0 & \cdots & 0 \\ 0 & j r_2 & 0 & \vdots \\ \vdots & \cdots & 0 \\ 0 & \cdots & 0 & j r_{N-1} \end{bmatrix}
\]

\[
A_{21} = \begin{bmatrix} j r_2 & 0 & \cdots & 0 \\ 0 & j r_3 & 0 & \vdots \\ \vdots & \cdots & 0 & j r_N \end{bmatrix}
\]

\[
A_{22} = \begin{bmatrix} 0 & t_2 & 0 & \cdots & 0 \\ \vdots & \cdots & 0 & 0 & t_{N-1} \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}
\]

\[
C = j \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \left[ \begin{array}{cc} \delta I_{N-1} & \delta I_{N-1} \\ \delta I_{N-1} & \delta I_{N-1} \end{array} \right]
\]

(10)
where $C_1$ and $C_2$ are given by:

$$C_1 = \begin{bmatrix} g_1 & g_2 & \ldots & g_{N_1} & 0 & \ldots & 0 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0 & \ldots & 0 & g_{N_1+T+1} & \ldots & g_N \end{bmatrix}$$

and

$$B^T = \begin{bmatrix} t_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & t_N \end{bmatrix} \quad (11)$$

The controllability gramian of $(B, A)$ is the identity.

**Corollary 1** Let $\alpha^\sharp$ have realization

$$\alpha^\sharp = \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} B \\ 0 \end{pmatrix}$$

as in Theorem 2, and suppose $Y + Y^\sharp = \alpha^\sharp \alpha$ and $S = (Y + I)^{-1}(Y - I)$. Then $Y^\sharp$ has realization:

$$Y^\sharp = \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A C^* \\ \frac{1}{2} C C^* \end{pmatrix} \quad (12)$$

and $S^\sharp$ has realization:

$$S^\sharp = \begin{pmatrix} A_S & B_S \\ C_S & D_S \end{pmatrix},$$

where

$$\begin{cases} A_S & = A(I - C^*(\frac{1}{2} C C^* + I)^{-1} C) \\ B_S & = \sqrt{2} A C^*(\frac{1}{2} C C^* + I)^{-1} \\ C_S & = \sqrt{2}(\frac{1}{2} C C^* + I)^{-1} C \\ D_S & = (\frac{1}{2} C C^* - I)(\frac{1}{2} C C^* + I)^{-1} \end{cases} \quad (13)$$

### 5 Optimization

We tackle our parameters determination problem has an approximation problem in $L^2$ norm: given a reference filter satisfying the specifications, we minimize the distance from it to the set of transfer functions of the form $S_{12}$. The criterion is expressed in terms of state space realizations. We present some results on figure 7. The reference filter is a Chebyshev filter of degree 3 (dotted line) and the model we obtain contains $N_1 = 6$ cells in the left transducer and $N_2 = 6$ cells in the right transducer. Thought this model matches quite well the specifications, the values of the parameters are not realistic from a physical point of view and at this time we are not able to improve this result. Being confident in this approach, we think that we lack for a good reference filter taking into account the particular form of the functions $S_{12}$. Observe that $S_{12}$ is a rational function of degree $2(N - 1)$ described by only $2(N_1 + N_2)$ parameters $r$ and $g$, so that it cannot be any rational function. The characterization of these functions or at least their asymptotic behavior is under study and would enable us to obtain a good reference filter.
References


