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Convex optimisation method for matching filters synthesis

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Abstract

In this paper, a certified synthesis technique is presented for the design of matching filters that combines convex optimisation with the Fano-Youla matching theory. This technique is applicable to any rational load and provides lower hard bounds for the best matching level, as well as a practical synthesis of a matching filter approaching those bounds. Furthermore, if the load is a rational function of degree 1, the optimal matching filter is synthesized, yielding in this case an extension of the classical filter synthesis for resistive loads. As example, a dual-band matching filter is conceived for a dual-band antenna.

1 Introduction

There exists a remarkable literature in the field of broad-band matching. In [1–3] broad-band matching was first introduced based on the use of the Darlington two port equivalent and extraction procedures. The theory was first reviewed in [1] where the problem of matching an *RC*-load is considered as the design of a lowpass filtering network where an *RC*-element is fixed. In [2] this problem was extended to the case of a generic load by using the *Darlington equivalent* and reformulated in [3] as a complex interpolation problem. The theory was, for example, used to synthesize matching networks with a Tchebychev type power gain transducer [4], nevertheless this type of responses are known to be non optimal in terms of matching performances unless the load is a constant impedance. This approach was therefore progressively replaced by the optimization based *real frequency* technique of Carlin [5] which is more oriented to practical applications. Additionally in [6] the matching problem was solved optimally by considering the broader class of infinite dimension functions H^∞ and therefore providing hard bounds for the matching problem in finite dimension.

In this work we use the Fano-Youla matching theory combined with convex optimisation to formulate the matching problem in a generalised form. Within this framework, we introduce in sec. 2 a convex relaxation of the general matching problem providing hard lower bounds for the original problem when rational filters of finite degree are considered. Finally, in sec. 3 an illustrative example is presented.

2 Theory

The matching problem aims to minimise the reflection of the power transmitted to a given load within a specified frequency band. The load is represented as a 2-port device A in Fig. 1. Usually the power is transmitted to the load through a filter F that rejects out of band signals. Both devices, the filter together with the load compose the system S.

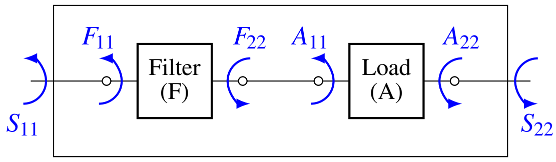

Figure 1. Global system composed of the cascade of the matching filter with the load and reflection coefficients.

It is important to specify that if only the input reflection of the load A_{11} is known, a *Darlington equivalent* of the load (see [7]) yields a loss-less two port network with the same input reflection A_{11}. Following the Fano-Youla approach to the matching problem, the system S is conceived first, followed by the de-embedding of the load. Let us introduce first some notations and definitions. Consider the complex variable $\lambda = \omega + j\sigma$ where ω is the frequency variable. We denote by $\mathbb{C}^+$ the open upper half plane, $\mathbb{C}^+ = \{\lambda : \Im(\lambda) > 0\}$ and by $\mathbb{C}^-$ the open lower half plane; $\overline{\mathbb{C}^-}$ denotes the closed lower half plane $\overline{\mathbb{C}^-} = \{\lambda : \Im(\lambda) \leq 0\}$. In this work we consider $\mathbb{C}^-$ as the analyticity domain.

Definition 2.1 (Scattering matrix)**.** *We call* scattering matrix *a rational* 2x2 *matrix of the complex variable* λ*, unitary for* $\lambda \in \mathbb{R}$ *and analytic in* $\mathbb{C}^-$*. Its elements are scalar rational functions contractive in* $\mathbb{C}^-$*, namely* Schur functions.

The matrices S, F and A are scattering matrices. Such matrices are represented in the *Belevitch* form [8]

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{1}{q} \begin{pmatrix} \varepsilon p^* & -\varepsilon r^* \\ r & p \end{pmatrix} \quad (1)$$

with ε a uni-modular constant, and q, p, r polynomials satisfying $qq^* = pp^* + rr^*$ with $p^*(\lambda) = \overline{p(\overline{\lambda})}$. Note that q is a *stable* polynomial, that is, with all roots in $\mathbb{C}^+$.

Definition 2.2 (Chaining)**.** *We define the chaining* $F_{22} \circ A$ *of a* Schur *function* F_{22} *and a matrix* A *in the form (1) as* *the output reflection of the global system S_{22} composed of the cascade of F and A (see Fig. 1).*

$$S_{22} = F_{22} \circ A = A_{22} + A_{21} F_{22} A_{21} \left(1 - A_{11} F_{22}\right)^{-1} \qquad (2)$$

Definition 2.3 (Feasibility)**.** *We define a function S_{22} as feasible for a given load A if there exists a* Schur *function F_{22}, such that $F_{22} \circ A = S_{22}$. Additionally we denote by $\mathbb{F}$ the set of feasible functions S_{22} for a given load A.*

Note that $\mathbb{F}$ is the image of the set of Schur functions under the application $f \to f \circ A$. If S_{22} is admissible for a load A, then the function F_{22} such that $F_{22} \circ A = S_{22}$ expresses as:

$$F_{22} = \left(A_{22} - S_{22}\right)\left(\det A - A_{11} S_{22}\right)^{-1} \qquad (3)$$

Next we present a characterisation of $\mathbb{F}$ by a set of interpolation conditions at the transmission zeros of A.

Definition 2.4 (Transmission zeros)**.** *We define the transmission zeros associated to a matrix function S in the form (1) as the zeros in $\overline{\mathbb{C}^-}$ (possibly at ∞) of $S_{12}S_{21}(\lambda)$:*

$$\mathrm{tz}\,[S] = \left\{ \lambda \in \overline{\mathbb{C}^-} : S_{12} S_{21}(\lambda) = 0 \right\} \qquad (4)$$

where we consider the classical multiplicity of the transmission zeros in $\mathbb{C}^-$ and half of the multiplicity for the transmission zeros in $\mathbb{R}$. Also remark that the transmission zeros, being in $\overline{\mathbb{C}^-}$, cannot simplify with the zeros of q.

Note that, if S_{12} is assumed to be *minimum phase*[1] (i.e. has no zeros in $\mathbb{C}^-$), then, the finite transmission zeros of S_{22} are the zeros of r. In that case the matrix S is recovered from S_{22}, up to the unimodular constant ε.

A core result of Fano's-Youla's matching theory is the necessary and sufficient conditions for F_{22} to be Schur in $\mathbb{C}^-$. These conditions represent the characterisation of the set $\mathbb{F}$.

Proposition 2.5 (Characterisation of $\mathbb{F}$)**.** *Consider a lossless load A with transmission zeros α_i, and $\mathbb{F}$ its feasible set. A rational schur function S_{22} belongs to $\mathbb{F}$ iff* [2]

- *At each transmission zero α_i of multiplicity m_i of A, the following interpolation conditions hold* [3]*:*

$$\left(\mathrm{D}^k S_{22}\right)[\alpha_i] = \xi_{i,k} \qquad 0 \le k \le m_i - 1 \qquad \forall \alpha_i \in \mathbb{C}^- \qquad (5a)$$

$$\left(\mathrm{D}^k S_{22}\right)[\alpha_i] = \xi_{i,k} \qquad 0 \le k \le 2m_i - 2 \qquad \forall \alpha_i \in \mathbb{R} \qquad (5b)$$

$$\left(\mathrm{D}^k j \ln S_{22}\right)[\alpha_i] \le \psi_{i,2m_i-1} \qquad \forall \alpha_i \in \mathbb{R} \qquad (5c)$$

with $\xi_{i,k} = \left(\mathrm{D}^k A_{22}\right)[\alpha_i]$ and $\psi_{i,k} = \left(\mathrm{D}^k j \ln A_{22}\right)[\alpha_i]$.[4]

2.1 Matching Problem

With these definitions, we can state the general form of the matching problem. Notice that in [6] the matching problem is stated as the minimisation of the reflection level without any additional constraints on $S_{22} \in \mathbb{F}$. It is only supposed that S_{22} belong to the infinite dimensional class of functions H^∞. In this work however, we constrain $S_{22} \in \mathbb{F}$ to be rational in the form (1) with $p, r \in \mathbb{P}^N$ (the set of polynomials of degree N). Additionally we suppose that the polynomial r is fixed as it is customary in classical filter synthesis. Furthermore we assume that the transmission zeros α_i are also roots of r. Thus r will have roots at the transmission zeros α_i as well as any other possible transmission zeros fixed in advance. Applying the change of variable: $pp^* = P$, $rr^* = R$ we denote with $\mathbb{F}_R^N$ the set of rational functions $S_{22} \in \mathbb{F}$ of degree N with the *transmission polynomial* $R \in \mathbb{P}_+^{2N}$.

$$\mathbb{F}_R^N = \left\{ f \in \mathbb{F} \mid \exists P \in \mathbb{P}_+^{2N} : |f(\omega)|^2 = \frac{P}{P+R}(\omega); \forall \omega \in \mathbb{R} \right\} \qquad (6)$$

where $\mathbb{P}_+^{2N}$ denotes the set of positive polynomials of degree at most $2N$. We state the problem as

Problem ($\mathscr{P}$)**.** *Find* $l = \min\limits_{S_{22} \in \mathbb{F}_R^N} \max\limits_{\omega \in \mathbb{I}_1} |S_{22}(\omega)|^2$

$$\textit{Subject to} \qquad |S_{22}(\omega)|^2 \ge \gamma \qquad \forall \omega \in \mathbb{I}_2$$

where $\mathbb{I}_1$ represents the passband, $\mathbb{I}_2$ the stopband and γ the desired rejection level in the interval $\mathbb{I}_2$.

We introduce now a convex relaxation of problem $\mathscr{P}$ by considering the notion of *admissibility*.

Definition 2.6 (Admissibility)**.** *A minimum phase Schur function U is admissible for a load A iff there exists a function $S_{22} \in \mathbb{F}$ such that for all $\omega \in \mathbb{R}$, $|S_{22}(\omega)| \le |U(\omega)|$. We denote by $\mathbb{G}$ the set of admissible U.*

For every admissible U there exist $S_{22} \in \mathbb{F}$ such that $S_{22}U^{-1}$ is a Schur function. Thus $\mathbb{G}$ can be characterised by reformulating (5a) to (5c) on B where $B = S_{22}U^{-1}$.

Proposition 2.7 (Characterisation of $\mathbb{G}$)**.** *A minimum phase rational Schur reflection U is admissible for a load A with transmission zeros α_i of multiplicity m_i if and only if*

- *There exists a Schur function B satisfying (7a) to (7c) at every transmission zero α_i of A.*

$$\left(\mathrm{D}^k B\right)[\alpha_i] = \Xi_{i,k} \qquad 0 \le k \le m_i - 1 \qquad \forall \alpha_i \in \mathbb{C}^- \qquad (7a)$$

$$\left(\mathrm{D}^k B\right)[\alpha_i] = \Xi_{i,k} \qquad 0 \le k \le 2m_i - 2 \qquad \forall \alpha_i \in \mathbb{R} \qquad (7b)$$

$$\left(\mathrm{D}^k j \ln B\right)[\alpha_i] \le \Psi_{i,2m_i-1} \qquad \forall \alpha_i \in \mathbb{R} \qquad (7c)$$

where we denote $\Xi_{i,k} = \left(\mathrm{D}^k A_{22} U^{-1}\right)[\alpha_i]$ and also $\Psi_{i,k} = \left(\mathrm{D}^k j \ln A_{22} U^{-1}\right)[\alpha_i]$.

[1] Minimum phase functions, also called *outer*, have many useful properties for our purpose, see e.g. [9, Th.4.6]

[2] Equivalent forms of (5b) and (5c) can be used if the transmission zeros α_i occurs at $\lambda = \infty$.

[3] Note this condition does not require the transmission zeros of A to be present in the system S as long as the interpolation conditions are satisfied. Nevertheless if the transmission zeros of A are not present in S, the matching filter obtained after deembedding will include those transmission zeros α_i at the expense of not being of minimal degree. In this paper, we assume that the transmission zeros of the load A are also present in S with at least the same multiplicity, thus obtaining a matching filter F of minimal degree.

[4] The symbol D^k stands for the k-th derivative.

2.1.1 Generalised Matching Problem

Define $U_P(\omega)$ as the outer spectral factor of $(1+R(\omega)/P(\omega))^{-1}$ with $P,R \in \mathbb{P}_+^{2N}$.

Definition 2.8 (Admissible polynomials)**.** *We denote by* $\mathbb{H}_R^N$ *the set of* $P \in \mathbb{P}_+^{2N}$ *such that* $U_P \in \mathbb{G}$.

Problem ($\mathscr{P}_C$)**.** *Find* $L = \min_{P\in\mathbb{H}_R^N} \max_{\omega\in\mathbb{I}_1} \frac{P}{R}(\omega)$

Proposition 2.9 (Convexity)**.** *The set* $\mathbb{H}_R^N$ *is a convex set. Problem* $\mathscr{P}_C$ *is convex and admits a unique solution.*

For simplicity we consider here no rejection constraints. However, as it is known in classical filter synthesis, linear constraints on the modulus of U_P can be transformed to linear constraints on the filtering function P/R [10].

$$|U_P(\omega)|^2 = \frac{P}{R+P}(\omega) \geq \gamma \quad \Leftrightarrow \quad P(\omega) \geq \Gamma \cdot R(\omega) \qquad (8)$$

with $\Gamma = (1/\gamma - 1)^{-1}$

2.1.2 Nevanlinna-Pick interpolation

In order to assure the admissibility of U_P in problem $\mathscr{P}_C$ it is necessary to guarantee the existence of B verifying (7a) to (7c). If we consider the simpliest case, where the load has only simple transmission zeros in $\mathbb{C}^-$, the problem is equivalent to the classical *Nevanlinna-Pick interpolation* problem. The Nevanlinna-Pick theorem states [9]:

Theorem 2.10 (Nevanlinna-Pick Interpolation Theorem)**.** *Given* $\gamma_1, \gamma_2 \ldots \gamma_n \in \overline{\mathbb{D}}$ *and* $\alpha_1, \alpha_2 \ldots \alpha_n \in \mathbb{C}^-$. *There exist a schur function* $B : \mathbb{C}^- \to \overline{\mathbb{D}}$ *satisfying* $B(\alpha_i) = \gamma_i$ *if and only if the* Pick matrix

$$\Delta_{i,k} = j\left(\frac{1-\gamma_i\overline{\gamma_k}}{\alpha_i - \overline{\alpha_k}}\right) \qquad (9)$$

is positive semidefinite. Furthermore, B is unique iff Δ *is singular. In this case B is a* Blaschke *product.*

The *Nevanlinna-Pick Interpolation Theorem* is generalised in [11] to consider interpolating points $\alpha_i \in \overline{\mathbb{C}^-}$ and interpolation conditions on the derivatives. The generalised form of the *Nevanlinna-Pick Interpolation Theorem* states the necessary and sufficient condition for the existence of a Schur function B satisfying (7a) to (7c). Additionally *Nevanlinna's* theory also provides a parametrisation of all possible functions B. For simplicity we consider here a load A having only simple transmission zeros $\alpha_i \in \mathbb{C}^-$.

Proposition 2.11 (Admissibility condition)**.** *Consider a load A with simple transmission zeros* $\alpha_i \in \mathbb{C}^-$ *and the transmission polynomial* $R \in \mathbb{P}_+^{2N}$. *A polynomial* $P \in \mathbb{P}_+^{2N}$ *is admissible iff* $\Delta(P) \succeq 0$ *where*

$$\Delta(P)_{i,k} = j\left(\frac{1-\left(A_{22}U_P^{-1}\right)[\alpha_i]\cdot\overline{\left(A_{22}U_P^{-1}\right)[\alpha_k]}}{\alpha_i - \overline{\alpha_k}}\right) \qquad (10)$$

Considering a load A of degree M with simple transmission zeros $\alpha_i \in \mathbb{C}^-$, we can use the previous theorem to state $\mathscr{P}_C$ as the minimisation of P/R on the passband over all $P \in \mathbb{P}_+^{2N}$ under the condition $\Delta(P) \succeq 0$ to ensure that $P \in \mathbb{H}_R^N$.

2.1.3 Bounds for the solution of problem $\mathscr{P}$

For each $U_P \in \mathbb{G}$, there exist a function B_P such that $S_{22} = U_P \cdot B_P \in \mathbb{F}$. This function verifies

1. The degree of B_P equals the rank of $\Delta(P)$.
2. Consider $\widehat{P}$ the optimal P of $\mathscr{P}_C$. Denote $\widehat{U} = U_{\widehat{P}}$, $\widehat{B} = B_{\widehat{P}}$ and $\widehat{S_{22}} = \widehat{U}\cdot\widehat{B}$. The matrix $\Delta(\widehat{P})$ is singular. Therefore the unique function $\widehat{B}$ verifying (7a) to (7c) is a *Blaschke* product of degree $L \leq M-1$.

$$\widehat{B}(\omega) = \frac{\prod_{i=1}^{L}(\omega-\beta_i)}{\prod_{i=1}^{L}(\omega-\overline{\beta_i})} \qquad (11)$$

3. The degree of $\widehat{S_{22}}$ is bounded between N and *N+M-1*.

From a formal point of view, problem $\mathscr{P}_C$ provides hard lower bounds for the attainable matching level in problem $\mathscr{P}$. However for a given degree N the provided bound is not always sharp since a system of degree $N+L$ with $0 \leq L \leq M-1$ could be necessary to attain such a bound. Additionally note that for a load of degree $M=1$, the obtained Blaschke product is always of degree 0. Therefore in this case the relaxation made in problem $\mathscr{P}_C$ is exact providing the optimal solution to problem $\mathscr{P}$.

2.1.4 Implementation as a Semi-definite Program

Problem $\mathscr{P}_C$ can be solved optimally by nonlinear semidefinite programming techniques. Indeed, the constraint on the positivity of P in $\mathscr{P}_C$ can be recasted by means of *linear matrix inequalities* by imposing the positive semidefiniteness of a matrix A [12]. We obtain then a *semidefinite program* with one non-linear constraint $\Delta(P) \succeq 0$ that ensures the admissibility of P. Those constraints are implemented by using a barrier/penalty function. Linear matrix inequalities are handled by the standard logarithmic barrier meanwhile the non-linear matrix inequality is ensured by the penalty function presented in [13]. Note that adding more passbands or some rejection constraints amounts to add some extra positive definite matrices to ensure the positivity of $LR(\omega) - P(\omega)$ in the i$^{\text{th}}$-passband, or to guarantee that $P(\omega) \geq \Gamma \cdot R(\omega)$ is satisfied in the j$^{\text{th}}$-stopband.

3 RESULTS

We present a synthesis example for a GNSS receiver matching a dualband antenna in the GPS/GALILEO bands: L2 (1.21-1.24GHz), E6 (1.26-1.3GHz), L1 (1.55-1.6GHz). Fig. 2 shows a comparison between the reflection of the antenna (A_{11}) and the results of $\mathscr{P}_C$ (S_{22}) by taking $N=7$.

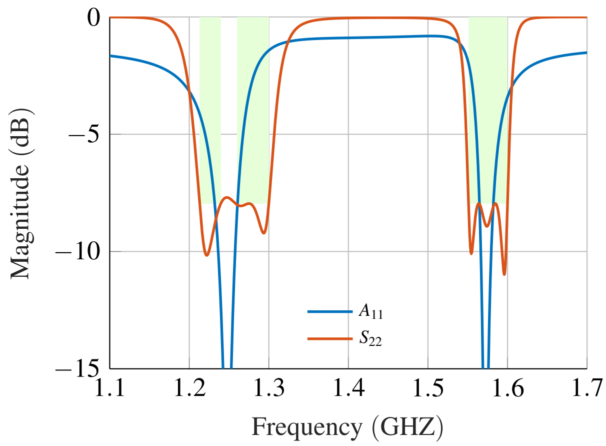

Figure 2. Result of $\mathscr{P}_C$ with a load of degree 2 and a system of degree 6.

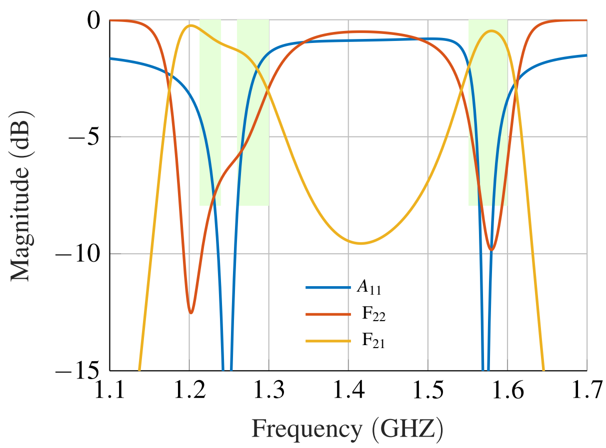

Figure 3. Matching filter providing the response in Fig. 2.

By using the matching filter, the reflection at the right edge of the band E6 (1.3GHz) has been improved from $-1.4dB$ to $-7.95dB$ representing an improvement of 450%. Parameters F_{22} and F_{21} of the matching filter that provides this result are shown in Fig. 3 together with the load reflection A_{11}. Furthermore we show in Fig. 4 the bounds for the optimal reflection level attainable in $\mathscr{P}$.

4 Conclusion

A practical imprementation of the *Fano-Youla* matching theory by means of convex optimisation has been presented. This approach provides hard lower bounds for the best achievable matching level in a set of frequency bands. Furthermore, if the load to be matched is of degree 1, our algorithm yields the guaranteed best matching response. In this case it is the generalization of the classical quasi-elliptic synthesis technique considering a resistive load to the case of a frequency varying load. Otherwise, for loads of higher degree, our algorithm allows to compute hard lower bounds for the attainable toss level when system N is considered and provides a rational filter aproaching the bound.

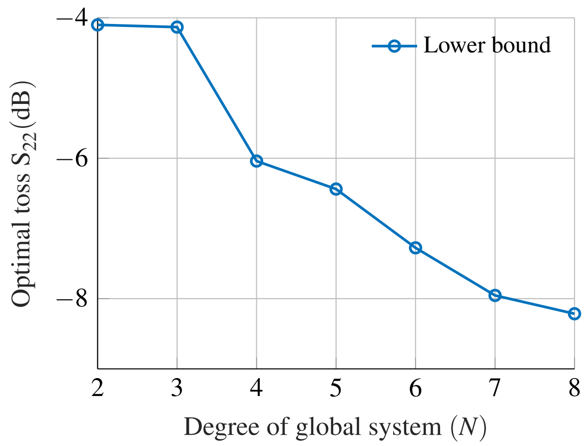

Figure 4. Bounds for the optimal solution of $\mathscr{P}$.

Acknowledgements

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