A Schur algorithm for symmetric inner functions

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Symmetric inner functions

Schur function (discrete-time): $S(z)$ square rational, analytic and contractive in the open unit disk,

$$S(z) S(z) \leq I, |z| < 1$$

Inner function: $Q(z)$ is Schur and

$$Q(z) Q(z) = I, |z| = 1$$

Symmetric: $Q(z)^T = Q(z)$

Symmetric inner rational functions arise in the description of physical systems which satisfy the conservation and reciprocity laws.
Nevanlinna-Pick interpolation problem

Find a $p \times p$ Schur function $S$ such that

$$S(w)u = v, \quad u, v \in \mathbb{C}^p, \quad |w| < 1, \quad \|u\| = 1, \quad \|v\| < 1$$

The solutions can be parametrized via a linear fractional transformation

$$S = T_\Theta(R) = (\Theta_1 R + \Theta_2)(\Theta_3 R + \Theta_4)^{-1}$$

in which $R$ is a $p \times p$ Schur function.

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{bmatrix}$$

is built from the interpolation data $(w, u, v)$.

It is $J$-inner:

$$\Theta(z)^* J \Theta(z) \leq J, \quad |z| < 1,$$

$$\Theta(z)^* J \Theta(z) = J, \quad |z| = 1$$

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}$$
Parametrizations via a Schur algorithm

\( \mathcal{T}_n^p \) manifold of \( p \times p \) inner functions of McMillan degree \( n \)

**Lemma:** \( w \in \mathbb{C}, \ |w| < 1, \ \exists u \) such that \( \|Q(w)u\| < 1. \)

**Schur algorithm:** \( Q = Q_n, Q_{n-1}, \ldots, Q_k \xrightarrow{\text{LFT}} Q_{k-1}, \ldots, Q_0 \)

\( Q_k \) has McMillan degree \( k \) and \( Q_0 \) is constant unitary. The LFT is built from an interpolation condition

\[ Q_k(w_k)u_k = v_k, \ |w_k| < 1, \ \|v_k\| < \|u_k\| = 1 \]

**Atlas of charts** (local parametrizations):
The \( w_i \)'s and the \( u_i \)'s define the chart while the \( v_i \)'s are the Schur parameters of the function \( Q \) in the chart.
Choice of the LFT

\[ Q(w)u = v \Rightarrow Q = T_{\Theta H}(R) = T_{\Theta}(T_{H}(R)) \]

\[ \Theta(z) = I_{2p} + (z - 1) \frac{1 - |w|^2}{1 - \|v\|^2} \frac{1}{(z - w)(1 - \bar{w})} \]

\[ \begin{bmatrix} v \\ u \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}^* \]

\[ J \]

\( H \) arbitrary constant \( J \)-unitary, \( H^*JH = J \)

Kimura, IEEE trans. autom. control, 1987

\( H = H(\nu u^*) \) circuit theoretical interpretation


\( H = I_{2p} \) function space approach \( Q_0 = Q(1) \).

Hanzon, Olivi, Peeters, ECC99

\( H = H(w, u, v) \rightarrow \) nice construction of balanced realizations
LFT’s which preserve symmetry

Let \( \Theta(z) \) be a \( J \)-inner function such that

\[
\bar{\Theta}(1/z) = K \Theta(z) K, \quad K = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}
\]

Then, the linear fractional transformation \( T_\Theta \) preserves symmetry

Every \( H \) constant \( J \)-unitary is of the form

\[
H = H(E) \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad H(E) = \begin{bmatrix} (I_p - EE^*)^{-1/2} & E(I_p - E^* E)^{-1/2} \\ E^* (I_p - EE^*)^{-1/2} & (I_p - E^* E)^{-1/2} \end{bmatrix}
\]

\( P, Q \) unitary matrices, \( H(E) \) Halmos extension (\( E \) strictly contractive)

\[
\overline{H(E)} = KH(E)K \iff E \text{ symmetric}
\]
Interpolation data for symmetric inner functions.

\( Q: p \times p \) symmetric inner function, McMillan degree \( n \geq 2 \)

To take into account symmetry:

\[
\begin{aligned}
Q(w)u & = v \\
u^T Q(w) & = v^T
\end{aligned}
\]

→ two-sided Nevanlinna-Pick problem ... same interpolation point.

To be well-posed:

\[ u^T Q'(w)u = \rho \]

Interpolation data: \( \delta = (w, u, v, \rho) \) → interpolation conditions \( C(\delta) \)

→ two-sided Nudelman problem Ball, Gohberg, Rodman, 1990
The solutions

Pick matrix: \( \Lambda_\delta = \begin{bmatrix} \sigma & \bar{\rho} \\ \rho & \sigma \end{bmatrix} \)

\( J \)-inner function:

\[ \Theta_\delta(z) = \]

\[ I_{2p} + (z - 1)C \begin{bmatrix} (z - w)^{-1} & 0 \\ 0 & (1 - z\bar{w})^{-1} \end{bmatrix} \Lambda_\delta^{-1} \begin{bmatrix} (1 - \bar{w})^{-1} & 0 \\ 0 & (1 - w)^{-1} \end{bmatrix} C^* J \]

\[ C = \begin{bmatrix} v & -\bar{u} \\ u & -\bar{v} \end{bmatrix} \]

There exists an inner function satisfying \( C(\delta) \) if and only if \( \Lambda_\delta \) is positive definite. Then

\[ Q = T_{\Theta_\delta}(R), \]

for some inner function \( R \) of McMillan degree \( n - 2 \)
Sketch of the proof (1)

\[ \delta = (w = 0, u, v = 0, \rho), \quad |\rho| < 1 \]

Taylor series: \[ Q(z) = Q(0) + zQ'(0) + \ldots \]

SVD: \[ Q(0) = V \text{diag}(0, \ldots, 0, \lambda_1, \ldots, \lambda_r)U^* \]

Since \( Q(0)u = 0 \), choose \( U = [u \ \cdots] \) and \( B(z) = U \text{diag}(z, 1, \ldots, 1)U^* \)

\[ Q_1(z) = Q(z)B(z)^{-1} \]

\[ = Q(0)U \text{diag}(1/z, 1, \ldots, 1)U^* + Q'(0)U \text{diag}(1, z, \ldots, z)U^* \ldots \]

\[ = Q(0) + Q'(0)U \text{diag}(1, z, \ldots, z)U^* + z \times \ldots \]

and \( Q_1 \) satisfies the Nevanlinna-Pick interpolation condition

\[ u^TQ_1(0) = u^TQ'(0)uu^* = \rho u^* \]
Sketch of the proof (2)

\[ \delta = (w, u, v = 0, \rho) \]

\[ \beta_w(z) = \frac{(1 - \bar{w})(z - w)}{(1 - w)(1 - \bar{w}z)}, \quad \beta_w(w) = 0, \]

Then \( \tilde{Q}(\beta_w(z)) = Q(z) \) satisfies \( C(\tilde{\delta}) \)

\[ \tilde{\delta} = (0, u, v = 0, \tilde{\rho}), \quad \tilde{\rho} = \rho(1 - |w|^2) \frac{1 - w}{1 - \bar{w}} \]

\[ \delta = (w, u, v, \rho) \]

Assume \( E = Q(w) \) stricktly contractive (symmetric)

Then \( \hat{Q} = T_{H(-E)}(Q) \) satisfies \( C(\hat{\delta}) \)

\[ \hat{\delta} = (w, \hat{u}, v = 0, \hat{\rho}), \quad \hat{u} = (I - EE^*)^{1/2} \frac{u}{\sqrt{1 - \|v\|^2}}, \quad \hat{\rho} = \frac{\rho}{1 - \|v\|^2} \]
Symmetric Potapov factorization

If both $v$ and $\rho$ are zero: $\delta = (w, u, 0, 0)$, the linear fractional representation $Q = T_{\Theta_\delta}(R)$ is a symmetric Potapov factorization

$$Q(z) = B_w(z)^T R(z) B_w(\bar{z})$$

$B_w$ Blaschke factor:

$$B_w(z) = I_p + (\beta_w(z) - 1)uu^*, \quad \beta_w(z) = \frac{(1 - \bar{w})(z - w)}{(1 - w)(1 - \bar{w}z)}$$
Schur algorithm

\[ Q = Q_n \xrightarrow{\delta_n} Q_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_0} Q_0 \]

where \( Q_0 \) is inner symmetric of degree 1 or 0.

degree 1:
\[ Q_0(z) = Y (I_p + (\beta_w(z) - 1)yy^*) Y^T, \quad y \in \mathbb{R}^p, \|y\| = 1, \quad Y \text{ unitary} \]

degree 0:
\[ Q_0 = O\Lambda O^T, \quad O \text{ real orthogonal}, \quad \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \quad |\lambda_i| = 1. \]

Choice of the LFT : \( \Theta = \Theta_\delta H(E) \), \( E \) symmetric
\[ \Theta_\delta(1) = I_{2p} \Rightarrow Q(1) = Q_0 \]

Application to SAW filters: chaining acoustic matrices corresponds to a Schur algorithm in which \( \Theta = \Theta_\delta H(vu^*) \) where \( H(vu^*) \) symmetric \( E \) symmetric ? \( \rightarrow \) nice construction of realizations