

Surface Acoustic Wave Filters, Unitary Extensions and Schur Analysis.

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Abstract. We study the concept of a mixed matrix in connection with linear fractional transformations of lossless and passive matrix-valued rational functions, and show that they can be parametrized by sequences of elementary chain matrices. These notions are exemplified on a model of a Surface Acoustic Wave filter for which a state-space realization is carried out in detail. The issue of optimal design of such filters –as yet unsolved– naturally raises a Darlington synthesis problem with both symmetry and interpolation constraints with control on the McMillan degree. As a partial answer, we give necessary and sufficient conditions for symmetric Darlington synthesis to be possible without increasing the McMillan degree for a symmetric rational contractive matrix which is strictly contractive in at least one point of the unit circle .

1 Introduction

Surface Acoustic Wave (SAW) devices have been a spectacular application of piezo-electric components to signal processing [9, 5], the design and synthesis

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of which has been the object of much attention (see for example [21, 18] and the bibliography therein). Despite these efforts a number of fundamental issues are still not well-understood, for instance, the optimal tuning of a passband filter is still largely an open problem. SAW filters have a rich mathematical structure that results in several different formalisms to describe them. Among those, the concept of a *mixed matrix* was advocated in [23] as a natural mean to express the fact that two types of energies interact within such systems: Electric energy and acoustic energy. Using the classical equivalence between waves and currents, the passage from the mixed matrix to the more classical scattering and admittance matrices has been explicitly carried out at several places in the literature, see for instance [21], where the projective viewpoint is emphasized.

Our first concern in this paper is to give a somewhat abstract treatment of the correspondence between mixed matrices on the one hand, and lossless or passive matrices on the other hand, based on linear fractional transformations. This is done in Section 3, for which Section 2 provides the necessary background.

Our second goal is to lay some ground for optimization of such systems. There, it is of primary interest to parametrize the mixed matrices associated to the device under study, and the parametrization may take place either in the time domain or in the frequency domain, that is, either in terms of state-space realizations or in terms of transfer functions. Both are equally important: the physical parameters live in the time domain, and they are the relevant quantities for synthesis, whereas the parametrization of the transfer function is to be used for optimization purposes, since the specifications on a device are formulated in the frequency domain (*e.g.* a SAW filter should transmit a substantial amount of signal in the pass-band and roll-off at other frequencies).

Our contribution to time domain parametrization will be to derive a state-space realization of the mixed matrix associated to a general class of SAW filters which is carried out in detail in Section 5. The derivation uses the relation to the chain matrix obtained in Section 3, and benefits from the fact that the chain matrix of the whole SAW filter is the usual matrix product of the chain matrices for each of the simple transducers that are cascaded to

form the filter. The underlying recursion formulas are of the Levinson type to construct the Szegő orthogonal polynomials associated to a positive measure on the circle. The reflection coefficients and a set of additional parameters representing the intensity of the sources do parameterize all the realizable filters.

The frequency domain parametrization is not nearly as complete. In the present paper, we focus on one aspect which is both important and connected to classical Network Theory. More precisely, the lower-right block of a mixed matrix, which corresponds to the electric impedance function in the case of a SAW filter, is such that its Cayley transform has a lossless extension of prescribed McMillan degree which is symmetric. It is well-known from Darlington synthesis [8, 3, 10, 2] that any contractive rational function can be extended to a (generally non-symmetric) lossless one without increasing the McMillan degree of the function. Here we are interested in extensions that in addition are symmetric, and in Section 4 we give necessary and sufficient conditions for a symmetric extension of the same degree to exist for contractive functions that are strictly contractive in at least one point of the unit circle. More generally, it would be interesting to know when there exists a symmetric extension possibly of higher degree, and if it exists what is its the minimal degree. For conjugate symmetric functions (*i.e.* those having real Fourier coefficients) such a symmetric extension is known to exist [2], but no control on the McMillan degree seems to be available. These questions will be left for further study.

In the Appendix we give an example of a particular class of SAW filters [18] that illustrates other constraints beyond symmetry such as interpolation conditions at infinity, in order to motivate further studies in the direction of frequency domain parametrization of mixed matrices. The filters in question span a rather large subset of the feasible mixed matrices parameterized by the reflection coefficients and the electroacoustic intensities. The latter give a physical interpretation of the parameters a_ℓ^k introduced in Section 5.

2 Preliminaries and notations

We shall denote by \mathbb{E} the complement in $\overline{\mathbb{C}}$ of the closed unit disk $\overline{\mathbb{D}}$ and by \mathcal{H}_2 the corresponding Hardy space of vector or matrix valued functions (the proper dimension will be understood from the context). The space \mathcal{H}_2 is naturally endowed with the scalar product:

$$\langle F, G \rangle = \frac{1}{2\pi} \text{Tr} \int_{-\pi}^{\pi} F(e^{j\omega}) G(e^{j\omega})^* d\omega, \quad (1)$$

where j is the square root of -1 ; we shall denote by $\| \cdot \|_2$ the associated norm. Throughout, if A is a complex matrix, $\text{Tr}(A)$ stands for its trace, A^T for its transpose and A^* for the transposed conjugate; if $A(z)$ is a matrix valued function, then

$$A^*(z) := A(1/\bar{z})^*, \quad (2)$$

is the *para-Hermitian conjugate* of A .

We say that a $p \times p$ matrix valued function Q is *contractive* (or a Schur function) in \mathbb{E} if it satisfies

$$Q(z)Q(z)^* \leq I_p, \quad z \in \mathbb{E}. \quad (3)$$

Note that a rational matrix which is contractive in \mathbb{E} is automatically analytic there. A contractive function Q is said to be *lossless* if, in addition,

$$Q(e^{j\omega})Q(e^{j\omega})^* = I_p, \quad \omega \in [-\pi, \pi]. \quad (4)$$

When Q is a rational matrix which is lossless, we use the standard notation

$$\mathcal{H}(Q) := \mathcal{H}_2 \ominus \mathcal{H}_2 Q. \quad (5)$$

to denote the orthogonal complement in \mathcal{H}_2 of the left multiples of Q . Let J, K, L be the following $p + q$ block matrices:

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad K = \begin{bmatrix} 0 & I_q \\ I_p & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & I_q \\ -I_p & 0 \end{bmatrix}. \quad (6)$$

A matrix valued function $R(z)$ is said to be *J-contractive* if it satisfies

$$R(z)JR(z)^* \leq J, \quad z \in \mathbb{E},$$

and if in addition equality holds on the circle we say that it is J -lossless.

Linear fractional transformations will be important in this context. Let us briefly recall some basic facts about them.

For $\Theta(z)$ a $(p+q) \times (p+q)$ invertible rational matrix, block partitioned as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

with Θ_{11} of size $p \times p$, and Θ_{22} of size $q \times q$, the associated linear fractional transformation

$$T_{\Theta}(R) = (\Theta_{11}R + \Theta_{12})(\Theta_{21}R + \Theta_{22})^{-1} \quad (7)$$

is defined for every $p \times q$ rational matrix function R such that $\Theta_{21}R + \Theta_{22}$ is invertible as a rational matrix. The linear fractional transformation (7) enjoys the following properties:

$$(i) \quad T_{\Theta} \circ T_{\Phi} = T_{\Theta\Phi} \quad (8)$$

$$(ii) \quad T_{\Theta W}(R) = T_{\Theta}(URV^*), \quad (9)$$

$$(iii) \quad T_{W\Theta}(R) = UT_{\Theta}(R)V^*, \quad (10)$$

where

$$W \triangleq \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

for all $(p+q) \times (p+q)$ rational invertible matrix functions Θ and Φ , all $p \times p$ rational matrix functions R , and all $p \times p$ (resp. $q \times q$) unitary matrices U and V .

Define

$$\Pi_1 = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \quad \Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2 & \Pi_1 \end{bmatrix}. \quad (11)$$

The following lemma, a physical interpretation of which will be mentioned in Section 5.1, is easy to establish.

Lemma 1 [12, chap.1] *Let*

$$\begin{bmatrix} a_2 \\ b_1 \end{bmatrix} = \Sigma \begin{bmatrix} a_1 \\ b_2 \end{bmatrix},$$

be an input-output relation for some invertible matrix valued transfer function Σ . Then

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = T_{\Pi}(\Sigma) \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}.$$

If the transfer function Σ is contractive (resp. lossless), then $T_{\Pi}(\Sigma)$ is $J_{p,q}$ -contractive (resp. $J_{p,q}$ -lossless).

3 Scattering, admittance and mixed matrices.

We introduce below the so-called mixed matrices (or P -matrices) which arise naturally when modelling physical devices where electric networks interact through wave propagating phenomena. Typical examples are the so-called Surface Acoustic Wave (SAW) filters described in Section 5 and the subsequent appendix, see also [21] and the references therein. Mixed matrices have their own mathematical interest which justifies the following definition.

Definition 3.1 *A mixed matrix is a rational matrix function $\mathcal{M} \in \mathcal{H}_2$ with block structure*

$$\mathcal{M} = \begin{bmatrix} M & \alpha \\ \beta & Y \end{bmatrix}, \quad (12)$$

where M and Y are square matrices and the blocks M, α, β , and Y satisfy the following relations:

$$MM^* = I_p, \quad (13)$$

$$\alpha^* \alpha = \beta \beta^* = Y + Y^*, \quad (14)$$

$$\alpha = M \beta^*. \quad (15)$$

In mathematical terms, (13) means that M is lossless, (15) is a so-called Douglas-Shapiro-Shields factorization of α [11] (that need not be coprime), while (14) means that α is a spectral factor of the spectral density $Y + Y^*$ associated to the positive real function Y .

In the appendix, it will be explained how a mixed matrix \mathcal{M} , in the special case where $p = q = 2$, can represent the transfer function of a two-ports device where a piezoelectric plane medium in which acoustic waves propagate is used to connect two electrical circuits; in this case there will be two input waves (one at each port) stacked into a vector \mathbf{W}_i and two voltages stacked into a vector \mathbf{V} , together with two output waves and to currents stacked into vectors \mathbf{W}_o and \mathbf{I} respectively. Definition 3.1 allows for general and possibly unequal number of acoustical and electrical ports. In this analogy, the matrix M in (12) will be the scattering matrix (or diffraction matrix) of the acoustic waves, while the matrix Y is the electric admittance, and α , β are the electroacoustic matrices relating currents and waves. This can be capsulized as

$$\begin{bmatrix} \mathbf{W}_o \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} M & \alpha \\ \beta & Y \end{bmatrix} \begin{bmatrix} \mathbf{W}_i \\ \mathbf{V} \end{bmatrix}.$$

The incoming and outgoing waves \mathbf{W}_i and \mathbf{W}_o at the acoustical ports can be converted to equivalent “currents” and “voltages” \mathbf{J} and \mathbf{U} using the relations

$$\begin{bmatrix} \mathbf{W}_i \\ \mathbf{W}_o \end{bmatrix} = O_p \begin{bmatrix} \mathbf{U} \\ \mathbf{J} \end{bmatrix},$$

$$O_p = \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & I_p \\ I_p & -I_p \end{bmatrix}, \quad O_p = O_p^* = O_p^{-1}. \quad (16)$$

The *global admittance matrix* \mathcal{Z} relating electrical and acoustical currents and voltages is thus defined by:

$$\begin{bmatrix} \mathbf{J} \\ \mathbf{I} \end{bmatrix} = \mathcal{Z} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}. \quad (17)$$

Similarly, currents and voltages \mathbf{I} and \mathbf{V} can be converted to equivalent incoming and outgoing “waves” \mathbf{W}_i' and \mathbf{W}_o' through the transformation:

$$\begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = O_q \begin{bmatrix} \mathbf{W}_i' \\ \mathbf{W}_o' \end{bmatrix}, \quad (18)$$

and we may then define the *global scattering matrix* \mathcal{S} via the equation:

$$\begin{bmatrix} \mathbf{W}_o \\ \mathbf{W}_o' \end{bmatrix} = \mathcal{S} \begin{bmatrix} \mathbf{W}_i \\ \mathbf{W}_i' \end{bmatrix}. \quad (19)$$

The relation between the three matrices \mathcal{S} , \mathcal{Z} and \mathcal{M} is best expressed through the following factorization of the Cayley transform.

Proposition 1 *Let \mathcal{S} , \mathcal{Z} and \mathcal{M} be defined by (19), (17) and (12), respectively. Then \mathcal{S} and \mathcal{Z} are linear fractional transformations of \mathcal{M} :*

$$\mathcal{S} = T_{\Theta}(\mathcal{M}) = \begin{bmatrix} M - \alpha(Y + I_q)^{-1}\beta & \sqrt{2}\alpha(Y + I_q)^{-1} \\ -\sqrt{2}(Y + I_q)^{-1}\beta & (Y + I_q)^{-1}(I_q - Y) \end{bmatrix}, \quad (20)$$

$$\mathcal{Z} = T_{\Phi}(\mathcal{M}) = \begin{bmatrix} (M + I_p)^{-1}(I_p - M) & -\sqrt{2}(M + I_p)^{-1}\alpha \\ \sqrt{2}\beta(M + I_p)^{-1} & Y - \beta(M + I_p)^{-1}\alpha \end{bmatrix}, \quad (21)$$

and they are related by the Cayley transform Γ :

$$\mathcal{Z} = \Gamma(\mathcal{S}) = (I_{p+q} - \mathcal{S})(I_{p+q} + \mathcal{S})^{-1},$$

so that the diagram in Figure 1 commutes.

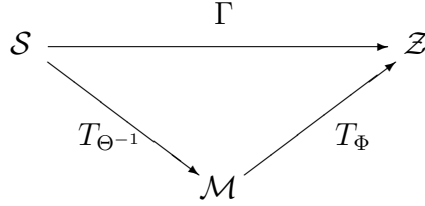


Figure 1: Factorization of the Cayley transform

Proof. Applying Lemma 1 to the relation

$$\begin{bmatrix} \mathbf{W}_o \\ \mathbf{I} \end{bmatrix} = \mathcal{M}K \begin{bmatrix} \mathbf{V} \\ \mathbf{W}_i \end{bmatrix},$$

we get upon using (18) and the properties (8), (9), (10) that

$$\begin{bmatrix} \mathbf{W}_o \\ \mathbf{W}_i \end{bmatrix} = T_{\Pi}(\mathcal{M}K) \begin{bmatrix} \mathbf{V} \\ \mathbf{I} \end{bmatrix} = T_{\Pi}(\mathcal{M}K) O_q \begin{bmatrix} \mathbf{W}_i' \\ \mathbf{W}_o' \end{bmatrix} = T_{\Sigma}(\mathcal{M}) \begin{bmatrix} \mathbf{W}_i' \\ \mathbf{W}_o' \end{bmatrix}$$

where

$$\Sigma = \begin{bmatrix} I_p & 0 \\ 0 & O_q \end{bmatrix} \Pi \begin{bmatrix} I_p & 0 \\ 0 & K \end{bmatrix}.$$

Applying Lemma 1 again yields

$$\begin{bmatrix} \mathbf{W}_{\circ}' \\ \mathbf{W}_{\circ} \end{bmatrix} = T_{\Pi}(T_{\Sigma}(\mathcal{M})) \begin{bmatrix} \mathbf{W}_{\mathbf{i}}' \\ \mathbf{W}_{\mathbf{i}} \end{bmatrix} = T_{\Pi\Sigma}(\mathcal{M})K \begin{bmatrix} \mathbf{W}_{\mathbf{i}} \\ \mathbf{W}_{\mathbf{i}}' \end{bmatrix} = T_{\Theta}(\mathcal{M}) \begin{bmatrix} \mathbf{W}_{\mathbf{i}} \\ \mathbf{W}_{\mathbf{i}}' \end{bmatrix},$$

where

$$\Theta = \begin{bmatrix} I_p & 0 \\ 0 & K \end{bmatrix} \Pi\Sigma.$$

The matrix \mathcal{S} is a linear fractional transformation of the mixed matrix \mathcal{M} :

$$\mathcal{S} = T_{\Theta}(\mathcal{M}), \quad (22)$$

where

$$\Theta = \left[\begin{array}{cc|cc} I_p & 0 & 0 & 0 \\ 0 & -I_q/\sqrt{2} & 0 & I_q/\sqrt{2} \\ \hline 0 & 0 & I_p & 0 \\ 0 & I_q/\sqrt{2} & 0 & I_q/\sqrt{2} \end{array} \right], \quad (23)$$

which gives (20).

Similarly, the matrix \mathcal{Z} turns out to be given by the linear fractional transformation

$$\mathcal{Z} = T_{\Phi}(\mathcal{M}), \quad (24)$$

where

$$\Phi = \left[\begin{array}{cc|cc} -I_p/\sqrt{2} & 0 & I_p/\sqrt{2} & 0 \\ 0 & I_q & 0 & 0 \\ \hline I_p/\sqrt{2} & 0 & I_p/\sqrt{2} & 0 \\ 0 & 0 & 0 & I_q \end{array} \right], \quad (25)$$

which gives (21).

Finally, it is immediately verified that

$$T_{\Phi} = \Gamma \circ T_{\Theta}.$$

□

Proposition 1 leads to the following equivalent characterizations of a mixed matrix. For a definition of the McMillan degree, we refer the reader for instance to [19, 4] and also to section 4 where we very briefly recall some basic facts.

Theorem 1 *Let \mathcal{M} be a rational block matrix as in (12). Then, the following assertions are equivalent.*

- (i) \mathcal{M} is a mixed matrix,
- (ii) the scattering matrix \mathcal{S} defined by (20) is lossless,
- (iii) the impedance matrix \mathcal{Z} defined by (21) is positive real and pure imaginary a.e. on the unit circle.

In this case the three matrices \mathcal{M} , \mathcal{S} and \mathcal{Z} have the same McMillan degree.

Proof. We first prove the equivalence between (i) and (ii). It is easily verified from (20) that if the blocks of \mathcal{M} are analytic outside the disc, then the blocks of \mathcal{S} are analytic in the same domain, and conversely. Let

$$\tilde{\mathcal{J}} = \begin{bmatrix} I_{p+q} & 0 \\ 0 & -I_{p+q} \end{bmatrix},$$

then

$$\mathcal{S}^* \mathcal{S} - I_{p+q} = \begin{bmatrix} \mathcal{S}^* & I_{p+q} \end{bmatrix} \tilde{\mathcal{J}} \begin{bmatrix} \mathcal{S} \\ I_{p+q} \end{bmatrix} = 0$$

on the unit circle. Using the relation

$$\begin{bmatrix} \mathcal{S} \\ I_{p+q} \end{bmatrix} = \Theta \begin{bmatrix} \mathcal{M} \\ I_{p+q} \end{bmatrix} X^{-1}, \quad (26)$$

where $X = \Theta_{21} \mathcal{M} + \Theta_{22}$, we obtain

$$\mathcal{S}^* \mathcal{S} - I_{p+q} = X^{-*} \begin{bmatrix} \mathcal{M}^* & I_{p+q} \end{bmatrix} \Theta^* \tilde{\mathcal{J}} \Theta \begin{bmatrix} \mathcal{M} \\ I_{p+q} \end{bmatrix} X^{-1},$$

and since

$$\Theta^* \tilde{\mathcal{J}} \Theta = \left[\begin{array}{cc|cc} I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_q \\ \hline 0 & 0 & -I_p & 0 \\ 0 & -I_q & 0 & 0 \end{array} \right]$$

we get

$$\mathcal{S}^* \mathcal{S} - I_p = X^{-*} \begin{bmatrix} M^* M - I_p & M^* \alpha - \beta^* \\ \alpha^* M - \beta & \alpha^* \alpha - Y - Y^* \end{bmatrix} X^{-1}.$$

It is now apparent that \mathcal{S} is lossless if, and only if, \mathcal{M} is a mixed matrix. Finally, the equivalence of (ii) and (iii) is a well-known property of the Cayley transform, see e.g. [22]. \square

The submatrix $S = (Y + I_q)(I_q - Y)^{-1}$ in the right-hand side of equation (20) is of particular significance from the point of view of signal processing, because it describes the electrical power transmission (see appendix) of the electro-acoustical device underlying the mixed matrix as described after Definition 3.1.

Now, the fact that the contractive matrix S embeds into the bigger lossless matrix \mathcal{S} means the latter is a solution to the *Darlington embedding* or *Darlington synthesis* problem for S [8, 3, 10, 2]. Given a contractive rational matrix, it is well known that a Darlington embedding always exists, and one can even preserve the McMillan degree in this extension process (in the non-rational case which is not a concern to us here, some extra conditions are needed).

In another connection, when the mixed matrix \mathcal{M} in (12) arises from such an electro-acoustical device (a typical example being a SAW filter as explained in Section 5 and its appendix), the physical law of reciprocity implies that the following additional relations hold [18]):

$$M = M^T \tag{27}$$

$$Y = Y^T \tag{28}$$

$$\beta = -\alpha^T. \tag{29}$$

This is equivalent to require the symmetry of the scattering matrix \mathcal{S} or equivalently of the admittance matrix \mathcal{Z} . To the author's knowledge, the

issue as to when and how a symmetric Darlington synthesis is possible for a given symmetric contractive rational matrix has not been much studied in the literature. From Network Theory it is known to be possible for rational functions with real Fourier coefficients, that are strictly contractive at infinity, but no sharp control on the McMillan degree [2] is apparently available. The forthcoming section is devoted to a particular aspect of this problem. It should be noted that the symmetry constraint is not the only one for electro-acoustical mixed matrices: for instance the delay in acoustic wave propagation is to the effect that the non-purely electrical transfer matrices M, α, β in (12) vanish at infinity (in the language of System Theory, one says they are *strictly proper*). Thus it seems that the specific Darlington embedding to be faced in this context includes both symmetry and interpolation constraints, although the latter will not be considered in the present paper.

4 Symmetric Darlington embedding.

In this section we will characterize, among those $(q \times q)$ symmetric contractive rational matrices S that are strictly contractive in at least one point of the unit circle, those who admit a $(p + q) \times (p + q)$ lossless extension \mathcal{S} of the same McMillan degree which is symmetric as well :

$$\mathcal{S} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S \end{pmatrix}, \quad \text{with } S_{1,1} = S_{1,1}^T \text{ and } S_{2,1} = S_{1,2}^T. \quad (30)$$

As an extra-piece of notation, we shall write

$$S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

whenever (A, B, C, D) is a *realization* of the rational matrix S , in other words whenever $S(s) = C(sI - A)^{-1}B + D$ where A, B, C, D are complex matrices of appropriate sizes. To emphasize that S in this case is the transfer function of the linear dynamical system:

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k,$$

with state x , input u , and output y , the matrix A is sometimes called a *dynamics matrix* of S .

Every rational matrix has infinitely many realizations, and a realization is said to be minimal if the matrix A has minimal size. As is well known [19, 4], this holds if and only if the so-called *Kalman criterion* is satisfied, that is if and only if the following two matrices are surjective:

$$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}, \quad \begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{n-1} C^T \end{bmatrix}, \quad (31)$$

where n denotes the size of A . The surjectivity of the first matrix is a property called *reachability*, and that of the second matrix is called *observability*. Any two minimal realizations can be deduced from each other by a linear change of coordinates:

$$(A, B, C, D) \mapsto (PAP^{-1}, PB, CP^{-1}, D), \quad P \text{ an invertible matrix,}$$

so that a dynamics matrix of minimal size is well defined up to similarity. As a consequence the size of A is independent of the particular minimal realization under consideration, and it can be taken as definition of the McMillan degree of S . The eigenvalues of A are likewise well-defined, and they are in fact the poles of S . If S is invertible as a rational matrix, the poles of its inverse are its zeros by definition (if S is not invertible, zeros have to be introduced differently, see [4]). Finally, a realization is called symmetric if $A = A^T$, $B^T = C$ and $D = D^T$. It is not too difficult to see that a symmetric transfer function (namely a rational matrix S analytic at infinity such that $S^T = S$) has a minimal symmetric realization [15].

For easier calculations we switch in this section to the right half-plane

$$\Pi^+ = \{s \in \mathbb{C}; \Re s > 0\}$$

rather than the complement of the disk. From the point of view of System Theory, the corresponding setting is that of continuous-time rather than discrete-time systems. The passage from one setting to the other is carried out through a simple Möbius transformation of the argument that maps contractive functions in \mathbb{E} to contractive functions in Π^+ (for a definition replace \mathbb{E} by Π^+ in equation (3)) and lossless functions in \mathbb{E} to lossless functions in

Π^+ (replace the unit circle by the imaginary axis in equation (4)). The next lemma ensures that this transformation preserves rationality and the McMillan degree, hence the results that we prove on the symmetric Darlington synthesis in Π^+ carry over immediately to \mathbb{E} .

Lemma 2 *A rational matrix valued function $S_d(z)$ which is contractive in \mathbb{E} gets mapped to a rational matrix valued function $S_c(z)$ which is contractive in Π^+ under the rule:*

$$S_c(s) = S_d\left(\frac{z-1}{z+1}\right).$$

In fact, this mapping defines a one-to-one correspondence between these classes of functions that preserves the McMillan degree and maps lossless functions onto lossless functions.

On the level of realizations, this correspondence becomes:

$$S_d \triangleq \left(\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array} \right) = \left(\begin{array}{c|c} (I + A_c)(I - A_c)^{-1} & \sqrt{2}(I - A_c)^{-1}B_c \\ \hline \sqrt{2}C_c(I - A_c)^{-1} & D_c + C_c(I - A_c)^{-1}B_c \end{array} \right),$$

and

$$S_c \triangleq \left(\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array} \right) = \left(\begin{array}{c|c} (A_d - I)(I + A_d)^{-1} & \sqrt{2}(I + A_d)^{-1}B_d \\ \hline \sqrt{2}C_d(I + A_d)^{-1} & D_d - C_d(I + A_d)^{-1}B_d \end{array} \right).$$

Moreover, (A_d, B_d, C_d, D_d) is a symmetric realization if and only if (A_c, B_c, C_c, D_c) is a symmetric realization.

Proof. This is a simple computation. □

We define the para-Hermitian conjugate in the half-plane by the formula:

$$B^\star(s) \triangleq B(-\bar{s})^\star. \tag{32}$$

Here we used \star instead of $*$ to avoid confusion with the para-Hermitian conjugate defined in (2). Note that $B^\star(j\lambda) = B(j\lambda)^\star$ on the imaginary axis

and that if B is a polynomial, then B^* is a polynomial whose zeros are reflected from those of B across the imaginary axis.

For getting an idea of the solution to our problem, we first consider the symmetric Darlington embedding of a scalar proper and rational contractive function to a 2×2 lossless function, that is we assume momentarily that $p = q = 1$. Put

$$S = \frac{P}{Q},$$

where P and Q are coprime polynomials such that $\deg\{P\} \leq \deg\{Q\}$, P is not identically zero, $|P(i\omega)| \leq |Q(i\omega)|$ for $\omega \in \mathbb{R}$, and Q has roots in the open left half-plane only. The McMillan degree of S is just the degree of Q in this case. As is easily checked, every lossless extension \mathcal{S} of S , where S and \mathcal{S} have the same degree, is of the form

$$\mathcal{S} = \frac{1}{Q} \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^* & -R^* \\ R & P \end{bmatrix} \begin{bmatrix} e^{j\theta_2} & 0 \\ 0 & 1 \end{bmatrix}$$

where $\theta_1, \theta_2 \in \mathbb{R}$ and R is a polynomial solution of degree at most $\deg\{Q\}$ to the spectral factorization problem:

$$PP^* + RR^* = QQ^* \tag{33}$$

whose solvability is ensured by the contractivity of S . Clearly the extension is symmetric if and only if $-e^{j\theta_1}R^* = e^{j\theta_2}R$, which is compatible with (33) if, and only if, all zeros of the polynomial

$$\mu \triangleq QQ^* - PP^* \tag{34}$$

have even multiplicities. As $1 - SS^* = \mu/(QQ^*)$, the zeros of μ are the zeros of $1 - SS^*$ augmented by the common zeros to P and Q^* and the common zeros to P^* and Q ; the latter of course are reflected from the former across the imaginary axis, counting multiplicities. Thus we see already in the scalar case that a degree-preserving symmetric Darlington embedding requires special conditions that can be rephrased as:

- (i) the zeros of $1 - SS^*$ have even multiplicities,
- (ii) each common zero to S and $(S^*)^{-1}$, if any, is common with even multiplicity.

Remark. Note that (i) is automatically fulfilled for those zeros located on the imaginary axis if any, so the condition really bears on the non-purely imaginary zeros. Note also that (ii) concerns those zeros of S , if any, whose reflexion across the imaginary axis is a pole of S ; by the coprimeness of P and Q , such zeros are never purely imaginary.

Our goal is to generalize this result to matrix valued contractive rational functions. Because μ in the preceding example was the formal denominator of the function $(1 - SS^*)^{-1}$, we start by constructing a realization of the latter. For this, it will be convenient to assume that S is strictly contractive at infinity, *i.e.* that $\|S(\infty)\| < 1$ where the norm of a matrix is the operator norm. If S is strictly contractive at some other frequency on the imaginary axis, then our results will apply after a Möbius transform. But if S is strictly contractive at no point of the imaginary axis, then our current approach runs into difficulties, and the corresponding investigations will be left for future work. So, let

$$S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \quad (35)$$

be a minimal realization of S . The strict contractivity at infinity means that $I - D^*D$ and $I - DD^*$ are positive definite. Therefore we may set

$$\Psi = A + BD^*(I - DD^*)^{-1}C, \quad (36)$$

$$\Xi = B(I - D^*D)^{-1}B^*, \quad (37)$$

$$\Upsilon = C^*(I - DD^*)^{-1}C, \quad (38)$$

and subsequently we define:

$$\mathcal{A} = \begin{bmatrix} -\Psi^* & -\Upsilon \\ \Xi & \Psi \end{bmatrix}. \quad (39)$$

Lemma 3 *Assuming S given by (35) is contractive and strictly contractive at infinity, then the matrix \mathcal{A} defined in (39) is a dynamics matrix of $(I - SS^*)^{-1}$. Furthermore \mathcal{A} is Hamiltonian, *i.e.* $\mathcal{A}^* = L\mathcal{A}L$ where L was defined in (6).*

Proof. By definition

$$S^* = \left(\begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array} \right),$$

then

$$\Phi = I - SS^* = \left(\begin{array}{cc|c} -A^* & 0 & C^* \\ \hline BB^* & A & -BD^* \\ \hline DB^* & C & I - DD^* \end{array} \right),$$

and if S is strictly contractive at infinity the inverse of $I - DD^*$ is well defined. Then a straightforward computation shows that

$$\Phi^{-1} = \left(\begin{array}{cc|c} -A^* - C^* \Delta_l DB^* & -C^* \Delta_l C & C^* \Delta_l \\ \hline B \Delta_r B^* & A + BD^* \Delta_l C & -BD^* \Delta_l \\ \hline -\Delta_l DB^* & -\Delta_l C & \Delta_l \end{array} \right),$$

where $\Delta_l = (I - DD^*)^{-1}$ and $\Delta_r = (I - D^*D)^{-1}$, and whose dynamics matrix is none but \mathcal{A} .

Finally, it is easy to check from the definitions (36)-(38) that \mathcal{A} is a Hamiltonian matrix, i.e. that the partition of \mathcal{A} defined in (39) satisfies $\mathcal{A}_{12}^* = \mathcal{A}_{12}$, $\mathcal{A}_{21}^* = \mathcal{A}_{21}$, and $\mathcal{A}_{22}^* = -\mathcal{A}_{11}$. \square

Remark. Because $L^2 = -I$, the Hamiltonian character of \mathcal{A} implies that it is conjugate to $-\mathcal{A}^*$. In particular the eigenvalues of \mathcal{A} are symmetric with respect to the imaginary axis, counting multiplicities. It must also be stressed that the realization of $\Phi^{-1} = (I - SS^*)^{-1}$ given in the proof of Lemma 3 may not be minimal. Because the McMillan degree is invariant upon taking the inverse for rational functions whose value at infinity is invertible, the realization in question will in fact be minimal if, and only if, the McMillan degree of SS^* is the sum of the McMillan degrees of S and S^* . This will hold in particular when no zero of S is a pole of S^* [4], in other words if no zero of S is reflected from one of its poles. Hence the characteristic polynomial of \mathcal{A} plays in the matrix valued case the role of the polynomial μ given by (34) in the scalar case (compare condition (ii) after (34)).

Our point of departure will be the solution to the Darlington embedding problem for rational functions in terms of realizations. Actually, the lossless extensions of S , without the symmetry condition, are characterized by the following theorem borrowed from [16] and adapted to our right half plane setting.

Theorem 2 (*Theorem 4.1 in [16]*)

Assuming S given by (35) is contractive and strictly contractive at infinity, then all minimal lossless extensions

$$\mathcal{S} = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S \end{pmatrix}$$

of the same McMillan degree as S are given by

$$\mathcal{S} = \begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix} \mathcal{S}_P \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \quad (40)$$

where U_1 and U_2 are arbitrary unitary matrices and where \mathcal{S}_P is given by

$$\mathcal{S}_P = \left(\begin{array}{c|c} A & \mathcal{B} \\ \hline C & \mathcal{D} \end{array} \right) = \left(\begin{array}{c|cc} A & b & B \\ \hline c & d_{11} & d_{12} \\ C & d_{21} & D \end{array} \right), \quad (41)$$

with

$$d_{21} = (I - DD^*)^{1/2}, \quad d_{12} = (I - D^*D)^{1/2}, \quad d_{11} = -D^*, \quad (42)$$

$$c = -(I - D^*D)^{-1/2}(B^*P^{-1} + D^*C), \quad (43)$$

$$b = -(PC^* + BD^*)(I - DD^*)^{-1/2}, \quad (44)$$

and P is a Hermitian solution to the algebraic Riccati equation:

$$\mathcal{R}(P) = P\Upsilon P + \Psi P + P\Psi^* + \Xi = 0. \quad (45)$$

Remark 4.1 We note that Theorem 3.4 in [16] guarantees that, under the assumptions of Theorem 2, all Hermitian solutions of (45) are invertible.

In the case we are interested in, the function S is symmetric i.e. $S = S^T$. Accordingly, we assume from now on that (35) is a symmetric realization, i.e. that $A = A^T$, $B = C^T$ and $D = D^T$. Such a realization always exists thanks to Theorem 5 in [15]. Then, the realization of \mathcal{S} arising from (40) and (41) is in turn symmetric if, and only if, we have that

$$\begin{pmatrix} U_2 d_{11} U_1 & U_2 d_{12} \\ d_{21} U_1 & D \end{pmatrix} \text{ is symmetric, and } (bU_1)^T = U_2 c. \quad (46)$$

Taking into account that $A = A^T$, $B = C^T$, and $D = D^T$, we deduce from (42)–(45) upon writing

$$(bU_1)^T = -U_1^T(I - D^*D)^{-1/2}(B^*P^T + D^*C)$$

that (46) is equivalent to:

$$U_1^T = U_2 \quad \text{and} \quad B^*P^T = B^*P^{-1}, \quad (47)$$

where P is given by (45).

Lemma 4 *Assume that P solves the Riccati equation (45) generated by the symmetric minimal realization (35) of S . Then $B^*P^T = B^*P^{-1}$ if and only if $P^T = P^{-1}$.*

Proof. From [16] it follows that P is invertible, see Remark 4.1, and then

$$(P^{-1}\mathcal{R}(P)P^{-1})^T - \mathcal{R}(P) = \Psi(P^{-T} - P) + (P^{-T} - P)\Psi^* + P^{-T}\Upsilon P^{-T} - P\Upsilon P = 0,$$

and using that $B^*P^T = B^*P^{-1}$ implies $CP^{-T} = CP$ we get $P\Upsilon P = P^{-T}\Upsilon P^{-T}$ and $\Psi(P^{-T} - P) = A(P^{-T} - P)$, so we end up with

$$A(P^{-T} - P) + (P^{-T} - P)A^* = 0. \quad (48)$$

We know that

$$C(P^{-T} - P) = 0.$$

Assuming that $CA^k(P^{-T} - P) = 0$, it follows from (48) that

$$CA^{k+1}(P^{-T} - P) = -CA^k(P^{-T} - P)A^* = 0,$$

and by induction we have that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} (P^{-T} - P) = 0.$$

By minimality (31) holds, hence the above equation implies that $P = P^{-T}$, as desired. \square

We can now state:

Proposition 4.1 *Given a contractive rational symmetric matrix S which is strictly contractive at infinity, there exists a symmetric lossless extension \mathcal{S} of S of the form (30) having the same McMillan degree as S if, and only if, given a minimal symmetric realization (A, B, C, D) of S , there is a Hermitian solution P to the algebraic Riccati equation (45) with coefficients given by (36)-(38) that satisfies $PP^T = I$. Moreover, all such extension are parameterized by (40), where $U_1 = U_2^T$, and where P is a Hermitian solution of the Riccati equation (45) such that $PP^T = I$.*

Proof. From Lemma 4 and the analysis above, it is clear that the conditions of the proposition are sufficient. To see that these conditions are also necessary, observe from Theorem 2 that any lossless extension \mathcal{S} of S as in (30) such that \mathcal{S} and S have the same McMillan degree is of the form (40) where U_1, U_2 are unitary and where \mathcal{S}_P is given by (41)-(44) where P is a Hermitian solution to (45). Since D is symmetric, it follows easily from (42) that \mathcal{D} is symmetric if, and only if $U_1 = U_2^T$; and since A is symmetric while $B^T = C$ and the realization (41) is minimal, \mathcal{S} is symmetric if, and only if, we have $c^T = b$. In view of (43)-(44) this is equivalent to $B^*P^T = B^*P^{-1}$. From Lemma 4 it now follows that $P^T = P^{-1}$ as desired. \square

Proposition 4.1 stands in analog to Theorem 2 for the case of lossless extensions of a symmetric matrix that are themselves symmetric. However, it is not satisfactory in that it gives no practical means to check the existence of a Hermitian solution to (45) satisfying $P^T = P^{-1}$. To obtain a criterion, we need to investigate more deeply the structure of the Riccati equation, and this is the object of the next section.

4.1 Solutions to the Riccati equation

Given a minimal realization (A, B, C, D) of the symmetric contractive matrix S which is strictly contractive at infinity, let us put

$$\mathcal{G} = j\mathcal{A}, \tag{49}$$

where \mathcal{A} was defined in (39). It is well-known, and easy to check, that the Riccati equation (45) has a solution P , if and only if the graph space of P is

\mathcal{G} invariant, i.e. there exists a matrix X such that

$$\mathcal{G} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} X. \quad (50)$$

This can be used to prove a necessary condition on the eigenvalues of \mathcal{G} for a lossless symmetric extension to exist, which parallels conditions (i)-(ii) after (34).

Lemma 5 *If there exists a symmetric lossless extension of S of the same McMillan degree, then the characteristic polynomial of \mathcal{G} is of the form*

$$\chi_{\mathcal{G}}(s) = \Pi(s)^2, \quad (51)$$

where Π is a polynomial with real coefficients. This is equivalent to saying that the eigenvalues of \mathcal{G} all have even algebraic multiplicities and occur in complex conjugate pairs.

Furthermore, \mathcal{G} is then similar to a matrix of the form:

$$\begin{bmatrix} X & Y \\ 0 & X \end{bmatrix}. \quad (52)$$

Proof. If there exists a symmetric lossless extension, we know from Proposition 4.1 that there exists a solution P to the Riccati equation (45) such that $P^T = P^{-1}$.

Translating \mathcal{G} to a new basis using

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix},$$

we get

$$T^{-1}\mathcal{G}T = \begin{bmatrix} X & -j\Upsilon \\ 0 & X^* \end{bmatrix} \quad (53)$$

where $X = -j(\Psi^* + \Upsilon P)$. From (53) we see that $\sigma(\mathcal{G}) = \sigma(X) \cup \sigma(X^*)$ where σ indicates the spectrum. Next we show that X^* is similar to X^T . Using $\mathcal{R}(P) = 0$ and $P^T = P^{-1}$, we obtain:

$$PXP^{-1} = -j(P\Psi^* + P\Upsilon P)P^{-1} = j(\Psi P + \Xi)P^{-1} = j(\Psi + \Xi P^T),$$

and using $\Psi = \Psi^T, \Xi^T = \Upsilon = \Upsilon^*$ we deduce that

$$(PXP^{-1})^T = j(\Psi^T + P\Xi^T) = j(\Psi + P\Upsilon) = X^*.$$

The blocktriangular form (52) now follows from (53), and since X is similar to X^* the roots of $\Pi(s) = \chi_X(s)$ appear in complex conjugate pairs so that (51) holds. \square

Remark. We knew already from the Remark after Lemma 3 that the eigenvalues of \mathcal{G} appear in conjugate pairs, thus the new fact that we learn from (5) is that they have even multiplicities.

To show that (51) is also sufficient for the existence of a lossless symmetric extension of the same degree, we need to consider not only the eigenvalues but also the spectral subspaces of \mathcal{G} . As is well-known, and easy to check from (50) (see *e.g.* Theorem 7.1.2 in [20]), the solutions of the Riccati equation (45) are in a one-one correspondence with the n -dimensional invariant subspaces of \mathcal{G} that are graph subspaces; here n is the McMillan degree of S and therefore the size of A , whereas \mathcal{A} and thus \mathcal{G} have dimension $2n$. Since we are looking for a particular solution of that equation, namely one that satisfies $P^* = P$ and $P^T = P^{-1}$, we need to find out which properties of the associated invariant subspace ensure this. The three relevant notions that we need are introduced below (recall the definition (6) of the matrices J, K, L).

A subspace $\mathcal{V} \subset \mathbb{C}^{2n}$ will be called *\mathcal{G} -invariant* if $\mathcal{G}\mathcal{V} \subset \mathcal{V}$.

A subspace $\mathcal{V} \subset \mathbb{C}^{2n}$ will be called *L -neutral* if $x^*Ly = 0$ for all $x, y \in \mathcal{V}$.

A subspace $\mathcal{V} \subset \mathbb{C}^{2n}$ will be called *symmetric* if for the partitioning $\mathbb{C}^{2n} \sim \mathbb{C}^n \times \mathbb{C}^n$, it holds that

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{V} \quad \text{implies} \quad \begin{bmatrix} \bar{v} \\ \bar{u} \end{bmatrix} \in \mathcal{V}. \quad (54)$$

As we shall see shortly, the key properties that a subspace must satisfy in order to generate a Hermitian solution to (45) satisfying $P^T = P^{-1}$ are the following.

Key properties:

- (i) \mathcal{G} -invariance,
- (ii) L -neutrality, (55)
- (iii) symmetry.

Property (i) is needed to associate a solution of (45) to \mathcal{V} as pointed out already. It leads us to study the Jordan chains of \mathcal{G} in Lemma 7 below.

Property (ii) ensures the Hermitian character of the solution to (45) and is studied in Lemma 9 and Lemma 10.

Property (iii) is connected to the orthogonal character of the solution to (45) (*i.e.* the fact that $P^T = P^{-1}$) and therefore to the symmetry of the lossless extension by Proposition 4.1. In this connection, we have the following:

Lemma 6 *Assume that P is a Hermitian matrix. Then the graph subspace of P is symmetric if and only if $P^T = P^{-1}$.*

Proof. Assume that the graph subspace of P is symmetric, then

$$\begin{bmatrix} \bar{P} \\ I \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} X,$$

for some X , *i.e.* $X = \bar{P}$ and $P\bar{P} = I$. Since P is Hermitian, $P^T = P^{-1}$.

In the other direction, assume $P^T = P^{-1}$ and

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} x.$$

Then

$$\begin{bmatrix} \bar{v} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \bar{P}\bar{x},$$

which concludes the proof. □

A Jordan chain of a matrix \mathcal{G} corresponding to an eigenvalue λ of \mathcal{G} is a set of vectors $\{w_k\}_{k=1}^\ell$ such that $w_1 \neq 0$ and

$$(\mathcal{G} - \lambda I) w_1 = 0, \quad \text{and} \quad (\mathcal{G} - \lambda I) w_k = w_{k-1}, \quad k = 2, \dots, \ell. \quad (56)$$

It is clear that every \mathcal{G} -invariant subspace is spanned by some Jordan chains of \mathcal{G} and conversely.

Lemma 7 *Assume \mathcal{G} defined by (36,37,38,39,49) arises from a symmetric realization (A, B, C, D) of S , and that $\{w_k\}_{k=1}^\ell$ is a Jordan chain of \mathcal{G} corresponding to the eigenvalue λ , where*

$$w_k = \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \quad u_k, v_k \in \mathbb{C}^n.$$

Then,

$$\left\{ \begin{bmatrix} \bar{v}_k \\ \bar{u}_k \end{bmatrix} \right\}_{k=1}^\ell$$

is a Jordan chain of \mathcal{G} corresponding to the eigenvalue $\bar{\lambda}$.

Proof. From the symmetric realization it follows that $\Psi^T = \Psi$, $\Xi^T = \Upsilon$, and thus

$$\mathcal{G}^T = J\mathcal{G}J. \quad (57)$$

By Lemma 3 \mathcal{A} is Hamiltonian, i.e. $\mathcal{A} = L\mathcal{A}^*L$, and therefore $\mathcal{G} = -L\mathcal{G}^*L$. Using this together with (57) and the fact that $JL + LJ = 0$, $K = JL$, we get $\mathcal{G} = K\bar{\mathcal{G}}K$.

From (56) and $K^2 = I$ we derive that

$$(K\mathcal{G}K - \lambda I)Kw_k = Kw_{k-1}, \quad W_0 = 0.$$

Taking the complex conjugate we obtain

$$(\mathcal{G} - \bar{\lambda}I)K\bar{w}_k = K\bar{w}_{k-1},$$

which proves the lemma. □

Lemma 8 *If (A, B, C, D) is a minimal realization of S and $I - DD^* > 0$, then (Ψ, Ξ) is reachable and (Υ, Ψ) is observable, where Ψ , Ξ , and Υ are defined by (36)-(38).*

Proof. By minimality, the Kalman criterion (31) say that (A, B) is reachable and (C, A) is observable. Feedback invariance of the reachability property ensures that $(\Psi, B) = (A + BD^*(I - DD^*)^{-1}C, B)$ is also reachable. Let $\tilde{B} = B(I - DD^*)^{-1/2}$, then (Ψ, \tilde{B}) is also reachable and $(\Psi, \Xi) = (\Psi, \tilde{B}\tilde{B}^*)$ is in turn reachable.

Observability of (C, A) implies reachability of (A^*, C^*) . Therefore, using the same arguments as before, $(\Psi^*, C^*) = (A^* + C^*(I - DD^*)^{-1}DB^*, C^*)$ and (Ψ^*, Υ) are reachable. Since $\Upsilon = \Upsilon^*$, (Υ, Ψ) is observable. \square

Granted the observability of (Υ, Ψ) from Lemma 8 and the fact that $\Upsilon \geq 0$; Lemma 7.2.2 in [20] guarantees that every \mathcal{G} -invariant and L -neutral subspace of \mathbb{C}^{2n} is a graph subspace.

We state two lemmatas about J -orthogonality and L -neutrality before we come to the main result in this section.

Lemma 9 *Assume that G is a J -symmetric matrix, i.e. $JG = G^T J$, and let $\{x_k\}_{k=1}^\mu$ and $\{y_k\}_{k=1}^m$ be Jordan chains corresponding to the eigenvalues λ_x and λ_y .*

Then the following J -orthogonality relations holds:

- (i) $y_k^T J y_\ell = 0$ for all k, ℓ such that $k + \ell \leq m$.
- (ii) if $\lambda_x \neq \lambda_y$, then $x_k^T J y_\ell = 0$ for $k = 1, \dots, \mu$ and $\ell = 1, \dots, m$.
- (iii) if $\lambda_x = \lambda_y$, then $x_k^T J y_\ell = 0$ if $k + \ell \leq \max\{m, \mu\}$.
- (iv) Moreover, a symmetric subspace \mathcal{V} is J -isotropic, i.e. $x^T J y = 0$ for all $x, y \in \mathcal{V}$, if and only if it is L -neutral.

Proof. By assumption; $Gx_1 = \lambda_x x_1$ and $(G - \lambda_x I)x_k = x_{k-1}$ for $k = 2, \dots, \mu$, and $Gy_1 = \lambda_y y_1$ and $(G - \lambda_y I)y_\ell = y_{\ell-1}$ for $\ell = 2, \dots, m$. Assume that $k + \ell \leq m$, then $y_k = (G - \lambda_y I)^\ell y_{k+\ell}$ and

$$y_k^T J y_\ell = y_{k+\ell}^T (G^T - \lambda_y I)^\ell J y_\ell = y_{k+\ell}^T J (G - \lambda_y I)^\ell y_\ell = 0,$$

which proves (i).

If we define $x_0 = y_0 = 0$, then

$$\begin{aligned}
\lambda_x(x_k^T J y_\ell) &= x_k^T G^T J y_\ell - x_{k-1}^T J y_\ell \\
&= x_k^T J G y_\ell - x_{k-1}^T J y_\ell \\
&= \lambda_y(x_k^T J y_\ell) + x_k^T J y_{\ell-1} - x_{k-1}^T J y_\ell.
\end{aligned} \tag{58}$$

Note that the last two terms vanish if ℓ and k are one. Hence x_1 and y_1 are J -orthogonal. The result now follows by induction: first let $k = 1$, then (58) shows that x_1 and y_ℓ are J -orthogonal. By symmetry x_k and y_1 are J -orthogonal. Induction on both k and ℓ now proves (ii).

Assume that $k + \ell \leq \mu$, then $x_k = (G - \lambda_x I)^\ell x_{k+\ell}$ and

$$x_k^T J y_\ell = x_{k+\ell}^T (G^T - \lambda_x I)^\ell J y_\ell = x_{k+\ell}^T J (G - \lambda_y I)^\ell y_\ell = 0,$$

and upon exchanging the roles of x_k and y_ℓ this proves (iii).

Assume finally that $x, y \in \mathcal{V}$ where \mathcal{V} is symmetric. Since \mathcal{V} is symmetric, we know that $y = K\bar{z}$ for some $z \in \mathcal{V}$. Then $x^* Ly = x^* LK\bar{z} = -x^T Jz$, and therefore \mathcal{V} is L -neutral if and only if it is J -isotropic.

□

Lemma 10 *Assume that G is a L -Hermitian matrix, i.e. $LG = G^*L$, and let $\{x_k\}_{k=1}^\mu$ and $\{y_k\}_{k=1}^m$ be Jordan chains corresponding to the eigenvalues λ_x and λ_y .*

If $\bar{\lambda}_x \neq \lambda_y$, then $x_k^ Ly_\ell = 0$, $k = 1, \dots, \mu$, $\ell = 1, \dots, m$.*

In particular, a spectral subspace $\mathcal{V} = \ker(G - \lambda I)^t$, where $t \in \{1, \dots, n\}$, corresponding to a non-real eigenvalue λ of G is L -neutral.

Proof. By assumption; $Gx_1 = \lambda_x x_1$ and $(G - \lambda_x I)x_k = x_{k-1}$ for $k = 2, \dots, \mu$, and $Gy_1 = \lambda_y y_1$ and $(G - \lambda_y I)y_\ell = y_{\ell-1}$ for $\ell = 2, \dots, m$.

If we define $x_0 = y_0 = 0$, then

$$\begin{aligned}
\bar{\lambda}_x(x_k^*Ly_\ell) &= x_k^*G^*Ly_\ell - x_{k-1}^*Ly_\ell \\
&= x_k^*LGy_\ell - x_{k-1}^*Ly_\ell \\
&= \lambda_y(x_k^*Ly_\ell) + x_k^*Ly_{\ell-1} - x_{k-1}^*Ly_\ell.
\end{aligned} \tag{59}$$

The result follows by induction as in the proof of Lemma 9 (ii).

□

The main technical step in characterizing symmetric lossless extensions is now the following proposition.

Proposition 2 *A symmetric contractive rational function S which is strictly contractive at infinity has a symmetric lossless extension of the same McMillan degree if, and only if, the characteristic polynomial of \mathcal{G} can be written as*

$$\chi_{\mathcal{G}}(s) = \Pi(s)^2 \tag{60}$$

where Π is a polynomial, and in addition all Jordan blocks corresponding to real eigenvalues of \mathcal{G} have even size.

Proof. The necessity of the conditions follows from Lemma 5 and Corollary 7.3.4. in [20].

To prove sufficiency, we will construct a \mathcal{G} -invariant, L -neutral and symmetric subspace \mathcal{V} of dimension n which is a graph subspace. As in the proof of Lemma 7, it holds that $\mathcal{G} = J\mathcal{G}^TJ$ and $L\mathcal{G} = \mathcal{G}^*L$, so we can use Lemmata 10 and 9 that are fundamental to the following construction.

Assume the necessary conditions of the proposition hold. The set of all Jordan chains of \mathcal{G} span \mathbb{C}^{2n} , and we will choose exactly half the vectors in the Jordan chains corresponding to each eigenvalue of \mathcal{G} so that these span a \mathcal{G} -invariant subspace \mathcal{V} of dimension n ; we shall make this choice in the following manner that keeps track of symmetry and L -neutrality.

First take a maximal Jordan chain $\{x_k\}_{k=1}^m$ corresponding to a real eigenvalue of \mathcal{G} if any. By assumption it has even length. By Lemma 7, there is either

another Jordan chain $\{K\bar{x}_k\}_{k=1}^m$ corresponding to the same eigenvalue, or $x_k = K\bar{x}_k$. In the first case, let $\{x_k\}_{k=1}^{m/2}$ and $\{K\bar{x}_k\}_{k=1}^{m/2}$ be basis vectors for \mathcal{V} , and in the second case let $\{x_k\}_{k=1}^{m/2}$ be basis vectors for \mathcal{V} . It is easy to check that these vectors form a symmetric subspace and by Lemma 9 (i), (iii) and (iv) we see that it is L -neutral.

Repeat this construction for all the other Jordan chains corresponding to real eigenvalues. The union of these are still symmetric and by Lemma 9 (ii), (iii) and (iv) we see that it is L -neutral.

If all eigenvalues are real, we are done. Otherwise take a maximal Jordan chain $\{x_k\}_{k=1}^m$ corresponding to an eigenvalue λ of \mathcal{G} , say with strictly positive imaginary part (remember the eigenvalues occur in conjugate pairs by the Remark after Lemma 3). By Lemma 7, there exists another Jordan chain $\{K\bar{x}_k\}_{k=1}^m$ corresponding to the eigenvalue $\bar{\lambda}$.

If m is even, let $\{x_k\}_{k=1}^{m/2}$ and $\{K\bar{x}_k\}_{k=1}^{m/2}$ be basis vectors for \mathcal{V} .

If m is odd, there exist another Jordan chain $\{y_k\}_{k=1}^\mu$ corresponding to λ where μ is also odd. (Since the algebraic degree of λ is even.) Now let $\{x_k\}_{k=1}^{(m+1)/2}$, $\{y_k\}_{k=1}^{(\mu-1)/2}$, $\{K\bar{x}_k\}_{k=1}^{(m+1)/2}$ and $\{K\bar{y}_k\}_{k=1}^{(\mu-1)/2}$ be basis vectors for \mathcal{V} . It is easy to check that these vectors form a symmetric subspace, by Lemma 9 (ii) and (iv) and Lemma 10 they span a L -neutral subspace and we see that adding these vectors to the basis of \mathcal{V} gives a L -neutral subspace.

Repeat this construction for all the remaining Jordan chains. The total span which is obtained is still symmetric, and by Lemma 9 (ii),(iv), and Lemma 10, we see that it is L -neutral.

Thus a \mathcal{V} with the desired properties (55) is now constructed. From Lemma 8 we know that (Υ, Ψ) is observable, and since $\Upsilon \geq 0$ we can use Lemma 7.2.2 in [20] to guarantee that \mathcal{V} is the graph space of some matrix P . Since \mathcal{V} is L -neutral,

$$\begin{bmatrix} I \\ P \end{bmatrix}^* L \begin{bmatrix} I \\ P \end{bmatrix} = -P^* + P = 0,$$

and P is Hermitian. Then, since \mathcal{V} is symmetric, it follows from Lemma 6 that $PP^T = I$, and the proposition now follows from Proposition 4.1. \square

It turns out that Proposition 2 is still not optimal in that the hypotheses are somewhat redundant. In fact, for those matrices \mathcal{G} generated through (35)-(39) and (49) by a contractive rational S (that may be symmetric or not), the Jordan blocks corresponding to real eigenvalues of \mathcal{G} will automatically have even size if these eigenvalues have even algebraic multiplicities. This we state as a final lemma in this section.

Lemma 11 *If S is a contractive, and strictly contractive at infinity, rational function of which (A, B, C, D) is a minimal realization, then all real eigenvalues of \mathcal{G} defined by (36,37,38,39,49) have even partial multiplicities.*

Proof. The proof is based on an arbitrary small translation of the right half plane along the real axis and a limiting argument.

Let $\Sigma_\lambda(s) = S(s + \lambda)$, where λ is real and positive. Then, Σ_λ is strictly contractive at infinity with minimal realization $(A - \lambda I, B, C, D)$ and we claim that the corresponding matrix \mathcal{G}_λ defined by (36,37,38,39,49) with A replaced by $A - \lambda I$ has no real eigenvalues for λ sufficiently small. Indeed, we know from (49) and the remark after Lemma 3 that the eigenvalues of $-j\mathcal{G}_\lambda$ are the zeros of $I - \Sigma_\lambda(s)\Sigma_\lambda^*(s) = (I - SS^*)(s + \lambda)$ augmented with the zeros of Σ_λ that are the reflexion of one of its poles if any. But the zeros of $I - SS^*$ form a discrete set because it is an invertible rational matrix (since we assumed it is strictly positive at infinity) that cannot intersect every strip $(0, \lambda)$ for all sufficiently small λ ; therefore $(I - SS^*)(j\omega + \lambda)$ has no zero, say for $0 < \lambda < \lambda_0$, that is to say $I - \Sigma_\lambda \Sigma_\lambda^*$ has no purely imaginary zero for $0 < \lambda < \lambda_0$. Also, the zeros of Σ_λ are translated from those of S by the quantity $-\lambda$ whereas the reflexions of the poles of Σ_λ are translated from those of S by the quantity λ . Since S is full rank as a rational matrix (because by continuity $I - SS^*$ is strictly contractive if we get close enough to infinity along the imaginary axis), its zeros are isolated points and so are its poles. Therefore, for all sufficiently small $\lambda > 0$, no zero of Σ_λ is reflected from one of its poles. Altogether, *this proves the claim.*

Now, by Corollary 7.3.4. in [20] there exists a Hermitian solution P_λ of the algebraic Riccati equation corresponding to Σ_λ for every $\lambda > 0$, and then by Theorem 2 there exists a lossless extension Σ_λ of Σ_λ corresponding to P_λ .

Since the class of lossless and rational functions of McMillan degree at most n is compact in the weak-* topology of H^∞ [1], the limit

$$\mathcal{S} = \lim_{\lambda \rightarrow 0} \Sigma_\lambda \tag{61}$$

is a lossless and rational extension of S with the same McMillan degree (because it cannot be strictly smaller). This extension defines a Hermitian solution to the algebraic Riccati equation (45), and then by using Corollary 7.3.4. in [20] again, all the partial multiplicities of the real eigenvalues of \mathcal{G} are even. \square

Note that the previous lemma did not mention symmetry, but it shows that in Proposition 2, the assumption on real eigenvalues is automatically satisfied. In the next section, we state our results in final form.

4.2 Symmetric lossless extensions.

The developments of the preceding section enable us to prove the following result.

Theorem 3 *A symmetric contractive rational function S of size $p \times p$ which is strictly contractive at infinity has a symmetric lossless extension of the same McMillan degree of size $(p + m) \times (p + m)$ if, and only if, the characteristic polynomial of \mathcal{G} can be written*

$$\chi_{\mathcal{G}}(s) = \Pi(s)^2 \tag{62}$$

for some polynomial Π , where \mathcal{G} is given by (49) and (36)-(39).

Proof. This follows from Proposition 2 and Lemma 11. \square

From the partial fraction expansion of a transfer function [4, 7] and the characterization of the multiplicity of a pole as the rank of the block Toeplitz matrix associated with the coefficients of the singular part of the Laurent expansion at that point [4], we obtain the following corollary to Theorem 3.

Corollary 1 *A symmetric contractive rational function S no zero of which is the reflection of a pole across the imaginary axis, and which is strictly contractive in at least one point on the imaginary axis, has a symmetric lossless extension of the same McMillan degree if, and only if, $(I - SS^*)^{-1}$ has a partial fraction expansion*

$$(I - SS^*)^{-1} = \sum_{k=1}^m \sum_{\ell=1}^{\ell_k} \frac{S_{k,\ell}}{(z - \lambda_k)^\ell} \quad (63)$$

where all $S_{k,\ell}$ are constant matrices, all λ_k are distinct, and

$$\text{rank} \begin{bmatrix} S_{k,\ell_k} & S_{k,\ell_k-1} & \cdots & S_{k,1} \\ 0 & S_{k,\ell_k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & S_{k,\ell_k-1} \\ 0 & \cdots & 0 & S_{k,\ell_k} \end{bmatrix} \quad (64)$$

is even for every λ_k that is not purely imaginary (if λ is imaginary the condition will be satisfied automatically).

Proof. If no zero of S is a pole of S^* , then no zero of S^* is a pole of S either (they are obtained by reflection across the imaginary axis), and therefore the McMillan degree of SS^* is the sum of the McMillan degree of S and the McMillan degree of S^* [4], namely it is equal to $2n$. Hence the realization of $(I - SS^*)^{-1}$ in Lemma 3 is minimal (see the remark after this lemma), and the poles of this transfer function are precisely the eigenvalues of \mathcal{A} . Therefore the evenness of the rank in (64) is equivalent to the assumptions of Theorem 3 in case S is strictly contractive a infinity. And if it is not, since we know it is strictly contractive at some other point of the imaginary axis, we can send the latter at infinity by an appropriate Möbius transform. \square

Let us conclude this section with some comments and open questions.

First of all, necessary and sufficient conditions for the existence of a degree-preserving symmetric Darlington embedding of a symmetric rational contractive S were given only when S is strictly contractive at some point of the imaginary axis. If S is never strictly contractive on the latter, $(I - SS^*)^{-1}$

is not defined, and the previous characterization is meaningless. The extension issue, though still makes perfectly good sense, but to tackle it we must characterize the zeros differently (*i.e.* not as the poles of the inverse), and keep the distinction between generic and non-generic zeros. This is left for future work.

Next, it is natural to ask whether a symmetric and lossless extension is always possible, at the cost perhaps of increasing the McMillan degree, and what is the minimal increase of the degree which is incurred in such a process. For conjugate symmetric functions (*i.e.* those whose Fourier coefficients are real) that are contractive and strictly contractive at infinity, it is known that a symmetric lossless extension exists [2] but the minimal degree is apparently not known, and the complex case seems to be open in every respect. It is to be hoped that the preceding results will help advancing towards the solution of this problem.

5 Mathematical structure of a SAW filter.

We consider a SAW (surface acoustic wave) filter with two acoustical ports and two electrical ports. A concrete example of which will be given in the appendix. As explained in Section 3, the filter is described by a mixed matrix

$$\mathcal{M} = \begin{bmatrix} M & \alpha \\ \beta & Y \end{bmatrix},$$

where M , α , β and Y are (2×2) matrices analytic in \mathbb{E} and satisfying equations (27,28,29,13,14,15). In this section, $p = q = 1$ so that $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

and $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5.1 Chain and Scattering matrices.

We first focus on the acoustic waves and determine an expression for the scattering matrix M . From the acoustic point of view, the filter can be considered as a collection of N cells, each containing a reflector, with reflection coefficients

$$r_1, r_2, \dots, \dots, r_N, \quad (65)$$

as depicted in Figure 2 (see also the appendix).

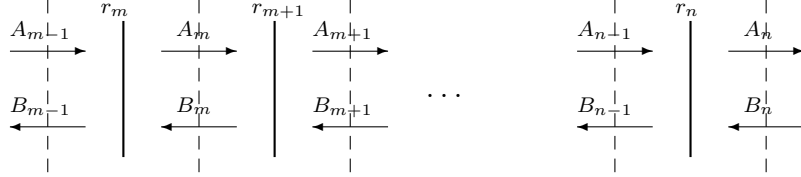


Figure 2: A set of cells

The scattering matrix associated to a set of $n - m + 1$ cells, relate incoming waves to outgoing waves,

$$\begin{bmatrix} B_{m-1} \\ A_n \end{bmatrix} = M_{m,n} \begin{bmatrix} A_{m-1} \\ B_n \end{bmatrix},$$

while the chain matrix is defined by

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = C_{m,n} \begin{bmatrix} A_{m-1} \\ B_{m-1} \end{bmatrix}.$$

The chain matrices satisfy the multiplicative property

$$C_{m,n} = C_{k+1,n} C_{m,k} \quad m \leq k < n. \quad (66)$$

Using Lemma 1 chain and scattering matrices are connected by the linear fractional transformation $M_{m,n} = KT_{\Pi}(C_{m,n})$, namely:

$$M_{m,n} = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} C_{m,n} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} C_{m,n} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1}. \quad (67)$$

The scattering matrix of a single cell is known to be

$$M_{n,n} = \frac{1}{z} \begin{bmatrix} -j r_n & t_n \\ t_n & -j r_n \end{bmatrix}, \quad t_n = \sqrt{1 - r_n^2}, \quad (68)$$

from which we deduce the chain matrix of a single cell:

$$C_{n,n} = \frac{1}{t_n} \begin{bmatrix} \frac{1}{z} & -j r_n \\ j r_n & z \end{bmatrix}. \quad (69)$$

Let ϕ_n and ψ_n be the polynomials of degree n defined by the Levinson recursions

$$\begin{bmatrix} \phi_{n+1}(\zeta) & \tilde{\psi}_{n+1}(\zeta) \\ \psi_{n+1}(\zeta) & \tilde{\phi}_{n+1}(\zeta) \end{bmatrix} = \begin{bmatrix} \zeta & r_{n+1} \\ r_{n+1}\zeta & 1 \end{bmatrix} \begin{bmatrix} \phi_n(\zeta) & \tilde{\psi}_n(\zeta) \\ \psi_n(\zeta) & \tilde{\phi}_n(\zeta) \end{bmatrix}, \quad (70)$$

with $\phi_0 = 1$ and $\psi_0 = 0$, and let

$$\tilde{\phi}_n(\zeta) = \zeta^n \phi_n(1/\zeta), \quad \tilde{\psi}_n(\zeta) = \zeta^n \psi_n(1/\zeta),$$

be the reciprocal polynomials. These polynomials are closely related to the Szegő polynomials, in fact $\zeta \phi_n(\zeta) - \tilde{\psi}_n(\zeta)$ and $\zeta \phi_n(\zeta) + \tilde{\psi}_n(\zeta)$ are the Szegő polynomials of the first and second kind associated to the sequence (65) of reflection coefficients.

Lemma 12 *The following relation is satisfied*

$$\phi_n(\zeta)\tilde{\phi}_n(\zeta) - \psi_n(\zeta)\tilde{\psi}_n(\zeta) = P_n^2 \zeta^n, \quad (71)$$

where

$$P_n = t_1 t_2 \dots t_n.$$

The polynomials ϕ_n are stable (roots inside the disk) and $\phi_n(0) = \psi_n(0) = 0$.

Proof. Relation (71) is proved by induction taking the determinants in the Levinson recursion. Then, we prove by induction that $\tilde{\phi}_n$ has no roots in \mathbb{D} . It is clearly true for $n = 0$. If it is true for n , then the function $\frac{\tilde{\psi}_n}{\tilde{\phi}_n}$ is analytic

in \mathbb{D} , and we deduce from (71) that $|\frac{\tilde{\psi}_n}{\tilde{\phi}_n}| < 1$ on the unit circle, and therefore also in \mathbb{D} . But then,

$$\tilde{\phi}_{n+1}(\zeta) = \tilde{\phi}_n(\zeta) \left(\zeta r_{n+1} \frac{\tilde{\psi}_n}{\tilde{\phi}_n}(\zeta) + 1 \right)$$

cannot have roots in \mathbb{D} . □

Proposition 3 *The chain matrix $C_{1,n}$ is unimodular, J -lossless, and it satisfies the relation*

$$\bar{C}_{1,n}(1/z) = K C_{1,n}(z) K. \quad (72)$$

It has the form

$$C_{1,n} = \frac{1}{P_n z^n} \begin{bmatrix} \tilde{\phi}_n(z^2) & -j z^{-1} \psi_n(z^2) \\ j z \tilde{\psi}_n(z^2) & \phi_n(z^2) \end{bmatrix}, \quad (73)$$

where ϕ_n and ψ_n are defined by (70).

Proof. The chain matrix of a single cell is clearly unimodular and J -lossless, since

$$J - C_{n,n}(z) J C_{n,n}(z)^* = \frac{1}{t_n^2} \begin{bmatrix} 1 - 1/|z|^2 & j r_n (1/z - \bar{z}) \\ -j r_n (1/\bar{z} - z) & |z|^2 - 1 \end{bmatrix}$$

is positive definite for $|z| > 1$, and it satisfies (72). Since $C_{1,n}$ is the product of such matrices, it is unimodular, J -lossless and satisfies (72). Formula (73) is easily established by induction. □

Corollary 2 *The matrix $M_{1,n}$ is lossless and has McMillan degree $2n$. It can be written as*

$$M_{1,n} = \frac{1}{\phi_n(z^2)} \begin{bmatrix} -j z \tilde{\psi}_n(z^2) & P_n z^n \\ P_n z^n & -j z^{-1} \psi_n(z^2) \end{bmatrix}. \quad (74)$$

The scattering matrix of the whole filter has degree $2N$ and is given by $M = M_{1,N}$.

Proof. The linear fractional transformation (67) applied to a J -lossless function gives a lossless function (see [12]). Thus, $C_{1,n}$ being J -lossless, $M_{1,n}$ is lossless. Applying (67) to the expression (73) of $C_{1,n}$ gives (74). Then, using (71),

$$\det M_{1,n} = -\tilde{\phi}_n(z^2)/\phi_n(z^2),$$

and $M_{1,n}$ has degree $2n$. □

5.2 The structure of the mixed matrix.

For $n = 0, \dots, N$, we define (see the Appendix) the row-vectors v_n and w_n by the recursion

$$\begin{bmatrix} v_n \\ w_n \end{bmatrix} = V_n = C_{n,n} V_{n-1}, \quad (75)$$

$$\begin{bmatrix} v_0 \\ w_0 \end{bmatrix} = V_0 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} M \right).$$

In order to obtain a nice representation of (75) we introduce another family of polynomials. The polynomials $\phi_{n+1,N}$ and $\psi_{n+1,N}$ of degree $N - n$ are defined by the backward recursion:

$$\begin{bmatrix} \phi_{n,N}(\zeta) & \tilde{\psi}_{n,N}(\zeta) \\ \psi_{n,N}(\zeta) & \tilde{\phi}_{n,N}(\zeta) \end{bmatrix} = \begin{bmatrix} \phi_{n+1,N}(\zeta) & \tilde{\psi}_{n+1,N}(\zeta) \\ \psi_{n+1,N}(\zeta) & \tilde{\phi}_{n+1,N}(\zeta) \end{bmatrix} \begin{bmatrix} \zeta & r_n \\ r_n \zeta & 1 \end{bmatrix}, \quad (76)$$

with $\phi_{N+1,N}(\zeta) = 1$ and $\psi_{N+1,N}(\zeta) = 0$.

Proposition 4 For $n = 1, \dots, N$, we have that

$$\begin{bmatrix} v_n \\ w_n \end{bmatrix} = V_n = C_{1,n} V_0, \quad (77)$$

and the row-vectors v_n and w_n have the explicit expressions

$$v_n = \frac{1}{\phi_N(z^2)} \begin{bmatrix} \phi_{n+1,N}(z^2) P_n z^n & -j z^{-1} \psi_n(z^2) \frac{P_N z^N}{P_n z^n} \end{bmatrix}, \quad (78)$$

$$w_n = \frac{1}{\phi_N(z^2)} \begin{bmatrix} -j z \tilde{\psi}_{n+1,N}(z^2) P_n z^n & \phi_n(z^2) \frac{P_N z^N}{P_n z^n} \end{bmatrix}. \quad (79)$$

Moreover, for $n = 0, \dots, N$, they satisfy

$$\begin{cases} v_n(z) &= \bar{w}_n(1/z) M, \\ w_n(z) &= \bar{v}_n(1/z) M. \end{cases} \quad (80)$$

The family $\boldsymbol{\nu} = (v_1, \dots, v_N, w_0, \dots, w_{N-1})$ forms an orthogonal basis of $\mathcal{H}(M)$.

Proof. For $n = 1, \dots, N$ the matrix V_n is given by (75), and (77) follows immediately from (66).

Using the expressions (73) and (74) of $C_{1,n}$ and $M = M_{1,N}$ respectively, we get

$$\begin{aligned} V_n(z) &= \frac{1}{\phi_N(z^2)P_n z^n} \begin{bmatrix} \tilde{\phi}_n(z^2) & -jz^{-1}\psi_n(z^2) \\ jz\tilde{\psi}_n(z^2) & \phi_n(z^2) \end{bmatrix} \begin{bmatrix} \phi_N(z^2) & 0 \\ -jz\tilde{\psi}_N(z^2) & P_N z^N \end{bmatrix}, \\ &= \frac{1}{\phi_N(z^2)P_n z^n} \begin{bmatrix} (\tilde{\phi}_n\phi_N - \psi_n\tilde{\psi}_N)(z^2) & -jz^{-1}\psi_n(z^2)P_N z^N \\ jz(\psi_n\phi_N - \phi_n\tilde{\psi}_N)(z^2) & \phi_n(z^2)P_N z^N \end{bmatrix} \end{aligned}$$

Using lemma 12, it is easy to see that the inverse of the chain matrix (73) is given by

$$C_{1,n}^{-1} = \frac{1}{P_n z^n} \begin{bmatrix} \phi_n(z^2) & jz^{-1}\psi_n(z^2) \\ -jz\psi_n(z^2) & \tilde{\phi}_n(z^2) \end{bmatrix},$$

also, it follows by induction that

$$C_{n+1,N} = \frac{P_n z^n}{P_N z^N} \begin{bmatrix} \tilde{\phi}_{n+1,N}(z^2) & -jz^{-1}\psi_{n+1,N}(z^2) \\ jz\tilde{\psi}_{n+1,N}(z^2) & \phi_{n+1,N}(z^2) \end{bmatrix}.$$

From the relation $C_{1,N}C_{1,n}^{-1} = C_{n+1,N}$, we now obtain the following equation

$$\begin{bmatrix} \tilde{\phi}_{n+1,N} & \psi_{n+1,N} \\ \tilde{\psi}_{n+1,N} & \phi_{n+1,N} \end{bmatrix} = \frac{1}{P_n^2 z^{2n}} \begin{bmatrix} \phi_n\tilde{\phi}_N - \tilde{\psi}_n\psi_N & \tilde{\phi}_n\psi_N - \psi_n\tilde{\phi}_N \\ \phi_n\tilde{\psi}_N - \tilde{\psi}_n\phi_N & \tilde{\phi}_n\phi_N - \psi_n\tilde{\psi}_N \end{bmatrix}. \quad (81)$$

Therefore,

$$V_n(z) = \frac{1}{\phi_N(z^2)} \begin{bmatrix} \tilde{\phi}_{n+1,N}(z^2) & -jz^{-1}\psi_{n+1,N}(z^2) \\ -jz\tilde{\psi}_{n+1,N}(z^2) & \phi_{n+1,N}(z^2) \end{bmatrix} \begin{bmatrix} P_n z^n & 0 \\ 0 & \frac{P_N z^N}{P_n z^n} \end{bmatrix},$$

in view of (81).

Since M is symmetric and lossless, for $n = 1, \dots, N$ we have using (72)

$$\begin{aligned} \bar{V}_n \left(\frac{1}{z} \right) M &= \bar{C}_{1,n}(1/z) \bar{V}_0(1/z) M \\ &= K C_{1,n}(z) K \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \\ &= K C_{1,n}(z) V_0(z) = K V_n(z). \end{aligned}$$

The same relation also holds for $n = 0$, so that (80) is proved.

It is easily checked from (78) and (79) that v_n for $n = 1, \dots, N$ and w_n for $n = 0, \dots, N - 1$, are strictly proper and stable rational functions and thus belong to \mathcal{H}_2 , while $v_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $w_N = \begin{bmatrix} 0 & 1 \end{bmatrix}$ doesn't. Let v_n in $\boldsymbol{\nu}$, then $v_n \in \mathcal{H}_2$, and $\bar{v}_n(1/z) \in \mathcal{H}_2$, so that (80) $v_n \in \mathcal{H}(M)$. The same is true for the w_n 's in $\boldsymbol{\nu}$. Since M has degree $2N$, the vector space $\mathcal{H}(M)$ has dimension $2N$ and the family $\boldsymbol{\nu}$ will form a basis if its elements are shown to be independent (this will be established in the proof of Theorem 5 to come). \square

Theorem 4 *The function β has a representation*

$$\beta = \begin{bmatrix} \sum_{n=0}^N a_n^{(1)} v_n - \bar{a}_n^{(1)} w_n \\ \sum_{n=0}^N a_n^{(2)} v_n - \bar{a}_n^{(2)} w_n. \end{bmatrix}. \quad (82)$$

If we assume that β is strictly proper, then $a_0^{(1)} = a_0^{(2)} = a_N^{(1)} = a_N^{(2)} = 0$, and the rows of β belong to $\mathcal{H}(M)$. In this case, the function β has degree $2(N-1)$, is strictly proper and admits the Douglas-Shapiro-Shields factorization:

$$\beta(z) = [-\bar{\beta}(1/z)/z][zM(z)].$$

Proof. From the properties of the mixed matrix, we have that $\alpha(z) = M(z)\beta^*(z)$ and $\beta(z) = -\alpha^T(z)$, so that β must satisfy

$$\beta(z) = -\bar{\beta}(1/z)M(z). \quad (83)$$

Each row β_l , $l = 1, 2$ of β is the sum of its value at infinity and an element of $\mathcal{H}(M)$, and thus admits a decomposition of the form

$$\beta_l(z) = \sum_{j=0}^N a_j^{(l)} v_j + b_j^{(l)} w_j.$$

Now,

$$\begin{aligned} \bar{\beta}_l(1/z)M &= \sum_{j=0}^N \bar{a}_j^{(l)} \bar{v}_j(1/z)M + \bar{b}_j^{(l)} \bar{w}_j(1/z)M, \\ &= \sum_{j=0}^N \bar{a}_j^{(l)} w_j(z) + \bar{b}_j^{(l)} v_j(z), \end{aligned}$$

by 80. Thus, (83) will be satisfied if and only if

$$b_j^{(l)} = -\bar{a}_j^{(l)}.$$

We get the following representation for β

$$\beta = \begin{bmatrix} \sum_{n=0}^N a_n^{(1)} v_n - \bar{a}_n^{(1)} w_n \\ \sum_{n=0}^N a_n^{(2)} v_n - \bar{a}_n^{(2)} w_n. \end{bmatrix}.$$

If we assume that β is strictly proper, then

$$\beta(\infty) = \begin{bmatrix} a_0^{(1)} & -\bar{a}_N^{(1)} \\ a_0^{(2)} & -\bar{a}_N^{(2)} \end{bmatrix},$$

is zero and

$$\beta = \begin{bmatrix} \sum_{n=1}^{N-1} a_n^{(1)} v_n - \bar{a}_n^{(1)} w_n \\ \sum_{n=1}^{N-1} a_n^{(2)} v_n - \bar{a}_n^{(2)} w_n \end{bmatrix},$$

and the other assertions directly follow from Proposition 4. \square

5.3 Realizations.

We are now ready to present realisations for each of the functions in the mixed matrix (12).

Theorem 5 *The matrix $M = M_{1,N}$ given by (74) has a realization*

$$M = \left(\begin{array}{c|c} A_M & B_M \\ \hline C_M & 0 \end{array} \right)$$

where

$$A_M = \left[\begin{array}{cccc|cccc} 0 & \dots & \dots & 0 & 0 & -jr_1 & 0 & \dots & 0 \\ t_2 & \ddots & & \vdots & \vdots & \ddots & -jr_2 & \ddots & \vdots \\ 0 & \ddots & & \vdots & \vdots & & & \ddots & 0 \\ \vdots & \ddots & t_{N-1} & \ddots & \vdots & \vdots & & \ddots & -jr_{N-1} \\ 0 & \dots & 0 & t_N & 0 & 0 & \dots & \dots & 0 \\ \hline 0 & \dots & \dots & 0 & 0 & t_1 & 0 & \dots & 0 \\ -jr_2 & \ddots & & \vdots & \vdots & \ddots & t_2 & \ddots & \vdots \\ 0 & -jr_3 & & \vdots & \vdots & & & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \ddots & t_{N-1} \\ 0 & \dots & 0 & -jr_N & 0 & 0 & \dots & \dots & 0 \end{array} \right], \quad (84)$$

and

$$B_M^T = \begin{bmatrix} t_1 & 0 & 0 & \dots & 0 & -jr_1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -jr_N & 0 & 0 & \dots & 0 & t_N \end{bmatrix}, \quad (85)$$

and

$$C_M = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (86)$$

The reachability Gramian of (A_M, B_M) is the identity. Thus ν is an orthonormal basis of $\mathcal{H}(M)$.

Proof. From the definition (75) of V_n and (69), we have the recurrence relations

$$\begin{aligned} zw_{n-1}(z) &= t_n w_n(z) - jr_n v_{n-1}(z), \\ zv_n(z) &= t_n v_{n-1}(z) - jr_n w_n(z), \end{aligned} \quad (87)$$

for $n = 1, \dots, N$, which gives

$$z \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \\ w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} - A_M \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \\ w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} = \begin{bmatrix} t_1 v_0 \\ 0 \\ \vdots \\ -j r_N w_N \\ -j r_1 v_0 \\ \vdots \\ 0 \\ t_N w_N \end{bmatrix} = B_M.$$

As for C_M , we have

$$M = \begin{bmatrix} w_0 \\ v_N \end{bmatrix} = C_M (z I_{2N} - A_M)^{-1} B_M.$$

It is easily checked that $P = I$ satisfies $P = A_M P A_M^* + B_M B_M^*$, so that the reachability Gramian is identity, as announced. By the way, this ties up a loose end in the proof of Proposition 4. \square

Theorem 6 *In case the function β is strictly proper, it has a realization*

$$\beta = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right)$$

where

$$A = \left[\begin{array}{cccc|cccc} 0 & \dots & \dots & 0 & -j r_1 & 0 & \dots & \dots & 0 \\ t_2 & \ddots & & \vdots & 0 & -j r_2 & \ddots & & \vdots \\ 0 & t_3 & & & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & t_{N-1} & 0 & \dots & \dots & 0 & -j r_{N-1} \\ \hline -j r_2 & 0 & \dots & \dots & 0 & 0 & t_2 & 0 & \dots & 0 \\ 0 & -j r_3 & \ddots & & \vdots & \vdots & \ddots & t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & & & & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & \vdots & & & \ddots & t_{N-1} \\ 0 & \dots & \dots & -j r_N & 0 & \dots & \dots & \dots & 0 & 0 \end{array} \right] \quad (88)$$

$$C = \begin{bmatrix} a_1^{(1)} & a_2^{(1)} & \dots & a_{N-1}^{(1)} & -\bar{a}_1^{(1)} & -\bar{a}_1^{(1)} & \dots & -\bar{a}_{N-1}^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \dots & a_{N-1}^{(2)} & -\bar{a}_1^{(2)} & -\bar{a}_1^{(2)} & \dots & -\bar{a}_{N-1}^{(2)} \end{bmatrix} \quad (89)$$

and

$$B^T = \begin{bmatrix} t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & t_N \end{bmatrix} \quad (90)$$

The controllability gramian of (B, A) is the identity.

Proof. Formula (82) can be rewritten

$$\beta(z) = C \begin{bmatrix} v_1(z) \\ v_2(z) \\ \vdots \\ v_{N-1}(z) \\ w_1(z) \\ w_2(z) \\ \vdots \\ w_{N-1}(z) \end{bmatrix},$$

where C is given by (89). The recurrence (87) gives

$$z \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \\ w_1 \\ w_2 \\ \vdots \\ w_{N-1} \end{bmatrix} - A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \\ w_1 \\ w_2 \\ \vdots \\ w_{N-1} \end{bmatrix} = \begin{bmatrix} t_1 v_0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ t_N w_N \end{bmatrix} = B.$$

□

Remark. In case β is not strictly proper, it has realization

$$\beta = \left(\frac{A \mid B}{C \mid D} \right),$$

where

$$D = \beta(\infty) = \begin{bmatrix} a_0^{(1)} & -\bar{a}_N^{(1)} \\ a_0^{(2)} & -\bar{a}_N^{(2)} \end{bmatrix},$$

$$C = \begin{bmatrix} a_1^{(1)} & a_2^{(1)} & \dots & a_N^{(1)} & -\bar{a}_0^{(1)} & -\bar{a}_1^{(1)} & \dots & -\bar{a}_{N-1}^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \dots & a_N^{(2)} & -\bar{a}_0^{(2)} & -\bar{a}_1^{(2)} & \dots & -\bar{a}_{N-1}^{(2)} \end{bmatrix},$$

$A = A_M$ and $B = B_M$.

Corollary 3 *The matrix valued function Y and its Cayley transform $S = (I_2 + Y)(I_2 - Y)^{-1}$ have realizations*

$$Y = \left(\frac{A}{C} \middle| \frac{AC^*}{\frac{1}{2}CC^*} \right), \quad (91)$$

$$S = \left(\frac{A(I_{2(N-1)} + C^*(I_2 - \frac{1}{2}CC^*)^{-1}C)}{\sqrt{2}(I_2 - \frac{1}{2}CC^*)^{-1}C} \middle| \frac{\sqrt{2}AC^*(I_2 - \frac{1}{2}CC^*)^{-1}}{(I_2 + \frac{1}{2}CC^*)(I_2 - \frac{1}{2}CC^*)^{-1}} \right). \quad (92)$$

Proof. If (91) is true, we have

$$\begin{aligned} Y(z) + Y^\sharp(z) &= C(zI_{2(N-1)} - A)^{-1}AC^* + CC^* + CA^*(z^{-1}I_{2(N-1)} - A^*)^{-1}C^* \\ &= C(zI_{2(N-1)} - A)^{-1} [I_{2(N-1)} - AA^*] (z^{-1}I_{2(N-1)} - A^*)^{-1}C^* \\ &= C(zI_{2(N-1)} - A)^{-1}BB^*(z^{-1}I_{2(N-1)} - A^*)^{-1}C^* \\ &= \beta(z)\beta^\sharp(z). \end{aligned}$$

The realization for S follows from a straightforward calculation, see for example [17, Lemma 5.2]. \square

Appendix. We now give a concrete example of SAW filter with two acoustical ports and two electrical ports which illustrates the developments of Section 5.

The filter (see Figure 3) is constituted of two transducers Σ_1 and Σ_2 . Each transducer is made of cells containing each a reflection center and an electroacoustic center. We assume the total number of cells of the filter is N ,

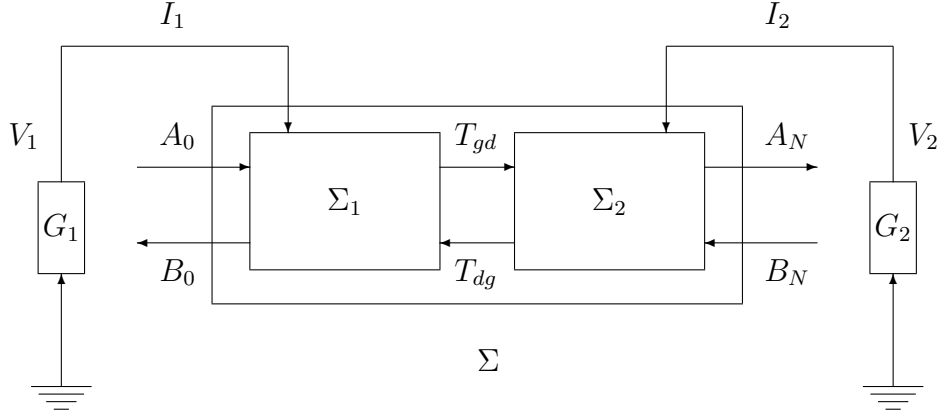


Figure 3: The filter.

including possibly some cells with only a reflection center, between the two transducers. The current and voltage are I_1 and U_1 at the electrical port of the transducer Σ_1 and I_2 and U_2 at the electrical port of the transducer Σ_2 . The incoming and outgoing waves are A_0 and B_0 at the acoustical port of Σ_1 and A_N and B_N at the acoustical port of Σ_2 . The filter is described by the mixed matrix

$$\begin{bmatrix} \mathbf{W}_o \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} M & \alpha \\ \beta & Y \end{bmatrix} \begin{bmatrix} \mathbf{W}_i \\ \mathbf{V} \end{bmatrix},$$

where

$$\mathbf{W}_i = \begin{bmatrix} A_0 \\ B_N \end{bmatrix}, \mathbf{W}_o = \begin{bmatrix} A_N \\ B_0 \end{bmatrix}, \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

Each cell has the same delay τ and the n th cell has reflection coefficient r_n and electroacoustic coefficient g_n . The position of the electroacoustic center is determined so that, near some given central frequency, say f_0 , Σ_1 is unidirectional to the right while Σ_2 is unidirectional to the left. This happens when the delay $\Delta\tau$ between the electroacoustic center and the right boundary of the cell in Σ_1 (resp. the left boundary of the cell in Σ_2), see Figures 4 and 5, is

$$\Delta\tau = \frac{1}{8f_0}.$$

We assume that

$$\delta = e^{j2\pi f \Delta\tau}$$

is constant near the central frequency and equal to $e^{j\pi/4}$.

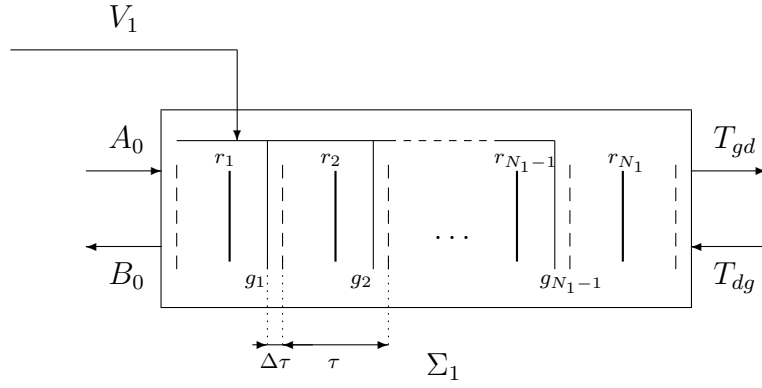


Figure 4: The left transducer.

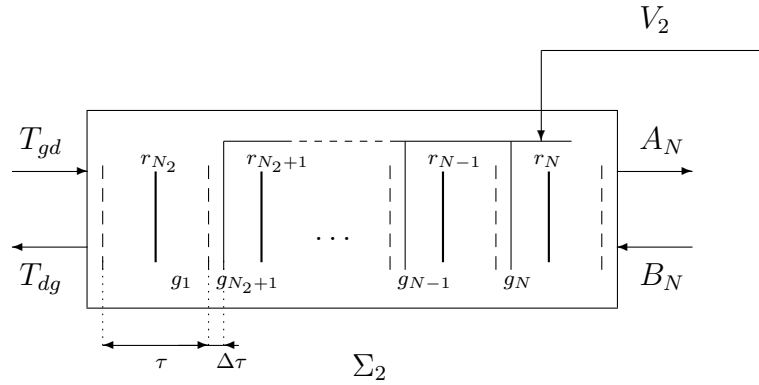


Figure 5: The right transducer.

Recall that β is given by

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \beta \begin{bmatrix} A_0 \\ B_N \end{bmatrix}.$$

The current I_n produced at the n th acoustic center of Σ_1 by the wave $\delta A_n + \bar{\delta} B_n$ is given by:

$$\begin{aligned}
I_n &= jg_n (\delta A_n + \bar{\delta} B_n), \\
&= jg_n \begin{bmatrix} \delta & \bar{\delta} \end{bmatrix} C_{1,n} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \\
&= jg_n \begin{bmatrix} \delta & \bar{\delta} \end{bmatrix} C_{1,n} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ A_N \end{bmatrix} \right) \\
&= jg_n \begin{bmatrix} \delta & \bar{\delta} \end{bmatrix} C_{1,n} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} M \right) \begin{bmatrix} A_0 \\ B_N \end{bmatrix}
\end{aligned}$$

while the current I_n produced at the n th acoustic center of Σ_2 by the wave $\bar{\delta} A_{n-1} + \delta B_{n-1}$ is given by:

$$\begin{aligned}
I_n &= jg_n (\bar{\delta} A_{n-1} + \delta B_{n-1}), \\
&= jg_n \begin{bmatrix} \bar{\delta} & \delta \end{bmatrix} C_{1,n-1} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \\
&= jg_n \begin{bmatrix} \delta & \bar{\delta} \end{bmatrix} C_{1,n-1} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} M \right) \begin{bmatrix} A_0 \\ B_N \end{bmatrix}.
\end{aligned}$$

These computations justify a posteriori the introduction of the matrices V_n (77). Since $I_1 = \sum_{n=1}^{N_1-1} I_n$, and $I_2 = \sum_{n=N_2+1}^N I_n$, we can specialize the general expression (82) of β to this particular case

$$\beta = j \begin{bmatrix} \sum_{n=1}^{N_1-1} g_n (\delta v_n + \bar{\delta} w_n) \\ \sum_{n=N_2}^{N-1} g_{n+1} (\bar{\delta} v_n + \delta w_n) \end{bmatrix}. \quad (93)$$

We may thus deduce from Section 5 the following realizations

$$\beta = \left(\frac{A \mid B}{C \mid 0} \right),$$

where A and B are given by (88) and (90), i.e. they are the same as in Theorem 6, and C is given by

$$C = j \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} \delta I_{N-1} & \bar{\delta} I_{N-1} \\ \bar{\delta} I_{N-1} & \delta I_{N-1} \end{bmatrix} \quad (94)$$

where C_1 and C_2 are given by:

$$\begin{aligned} C_1 &= [g_1 \ g_2 \ \dots \ g_{N_1-1} \ 0 \ \dots \ 0 \ \dots \ 0] \\ C_2 &= [0 \ 0 \ \dots \ 0 \ \dots \ 0 \ g_{N_2+1} \ \dots \ g_N]. \end{aligned}$$

and the realizations of Y and S are as in Corollary 3.

A typical problem here is to find the electroacoustic and reflection parameters of both transducers in order to produce a bandpass filter for some specified frequency in term of power transmission. The power transmission is represented by the electrical transfer function S , which is subject to five conditions at least: it is contractive, symmetric, it has a symmetric Darlington embedding of degree $n = \deg S + 2$, the extension \mathcal{S} satisfies at infinity a multiple interpolation condition of the form $\mathcal{S}_{1,1}^{(k)}(\infty) = 0$ for even $k \in \{0, n\}$, and $\mathcal{S}(\infty) = \text{diag}\{0, S(\infty)\}$. Considerable refinements of the results of the present paper will be needed in order to characterize completely the frequency behaviour of SAW filters.

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