6
Real-number calculus

6.1 What makes an honest function?

CALCULUS—or, according to its more sophisticated name, *mathematical analysis*—is built from two basic ingredients: *differentiation* and *integration*. Differentiation is concerned with velocities, accelerations, the slopes and curvature of curves and surfaces, and the like. These are rates at which things change, and they are quantities defined *locally*, in terms of structure or behaviour in the tiniest neighbourhoods of single points. Integration, on the other hand, is concerned with areas and volumes, with centres of gravity, and with many other things of that general nature. These are things which involve measures of *totality* in one form or another, and they are not defined merely by what is going on in the local or infinitesimal neighbourhoods of individual points. The remarkable fact, referred to as the *fundamental theorem of calculus*, is that each one of these ingredients is essentially just the *inverse* of the other. It is largely this fact that enables these two important domains of mathematical study to combine together and to provide a powerful body of understanding and of calculational technique.

This subject of mathematical analysis, as it was originated in the 17th century by Fermat, Newton, and Leibniz, with ideas that hark back to Archimedes in about the 3rd century BC, is called 'calculus' because it indeed provides such a body of calculational technique, whereby problems that would otherwise be conceptually difficult to tackle can frequently be solved 'automatically', merely by the following of a few relatively simple rules that can often be applied without the exertion of a great deal of penetrating thought. Yet there is a striking contrast between the operations of differentiation and integration, in this calculus, with regard to which is the 'easy' one and which is the 'difficult' one. When it is a matter of applying the operations to explicit formulae involving known functions, it is differentiation which is 'easy' and integration 'difficult', and in many cases the latter may not be possible to carry out at all in an explicit way. On the other hand, when functions are not given in terms of formulae, but are provided in the form of tabulated lists of numerical data, then it is integration which is 'easy' and differentiation 'difficult', and the latter may not, strictly speaking, be possible at all in the ordinary way. Numerical techniques are generally concerned with approximations, but there is also a close analogue of this aspect of things in the exact theory, and again it is integration which can be performed in circumstances where differentiation cannot. Let us try to understand some of this. The issues have to do, in fact, with what one actually means by a 'function'.

To Euler, and the other mathematicians of the 17th and 18th centuries, a 'function' would have meant something that one could write down explicitly, like x^2 or $\sin x$ or $\log(3 - x + e^x)$, or perhaps something defined by some formula involving an integration or maybe by an explicitly given power series. Nowadays, one prefers to think in terms of 'mappings', whereby some array A of numbers (or of more general entities) called the *domain* of the function is 'mapped' to some other array B, called the *target* of the function (see Fig. 6.1). The essential point of this is that the function would assign a member of the target B to each member of the domain A. (Think of the function as 'examining' a number that belongs to A and then, depending solely upon which number it finds, it would produce a definite number belonging to B.) This kind of function can be just a 'look-up table'. There would be no requirement that there be a reasonable-looking 'formula' which expresses the action of the function in a manifestly explicit way.

Let us consider some examples. In Fig. 6.2, I have drawn the graphs of three simple functions[1], namely those given by x^2, $|x|$, and $\theta(x)$. In each case, the domain and target spaces are both to be the totality of *real numbers*, this totality being normally represented by the symbol $\mathbb{R}$. The function that I am denoting by 'x^2' simply takes the square of the real number that it is examining. The function denoted by '$|x|$' (called the *absolute value*) just yields x if x is non-negative, but gives $-x$ if x is negative; thus $|x|$ itself is never negative. The function '$\theta(x)$' is 0 if x is negative, and 1 if x is positive; it is usual also to define $\theta(0) = \frac{1}{2}$. (This function is called the Heaviside *step function*; see §21.1 for another important mathematical influence of Oliver Heaviside, who is perhaps better known for first postulating the Earth's atmospheric 'Heaviside layer', so vital to radio transmission.) Each of these is a perfectly good

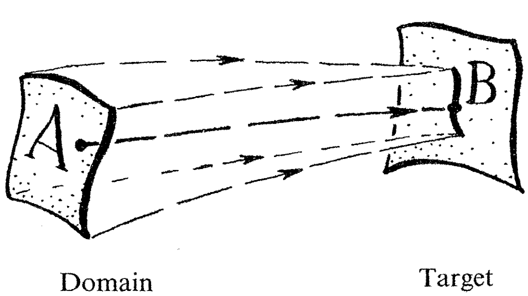

Fig. 6.1 A *function* as a 'mapping', whereby its *domain* (some array A of numbers or of other entities) is 'mapped' to its *target* (some other array B). Every element of A is assigned some particular value in B, though different elements of A may attain the same value and some values of B may not be reached.

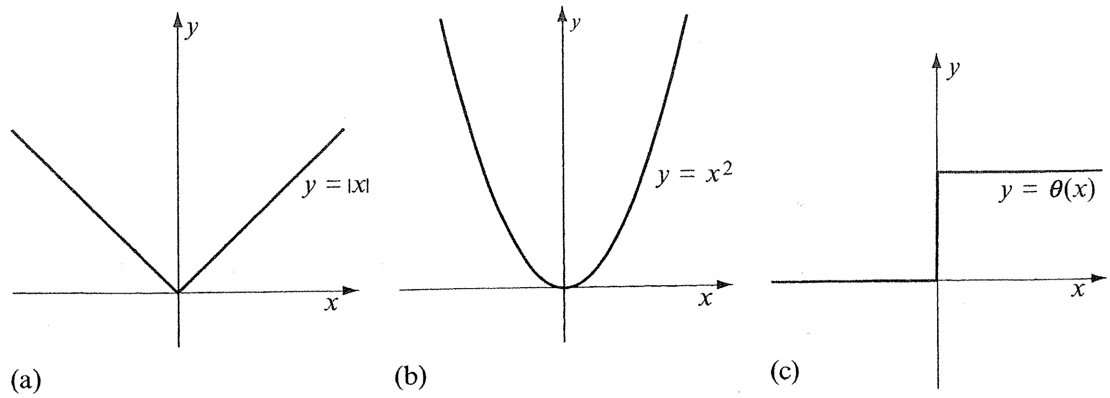

Fig. 6.2 Graphs of (a) $|x|$, (b) x^2, and (c) $\theta(x)$; the domain and target being the system of real numbers in each case.

function in this modern sense of the term, but Euler[2] would have had difficulty in accepting $|x|$ or $\theta(x)$ as a 'function' in *his* sense of the term.

Why might this be? One possibility is to think that the trouble with $|x|$ and $\theta(x)$ is that there is too much of the following sort of thing: 'if x is such-and-such then take so-and-so, whereas if x is...', and there is no 'nice formula' for the function. However, this is a bit vague, and in any case we could wonder what is really wrong with $|x|$ being counted as a formula. Moreover, once we have accepted $|x|$, we could write[6.1] a *formula* for $\theta(x)$:

$$\theta(x) = \frac{|x| + x}{2x}$$

(although we might wonder if there is a good sense in which this gets the right value for $\theta(0)$, since the formula just gives 0/0). More to the point is that the trouble with $|x|$ is that it is not 'smooth', rather than that its explicit expression is not 'nice'. We see this in the 'angle' in the middle of Fig. 6.2a. The presence of this angle is what prevents $|x|$ from having a well-defined *slope* at $x = 0$. Let us next try to come to terms with this notion.

6.2 Slopes of functions

As remarked above, one of the things with which differential calculus is concerned is, indeed, the finding of 'slopes'. We see clearly from the graph of $|x|$, as shown in Fig. 6.2a, that it does not have a unique slope at the

[6.1] Show this (ignoring $x = 0$).

origin, where our awkward angle is. Everywhere else, the slope is well defined, but not at the origin. It is because of this trouble at the origin that we say that $|x|$ is not *differentiable* at the origin or, equivalently, not *smooth* there. In contrast, the function x^2 has a perfectly good uniquely defined slope everywhere, as illustrated in Fig. 6.2b. Indeed, the function x^2 is differentiable everywhere.

The situation with $\theta(x)$, as illustrated in Fig. 6.2c, is even worse than for $|x|$. Notice that $\theta(x)$ takes an unpleasant 'jump' at the origin ($x = 0$). We say that $\theta(x)$ is *discontinuous* at the origin. In contrast, both the functions x^2 and $|x|$ are *continuous* everywhere. The awkwardness of $|x|$ at the origin is not a failure of continuity but of differentiability. (Although the failure of continuity and of smoothness are different things, they are actually interconnected concepts, as we shall be seeing shortly.)

Neither of these failings would have pleased Euler, presumably, and they seem to provide reasons why $|x|$ and $\theta(x)$ might not be regarded as 'proper' functions. But now consider the two functions illustrated in Fig. 6.3. The first, x^3, would be acceptable by anyone's criteria; but what about the second, which can be defined by the expression $x|x|$, and which illustrates the function that is x^2 when x is non-negative and $-x^2$ when x is negative? To the eye, the two graphs look rather similar to each other and certainly 'smooth'. Indeed, they both have a perfectly good value for the 'slope' at the origin, namely zero (which means that the curves have a horizontal slope there) and are, indeed, 'differentiable' everywhere, in the most direct sense of that word. Yet, $x|x|$ certainly does not seem to be the 'nice' sort of function that would have satisfied Euler.

One thing that is 'wrong' with $x|x|$ is that it does not have a well-defined *curvature* at the origin, and the notion of curvature is certainly something that the differential calculus is concerned with. In fact, 'curvature' is something that involves what are called 'second derivatives', which

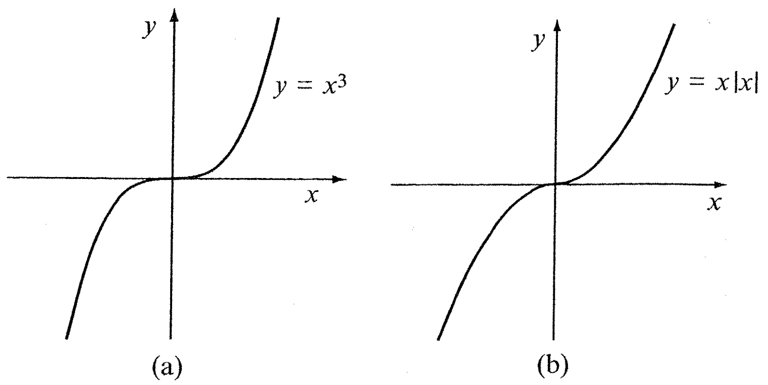

Fig. 6.3 Graphs of (a) x^3 and of (b) $x|x|$ (i.e. x^2 if $x \geq 0$ and $-x^2$ if $x < 0$).

means doing the differentiation twice. Indeed, we say that the function $x|x|$ is not *twice differentiable* at the origin. We shall come to second and higher derivatives in §6.3.

In order to start to understand these things, we shall need to see what the operation of differentiation really does. For this, we need to know how a *slope* is measured. This is illustrated in Fig. 6.4. I have depicted a fairly representative-looking function, which I shall call $f(x)$. The curve in Fig. 6.4a depicts the relation $y = f(x)$, where the value of the coordinate y measures the height and the value of x measures horizontal displacement, as is usual in a Cartesian description. I have indicated the slope of the curve at one particular point p, as the increment in the y coordinate divided by the increment in the x coordinate, as we proceed along the *tangent line* to the curve, touching it at the point p. (The technical definition of 'tangent line' depends upon the appropriate limiting procedures, but it is not my purpose here to provide these technicalities. I hope that the reader will find my intuitive descriptions adequate for our immediate purposes.[3]) The standard notation for the value of this slope is $\mathrm{d}y/\mathrm{d}x$ (and pronounced 'dy by dx'). We can think of 'dy' as a very tiny increase in the value of y along the curve and of 'dx' as the corresponding tiny increase in the value of x. (Here, technical correctness would require us to go to the 'limit', as these tiny increases each get reduced to zero.)

We can now consider another curve, which plots (against x) this slope at each point p, for the various possible choices of x-coordinate; see Fig. 6.4b. Again, I am using a Cartesian description, but now it is $\mathrm{d}y/\mathrm{d}x$ that is plotted vertically, rather than y. The horizontal displacement is still measured by x. The function that is being plotted here is commonly called $f'(x)$, and we can write $\mathrm{d}y/\mathrm{d}x = f'(x)$. We call $\mathrm{d}y/\mathrm{d}x$ the *derivative of y with respect to x*, and we say that the function $f'(x)$ is the *derivative*[4] *of* $f(x)$.

6.3 Higher derivatives; C^∞-smooth functions

Now let us see what happens when we take a *second* derivative. This means that we are now looking at the slope-function for the new curve of Fig. 6.4b, which plots $u = f'(x)$, where u now stands for $\mathrm{d}y/\mathrm{d}x$. In Fig. 6.4c, I have plotted this 'second-order' slope function, which is the graph of $\mathrm{d}u/\mathrm{d}x$ against x, in the same kind of way as I did before for $\mathrm{d}y/\mathrm{d}x$, so the value of $\mathrm{d}u/\mathrm{d}x$ now provides us with the slope of the second curve $u = f'(x)$. This gives us what is called the second derivative of the original function $f(x)$, and this is commonly written $f''(x)$. When we substitute $\mathrm{d}y/\mathrm{d}x$ for u in the quantity $\mathrm{d}u/\mathrm{d}x$, we get the *second derivative* of y with respect to x, which is

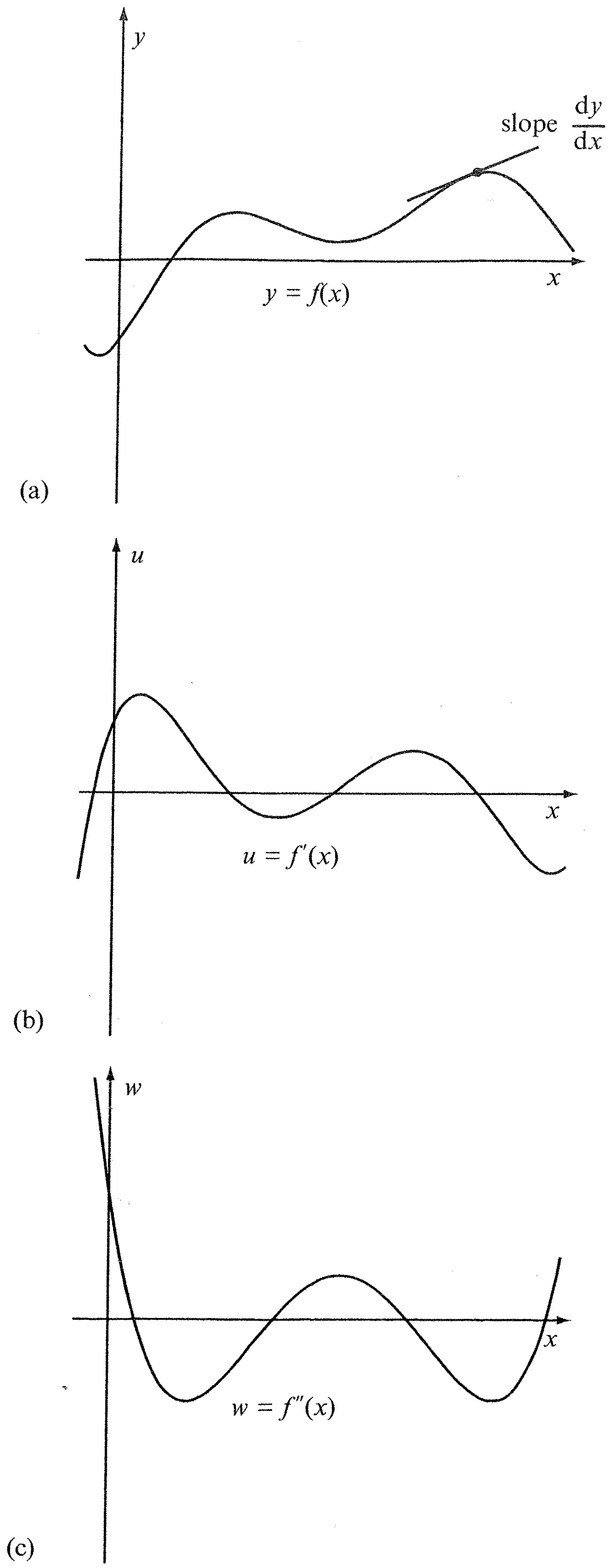

Fig. 6.4 Cartesian plot of (a) $y = f(x)$, (b) the derivative $u = f'(x)$ $(= \mathrm{d}y/\mathrm{d}x)$, and (c) the second derivative $f''(x) = \mathrm{d}^2y/\mathrm{d}x^2$. (Note that $f(x)$ has horizontal slope just where $f'(x)$ meets the x-axis, and it has an inflection point where $f''(x)$ meets the x-axis.)

(slightly illogically) written $\mathrm{d}^2y/\mathrm{d}x^2$ (and pronounced 'd-two-y by dx-squared').

Notice that the values of x where the original function $f(x)$ has a horizontal slope are just the values of x where $f'(x)$ meets the x-axis (so $\mathrm{d}y/\mathrm{d}x$ vanishes for those x-values). The places where $f(x)$ acquires a (local) maximum or minimum occur at such locations, which is important when we are interested in finding the (locally) greatest and smallest values of a function. What about the places where the second derivative $f''(x)$ meets the x-axis? These occur where the *curvature* of $f(x)$ vanishes. In general, these points are where the direction in which the curve $y = f(x)$ 'bends' changes from one side of the curve to the other, at a place called a *point of inflection*. (In fact, it would not be correct to say that $f''(x)$ actually 'measures' the curvature of the curve defined by $y = f(x)$, in general; the actual curvature is given by a more complicated expression[5] than $f''(x)$, but it involves $f''(x)$, and the curvature vanishes whenever $f''(x)$ vanishes.

Let us next consider our two (superficially) similar-looking functions x^3 and $x|x|$, considered above. In Fig. 6.5a,b,c, I have plotted x^3 and its first and second derivatives, as I did with the function $f(x)$ in Fig. 6.4, and, in Fig. 6.5d,e,f, I have done the same with $x|x|$. In the case of x^3, we see that

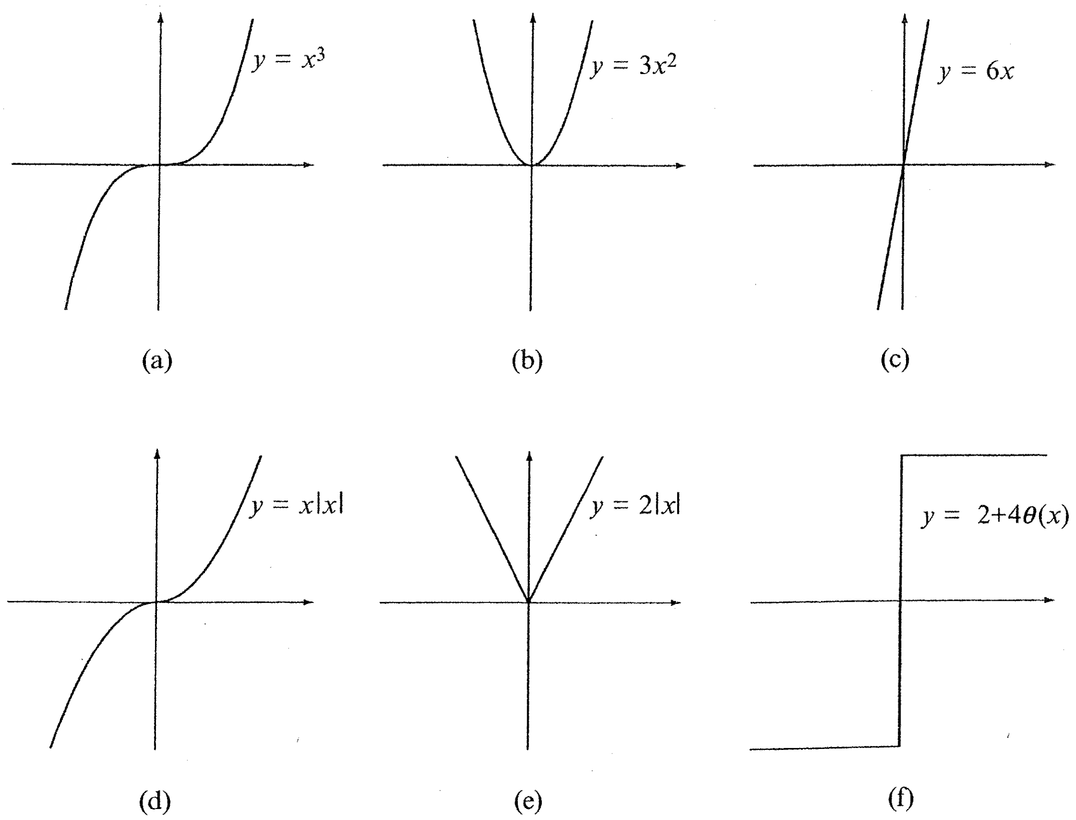

Fig. 6.5 (a), (b), (c) Plots of x^3, its first derivative $3x^2$, and its second derivative $6x$, respectively. (d), (e), (f) Plots of $x|x|$, its first derivative $2|x|$, and the second derivative $-2 + 4\theta(x)$, respectively.

there are no problems with continuity or smoothness with either the first or second derivative. In fact the first derivative is $3x^2$ and the second is $6x$, neither of which would have given Euler a moment of worry. (We shall see how to obtain these explicit expressions shortly.) However, in the case of $x|x|$, we find something very much like the 'angle' of Fig. 6.2a for the first derivative, and a 'step function' behaviour for the second derivative, very similar to Fig. 6.2c. We have failure of smoothness for the first derivative and failure of continuity for the second. Euler would not have cared for this at all. This first derivative is actually $2|x|$ and the second derivative is $-2 + 4\theta(x)$. (My more pedantic readers might complain that I should not so glibly write down a 'derivative' for $2|x|$, which is not actually differentiable at the origin. True, but this is just a quibble: full justification of this can be achieved using the notions that will be introduced at the end of Chapter 9.)

We can easily imagine that functions can be constructed for which such failure of smoothness or of continuity does not show up until many derivatives have been calculated. Indeed, functions of the form $x^n|x|$ will do the trick, where we can take n to be a positive integer which can be as large as we like. The mathematical terminology for this sort of thing is to say that the function $f(x)$ is C^n-*smooth* if it can be differentiated n times (at each point of its domain) and the nth derivative is continuous.[6] The function $x^n|x|$ is in fact C^n-smooth, but it is not C^{n+1}-smooth at the origin.

How big should n be to satisfy Euler? It seems clear that he would not have been content to stop at *any* particular value of n. It should surely be possible to differentiate the kind of self-respecting function that Euler would have approved of as many times as we like. To cover this situation, mathematicians refer to a function as being C^∞-*smooth* if it counts as C^n-smooth for *every* positive integer n. To put this another way, a C^∞-smooth function must be differentiable as many times as we choose.

Euler's notion of a function would, we presume, have demanded something like C^∞-smoothness. At least, we could imagine that he would have expected his functions to be C^∞-smooth at most places in the domain. But what about the function $1/x$? (See Fig. 6.6.) This is certainly not C^∞-smooth at the origin. It is not even *defined* at the origin in the modern sense of a function. Yet our Euler would certainly have accepted $1/x$ as a decent 'function', despite this problem. There is a simple natural-looking formula for it, after all. One could imagine that Euler would not have been so much concerned about his functions being C^∞-smooth at *every* point on its domain (assuming that he would have worried about 'domains' at all). Perhaps things going wrong at the odd point or so would not matter. But $|x|$ and $\theta(x)$ only went wrong at the same 'odd point' as does $1/x$. It seems that, despite all our efforts, we still have not captured the 'Eulerian' notion of a function that we have been striving for.

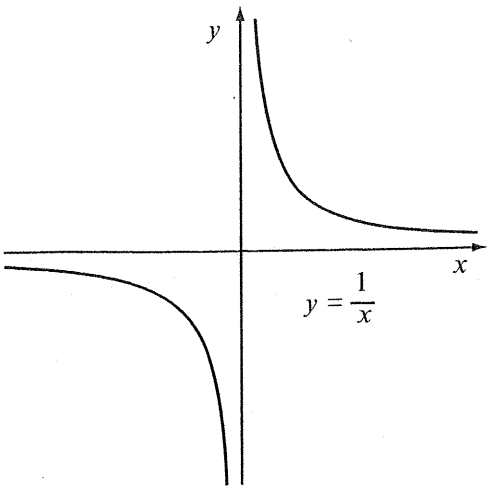

Fig. 6.6 Plot of $\frac{1}{x}$.

Let us take another example. Consider the function $h(x)$, defined by the rules

$$h(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

The graph of this function is depicted in Fig. 6.7. This certainly looks like a smooth function. In fact it is *very* smooth. It is C^∞-smooth over the entire domain of real numbers. (Proving this is the sort of thing that one does in a mathematics undergraduate course. I remember having to tackle this one when I was an undergraduate myself.[6.2] Despite its utter smoothness, one can certainly imagine Euler turning up his nose at a function defined in this kind of a way. It is clearly not just 'one function', in Euler's sense. It is 'two

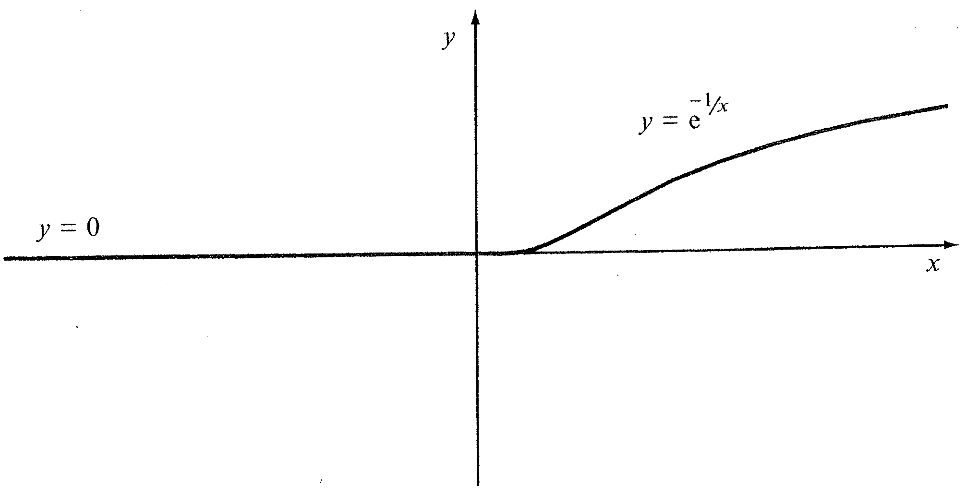

Fig. 6.7 Plot of $y = h(x)$ ($= 0$ if $x \le 0$ and $= e^{-1/x}$ if $x > 0$), which is C^∞-smooth.

[6.2] Have a go at proving this if you have the background.

functions stuck together', no matter how smooth a gluing job has been done to paste over the 'glitch' at the origin. In contrast, to Euler, $\frac{1}{x}$ *is* just one function, despite the fact that it is separated into two pieces by a very nasty 'spike' at the origin, where it is not even continuous, let alone smooth (Fig. 6.6). To our Euler, the function $h(x)$ is really no better than $|x|$ or $\theta(x)$. In those cases, we clearly had 'two functions glued together', though with much shoddier gluing jobs (and with $\theta(x)$, the glued bits seem to have come apart altogether).

6.4 The 'Eulerian' notion of a function?

How are we to come to terms with this 'Eulerian' notion of having just a single function as opposed to a patchwork of separate functions? As the example of $h(x)$ clearly shows, C^∞-smoothness is not enough. It turns out that there are actually two completely different-looking approaches to resolving this issue. One of these uses complex numbers, and it is deceptively simple to state, though momentous in its implications. We simply demand that our function $f(x)$ be extendable to a function $f(z)$ of the complex variable z so that $f(z)$ is smooth in the sense that it is merely required to be *once* differentiable with respect to the complex variable z. (Thus $f(z)$ is, in the complex sense, a kind of C^1-function.) It is an extraordinary display of genuine magic that we do not need more than this. If $f(z)$ can be differentiated once with respect to the complex parameter z, then it can be differentiated as many times as we like!

I shall return to the matter of complex calculus in the next chapter. But there is another approach to the solution of this 'Eulerian notion of function' problem using only real numbers, and this involves the concept of power series, which we encountered in §2.5. (One of the things that Euler was indeed a master of was manipulating power series.) It will be useful to consider the question of power series, in this section, before returning to the issue of complex differentiability. The fact that, locally, complex differentiability turns out to be equivalent to the validity of power series expansions is one of the truly great pieces of complex-number magic.

I shall come to all this in due course, but for the moment let us stick with real-number functions. Suppose that some function $f(x)$ actually has a power series representation:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots.$$

Now, there are methods of finding out, from $f(x)$, what the coefficients $a_0, a_1, a_2, a_3, a_4, \ldots$ must be. For such an expansion to exist, it is necessary (although not sufficient, as we shall shortly see) that $f(x)$ be C^∞-smooth, so we shall have new functions $f'(x), f''(x), f'''(x), f''''(x), \ldots,$

etc., which are the first, second, third, fourth, etc., derivatives of $f(x)$, respectively. In fact, we shall be concerned with the values of these functions only at the origin ($x = 0$), and we need the C^∞-smoothness of $f(x)$ only there. The result (sometimes called *Maclaurin's series*[7]) is that if $f(x)$ has such a power series expansion, then[6.3]

$$a_0 = f(0),\ a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!},\ a_4 = \frac{f''''(0)}{4!},\ldots.$$

(Recall, from §5.3, that $n! = 1 \times 2 \times \ldots \times n$.) But what about the other way around? If the a's are given in this way, does it follow that the sum actually gives us $f(x)$ (in some interval encompassing the origin)?

Let us return to our seemingly seamless $h(x)$. Perhaps we can spot a flaw at the joining point ($x = 0$) using this idea. We try to see whether $h(x)$ actually has a power series expansion. Taking $f(x) = h(x)$ in the above, we consider the various coefficients $a_0, a_1, a_2, a_3, a_4, \ldots$, noticing that they all have to vanish, because the series has to agree with the value $h(x) = 0$, whenever x is just to the left of the origin. In fact, we find that they all vanish also for $e^{-1/x}$, which is basically the reason why $h(x)$ is C^∞-smooth at the origin, with all derivatives coming from the two sides matching each other. But this also tells us that there is no way that the power series can work, because all the terms are zero (see Exercise [6.1]) and therefore do not actually sum to $e^{-1/x}$. Thus there *is* a flaw at the join at $x = 0$: the function $h(x)$ cannot be expressed as a power series. We say that $h(x)$ is not *analytic* at $x = 0$.

In the above discussion, I have really been referring to what would be called a power series expansion *about the origin*. A similar discussion would apply to any other point of the real-number domain of the function. But then we have to 'shift the origin' to some other particular point, defined by the real number p in the domain, which means replacing x by $x - p$ in the above power series expansion, to obtain

$$f(x) = a_0 + a_1(x-p) + a_2(x-p)^2 + a_3(x-p)^3 + \cdots,$$

where now

$$a_0 = f(p),\ a_1 = \frac{f'(p)}{1!},\ a_2 = \frac{f''(p)}{2!},\ a_3 = \frac{f'''(p)}{3!},\ldots.$$

This is called a power series expansion *about* p. The function $f(x)$ is called *analytic at* p if it can be expressed as such a power series expression in some interval encompassing $x = p$. If $f(x)$ is analytic at all points of its domain, we

[6.3] Show this, using rules given towards end of §6.5.

just call it an *analytic function* or, equivalently, a C^{ω}-smooth function. Analytic functions are, in a clear sense, even 'smoother' than C^{∞}-smooth functions. In addition, they have the property that it is not possible to get away with gluing two 'different' analytic functions together, in the manner of the examples $\theta(x)$, $|x|$, $x|x|$, $x^n|x|$, or $h(x)$, given above. Euler would have been pleased with analytic functions. These are 'honest' functions indeed!

However, all these power series are awkward things to be carrying around, even if only in the imagination. The 'complex' way of looking at things turns out to be enormously more economical. Moreover, it gives us a greater depth of understanding. For example, the function $\frac{1}{x}$ is not analytic at $x = 0$; yet it is still 'one function'.[6.4] The 'power series philosophy' does not directly tell us this. But from the point of view of complex numbers, $\frac{1}{x}$ is clearly just one function, as we shall be seeing.

6.5 The rules of differentiation

Before discussing these matters, it will be useful to say a little about the wonderful rules that the differential calculus actually provides us with—rules that enable us to differentiate functions almost without really thinking at all, but only after months of practice, of course! These rules enable us to see how to write down the derivative of many functions directly, particularly when they are represented in terms of power series.

Recall that, as a passing comment, I remarked above that the derivative of x^3 is $3x^2$. This is a particular case of a simple but important formula: the derivative of x^n is nx^{n-1}, which we can write

$$\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = nx^{n-1}.$$

(It would distract us too much, here, for me to explain why this formula holds. It is not really hard to show, and the interested reader can find all that is required in any elementary textbook on calculus.[8] Incidentally, n need not be an integer.) We can also express[9] this equation ('multiplying through by dx') by the convenient formula

$$\mathrm{d}(x^n) = nx^{n-1}\mathrm{d}x.$$

There is not much more that we need to know about differentiating power series. There are basically two other things. First, the derivative of a sum of functions is the sum of the derivatives of the functions:

$$\mathrm{d}[f(x) + g(x)] = \mathrm{d}f(x) + \mathrm{d}g(x).$$

[6.4] Consider the 'one function' e^{-1/x^2}. Show that it is C^{∞}, but not analytic at the origin.

This then extends to a sum of any finite number of functions.[10] Second, the derivative of a constant times a function is the constant times the derivative of that function:

$$\mathrm{d}\{a\,f(x)\} = a\;\mathrm{d}f(x).$$

By a 'constant' I mean a number that does not vary with x. The *coefficients* $a_0, a_1, a_2, a_3, \ldots$ in the power series are constants. With these rules, we can directly differentiate *any* power series.[6.5]

Another way of expressing the constancy of a is

$$\mathrm{d}a = 0.$$

Bearing this in mind, we find that the rule given immediately above this one is really a special case (with $g(x) = a$) of the 'Leibniz law':

$$\mathrm{d}\{f(x)\;g(x)\} = f(x)\;\mathrm{d}g(x) + g(x)\;\mathrm{d}f(x)$$

(and $\mathrm{d}(x^n)/\mathrm{d}x = nx^{n-1}$, for any natural number n, can also be derived from the Leibniz law[6.6]). A useful further law is

$$\mathrm{d}\{f(g(x))\} = f'(g(x))g'(x)\mathrm{d}x.$$

From the last two and the first, putting $f(x)[g(x)]^{-1}$ into the Leibniz law, we can deduce[6.7]

$$\mathrm{d}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\;\mathrm{d}f(x) - f(x)\;\mathrm{d}g(x)}{g(x)^2}.$$

Armed with these few rules (and loads and loads of practice), one can become an 'expert at differentiation' without needing to have much in the way of actual *understanding* of why the rules work! This is the power of a good calculus.[6.8] Moreover, with the knowledge of the derivatives of just a few special functions,[6.9] one can become even more of an expert. Just so that the uninitiated reader can become an 'instant member' of the club of expert differentiators, let me provide the main examples:[11],[6.10]

[6.5] Using the power series for e^x given in §5.3, show that $\mathrm{d}e^x = e^x\mathrm{d}x$.

[6.6] Establish this.

[6.7] Derive this.

[6.8] Work out $\mathrm{d}y/\mathrm{d}x$ for $y = (1 - x^2)^4$, $y = (1 + x)/(1 - x)$.

[6.9] With a constant, work out $\mathrm{d}(\log_a x)$, $\mathrm{d}(\log_x a)$, $\mathrm{d}(x^x)$.

[6.10] For the first, see Exercise [6.5]; derive the second from $\mathrm{d}(e^{\log x})$; the third and fourth from $\mathrm{d}e^{ix}$, assuming that the complex quantities work like real ones; and derive the rest from the earlier ones, using $\mathrm{d}(\sin(\sin^{-1} x))$, etc., and noting that $\cos^2 x + \sin^2 x = 1$.

$$
\begin{aligned}
d(e^x) &= e^x\, dx, \\
d(\log x) &= \frac{dx}{x}, \\
d(\sin x) &= \cos x\, dx, \\
d(\cos x) &= -\sin x\, dx, \\
d(\tan x) &= \frac{dx}{\cos^2 x}, \\
d(\sin^{-1} x) &= \frac{dx}{\sqrt{1-x^2}}, \\
d(\cos^{-1} x) &= \frac{-dx}{\sqrt{1-x^2}}, \\
d(\tan^{-1} x) &= \frac{dx}{1+x^2}.
\end{aligned}
$$

This illustrates the point referred to at the beginning of this section that, when we are given explicit formulae, the operation of differentiation is 'easy'. Of course, I do not mean by this that this is something that you could do in your sleep. Indeed, in particular examples, it may turn out that the expressions get very complicated indeed. When I say 'easy', I just mean that there is an explicit computational procedure for carrying out differentiation. If we know how to differentiate each of the ingredients in an expression, then the procedures of calculus, as given above, tell us how to go about differentiating the entire expression. 'Easy', here, really means something that could be readily put on a computer. But things are very different if we try to go in the reverse direction.

6.6 Integration

As stated at the beginning of the chapter, *integration* is the reverse of differentiation. What this amounts to is trying to find a function $g(x)$ for which $g'(x) = f(x)$, i.e. finding a solution $y = g(x)$ to the equation $dy/dx = f(x)$. Another way of putting this is that, instead of moving down the picture in Fig. 6.4 (or Fig. 6.5), we try to work our way upwards. The beauty of the 'fundamental theorem of calculus' is that this procedure is telling us how to work out areas under each successive curve. Have a look at Fig. 6.8. Recall that the bottom curve $u = f(x)$ can be obtained from the top curve $y = g(x)$ because it plots the slopes of that curve, $f(x)$ being the derivative of $g(x)$. This is just what we had before. But now let us start with the bottom curve. We find that the top curve simply maps out the areas beneath the bottom curve. A little more explicitly: if we take two vertical lines in the bottom picture given by $x = a$ and $x = b$, respectively, then the area bounded by these two lines, the x-axis, and the curve itself, will be the difference between the heights of the top curve at those two x-values. Of course, in matters such as this, we must

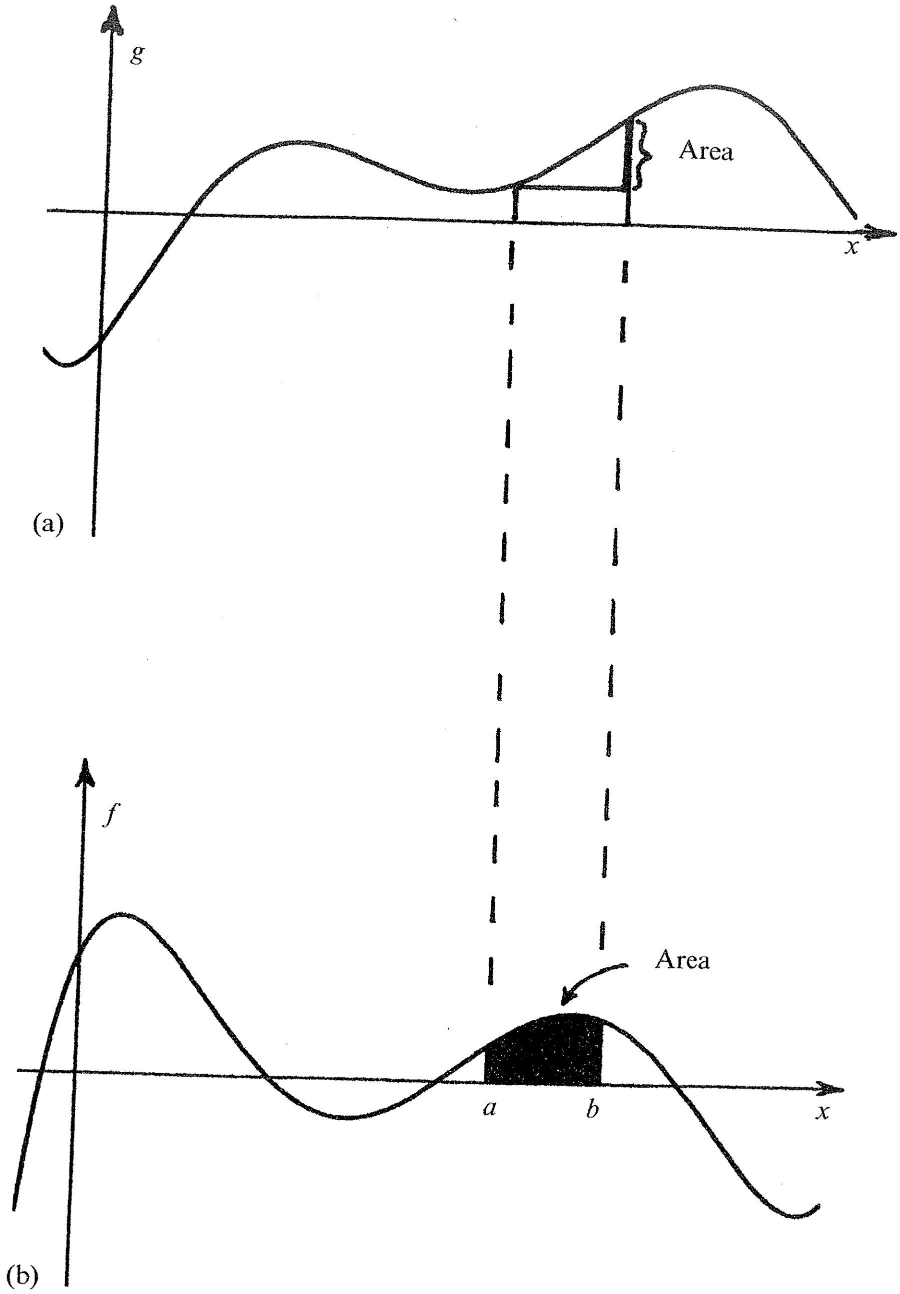

Fig. 6.8 Fundamental theorem of calculus: re-interpret Fig. 6.4a,b, proceeding upwards rather than downwards. Top curve (a) plots areas under bottom curve (b), where the area bounded by two vertical lines $x = a$ and $x = b$, the x-axis, and the bottom curve is the difference, $g(b) - g(a)$, of heights of the top curve at those two x-values (signs taken into account).

be careful about 'signs'. In regions where the bottom curve dips below the x-axis, the areas count negatively. Moreover, in the picture, I have taken $a < b$ and the 'difference between the heights' of the top curve in the form $g(b) - g(a)$. Signs would be reversed if $a > b$.

In Fig. 6.9, I have tried to make it intuitively believable why there is this inverse relationship between slopes and areas. We imagine b to be greater

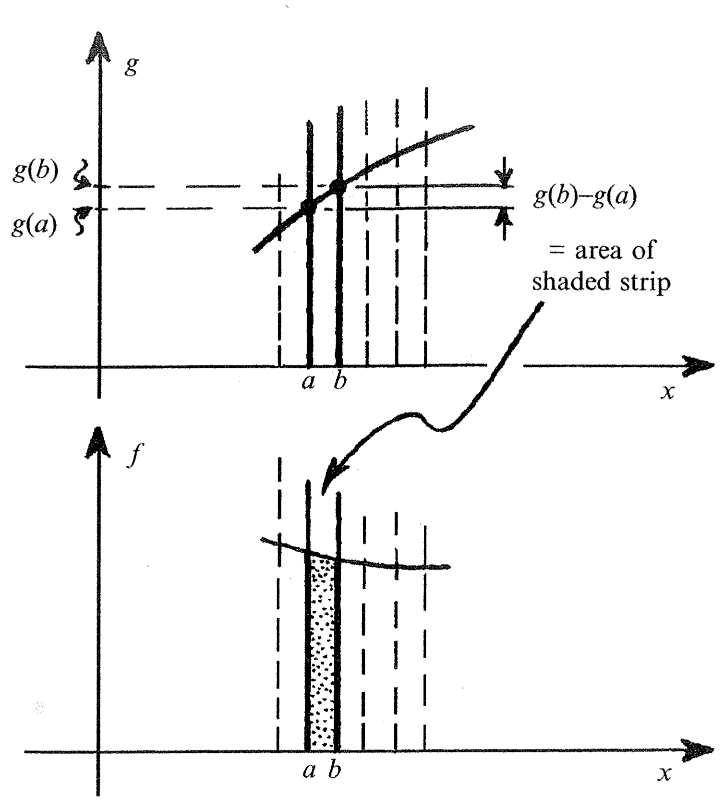

Fig. 6.9 Take $b > a$ by a tiny amount. In the bottom picture, the area of a very narrow strip between neighbouring lines $x = a$, $x = b$ is essentially the product of the strip's width $b - a$ with its height (from x-axis to curve). This height is the slope of top curve there, whence the strip's area is this slope × strip's width, which is the amount by which top curve rises from a to b, i.e. $g(b) - g(a)$. Adding many narrow strips, we find that the area of a broad strip under the bottom curve is the corresponding amount by which the top curve rises.

than a by just a very tiny amount. Then the area to be considered, in the bottom picture, is that of the very narrow strip bounded by the neighbouring lines $x = a$ and $x = b$. The measure of this area is essentially the product of the strip's tiny width (i.e. $b - a$) with its height (from the x-axis to the curve). But the strip's height is supposed to be measuring the slope of the top curve at that point. Therefore, the strip's area is this slope multiplied by the strip's width. But the slope of the top curve times the strip's width is the amount by which the top curve rises from a to b, that is, the difference $g(b) - g(a)$. Thus, for very narrow strips, the area is indeed measured by this stated difference. Broad strips are taken to be built up from large numbers of narrow strips, and we get the total area by measuring how much the top curve rises over the entire interval.

There is a significant point that I should bring out here. In the passage from the bottom curve to the top curve there is a non-uniqueness about how high the whole top curve is to be placed. We are only concerned with *differences* between heights on the top curve, so sliding the whole curve up or down by some constant amount will not make any difference. This is clear from the 'slope' interpretation too, since the slope at different points on the top curve will be just the same as before if we slide it up or down. What this amounts to, in our calculus, is that if we add a constant C to $g(x)$, then the resulting function still differentiates to $f(x)$:

$$d(g(x) + C) = dg(x) + dC = f(x)\,dx + 0 = f(x)\,dx.$$

Such a function $g(x)$, or equivalently $g(x) + C$ for some arbitrary constant C, is called an *indefinite integral* of $f(x)$, and we write

$$\int f(x)\,dx = g(x) + \text{const.}$$

This is just another way of expressing the relation $d[g(x) + \text{const.}] = f(x)dx$, so we just think of the '$\int$' sign as the inverse of the 'd' symbol. If we want the specific area between $x = a$ and $x = b$, then we want what is called the *definite integral*, and we write

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

If we know the function $f(x)$ and we wish to obtain its integral $g(x)$, we do not have nearly such straightforward rules for obtaining it as we did for differentiation. A great many tricks are known, a variety of which can be found in standard textbooks and computer packages, but these do not suffice to handle all cases. In fact, we frequently find that the family of explicit standard functions that we had been using previously has to be broadened, and that new functions have to be 'invented' in order to express the results of the integration. We have, in effect, seen this already in the special examples given above. Suppose that we were familiar just with functions made up of combinations of powers of x. For a general power x^n, we can integrate it to get $x^{n+1}/(n + 1)$. (This is just using our formula above, in §6.5, with $n + 1$ for n: $d(x^{n+1})/dx = (n + 1)x^n$.) Everything is fine until we worry about what to do with the case $n = -1$. Then the supposed answer $x^{n+1}/(n + 1)$ has zero in the denominator, so this won't work. How, then, do we integrate x^{-1}? Well, we notice that, by the greatest of good fortune, there is the formula $d(\log x) = x^{-1}dx$ sitting in our list in §6.5. So the answer is $\log x + \text{const.}$

This time we were lucky! It just happened that we had been studying the logarithm function before for a different reason, and we knew about some of its properties. But on other occasions, we might well find that there is no function that we had previously known about in terms of which we can express our answer. Indeed, integrals frequently provide the appropriate means whereby new functions are *defined*. It is in this sense that explicit integration is 'difficult'.

On the other hand, if we are not so interested in explicit expressions, but are concerned with questions of *existence* of functions that are the derivatives or integrals of given functions, then the boot is on the other foot. Integration is now the operation that works smoothly, and differentiation causes the problems. The same applies when performing these operations with numerical data. Basically, the problem with differentiation is that it depends very critically on the fine details of the function to be differentiated. This can present a problem if we do not have an explicit expression for the function to be differentiated. Integration, on the other hand, is relatively insensitive to such matters, being concerned with the broad overall nature of the function to be integrated. In fact, any continuous function (a C^0-function) whose domain is a 'closed' interval $a \leqslant x \leqslant b$ can be integrated,[12] the result being C^1 (i.e. C^1-smooth). This can be integrated again, the result being C^2, and then again, giving a C^3-smooth function, and so on. Integration makes the functions smoother and smoother, and we can keep on going with this indefinitely. Differentiation, on the other hand just makes things worse, and it may come to an end at a certain point, where the function becomes 'non-differentiable'.

Yet, there are approaches to these issues that enable the process of differentiation to be continued indefinitely also. I have hinted at this already, when I allowed myself to differentiate the function $|x|$ to obtain $\theta(x)$, even though $|x|$ is 'not differentiable'. We could attempt to go further and differentiate $\theta(x)$ also, despite the fact that it has an infinite slope at the origin. The 'answer' is what is called the Dirac[13] *delta function*—an entity of considerable importance in the mathematics of quantum mechanics. The delta function is not really a function at all, in the ordinary (modern) sense of 'function' which maps domains to target spaces. There is no 'value' for the delta function at the origin (which could only have been *infinity* there) and it is zero elsewhere. Yet the delta function does finds a clear mathematical definition within various broader classes of mathematical entities, the best known being *distributions*.

For this, we need to extend our notion of C^n-functions to cases where n can be a negative integer. The function $\theta(x)$ is then a C^{-1}-function and the delta function is C^{-2}. Each time we differentiate, we must decrease the differentiability class by unity (i.e. the class becomes more negative by one unit). It would seem that we are getting farther and farther from Euler's notion of a 'decent function' with all this and that he would tell us to have no truck with such things, were it not for the fact that they seem to be useful. Yet, we shall be finding, in due course, that it is here that complex numbers astound us with an irony—an irony that is expressed in one of their finest magical feats of all! We shall have to wait until the end of Chapter 9 to witness this feat, for it is not something that I can properly describe just yet. The reader must bear with me for a while, for the ground needs first to be made ready, paved with other superbly magical ingredients.