Advanced Logic

http://www-sop.inria.fr/members/Martin.Avanzini/teaching/2023/AL/

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MASTER INFORMATIQUE
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Last Lecture

- an alternating finite automata (AFA) is a tuple $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$ where all components are identical to an NFA except that

$$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$$

- AFAs are more concise but otherwise equi-expressive to NFAs

Theorem

For every AFA $\mathcal{A}$ there exist a DFA $\mathcal{B}$ with $O(2^{2^{|\mathcal{A}|}})$ states such that $L(\mathcal{A}) = L(\mathcal{B})$.

Corollary

AFAs recognize REG.

Today's Lecture

- infinite words
- regular languages over infinite words
- Büchi automata
- Monadic Second-Order Logic on Infinite Words

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  - note $|w|_a$ may be infinite
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  - in fact, $|w|_a = \infty$ holds for at least one $a \in \Sigma$
- the left-concatenation of $u \in \Sigma^*$ and $v \in \Sigma^\omega$, is denoted by $u \cdot v \in \Sigma^\omega$

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- the ω-iteration of $U \subseteq \Sigma^*$ is given by

$$U^\omega \triangleq \{w_0 \cdot w_1 \cdot w_2 \cdot \cdots \mid w_i \in U \text{ and } w_i \neq \epsilon \text{ for all } i \in \mathbb{N}\}$$

Generalising the Theory of Regular Languages to Infinite Words

Recall...

For a language $L \in \Sigma^*$, the following are equivalent:

1. L is regular
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Outlook...

For a language $L \in \Sigma^\omega$, the following are equivalent:

1. L is ω-regular
   - defined next
2. L is recognized by a Büchi Automaton
   - a finite automaton with a suitable acceptance condition for infinite words
3. L is defined through a MSO formula
   - we drop the requirement on finite models present in wMSO

Regular Languages over Infinite Words

ω-Regular Languages

- a language $L \subseteq \Sigma^\omega$ is ω-regular (or simply regular) if

$$L = \bigcup_{0 \leq i \leq n} U_i \cdot V_i^\omega$$

  for regular languages U_i, V_i $(0 \leq i \leq n)$

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Proof Outline.

- Union is obvious
- concerning left-concatenation $U \cdot L$ where L is as above

$$U \cdot L = U \cdot \left(\bigcup_{0 \leq i \leq n} U_i \cdot V_i^\omega \right) = \bigcup_{0 \leq i \leq n} U \cdot (U_i \cdot V_i^\omega) = \bigcup_{0 \leq i \leq n} (U \cdot U_i) \cdot V_i^\omega$$

Examples

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Büchi Automata

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- A non-deterministic (deterministic) Büchi Automaton $\mathcal{A}$, short NBA (DBA), is a tuple $(Q, \Sigma, q_I, \delta, F)$ identical to an NFA (DFA)
- a run on $w = \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\ldots$ is an infinite sequence

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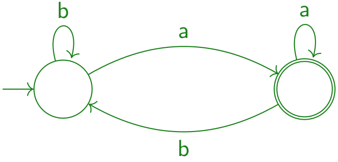

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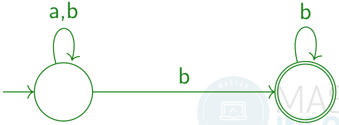

$\mathsf{L}(\mathcal{A}_2) = ?$

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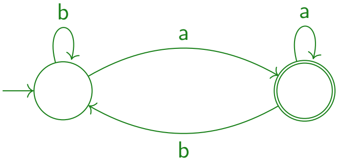

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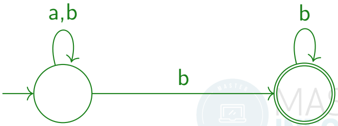

$\mathsf{L}(\mathcal{A}_2) = \{w \in \Sigma^\omega \mid |w|_\mathtt{a} \neq \infty\}$

Non-Determinisation

Theorem

There are NBAs without equivalent DBA.

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Proof Outline.

- ⋆ the NBA $\mathcal{A}_2$ with $\mathsf{L}(\mathcal{A}_2) = \{w \in \Sigma^\omega \mid |w|_{\mathtt{a}} \neq \infty\}$
- ⋆ it can be shown that $\mathsf{L}(\mathcal{A}_2)$ is not recognized by a DBA (exercise)

Closure Properties on NBAs

Theorem

For recognisable $U \in \Sigma^*$ *and* $V, W \in \Sigma^\omega$ *the following are recognisable:*

1. *union* $V \cup W$
2. *intersection* $V \cap W$
3. *left-concatenation* $U \cdot V$
4. ω*-iteration* U^ω
5. *complement* $\overline{V}$

Proof Outline.

- ⋆ (1) and (3). Identical to NFA construction

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Proof Outline.

- ⋆ (1) and (3). Identical to NFA construction
- ⋆ (2) Similar to NFA case. For Büchi condition, keep additional counter mod 2

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- ⋆ (4) exercise
- ⋆ (5) non-trivial, see next

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$L \in \omega REG(\Sigma)$ *if and only if* $L = \mathsf{L}(\mathcal{A})$ *for some NBA* $\mathcal{A}$

Proof Outline.

- ⋆ $\Longrightarrow$: consequence of closure properties

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Proof Outline.

- $\Rightarrow$: consequence of closure properties
- $\Leftarrow$:
  - for finite word $w = \mathrm{a}_1, \ldots, \mathrm{a}_n$ define

$$p \xrightarrow{w} q :\Longleftrightarrow p \xrightarrow{\mathrm{a}_1} \cdots \xrightarrow{\mathrm{a}_n} q \text{ and } L_{p,q} \triangleq \{w \mid p \xrightarrow{w} q\}$$

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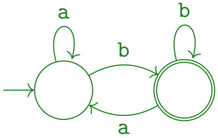
  - hence

$$\mathrm{L}(\mathcal{A}) = \bigcup_{q \in F} L_{q_I,q} \cdot L_{q,q}^\omega \in \omega REG(\Sigma)$$

Complementation of NBA (I)

even for DBAs, unlike for DFAs, complementation is non-trivial



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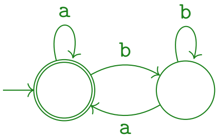
$\ni$ ababa$\cdots$



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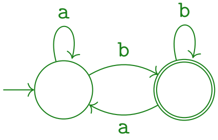

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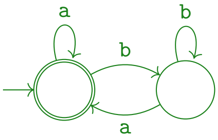


$(\mathtt{b}^*\mathtt{a})^\omega$

$\ni$ ababa···

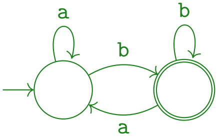
Idea

- ⋆ find a finite partition P of Σ^* of regular languages such that

$$(i) \text{ either } U \cdot V^\omega \subseteq \mathrm{L}(\mathcal{A}) \text{ or } U \cdot V^\omega \subseteq \overline{\mathrm{L}(\mathcal{A})} \text{ for } U, V \in P \qquad (ii)\ \Sigma^\omega = \bigcup_{U,V \in P} U \cdot V^\omega$$

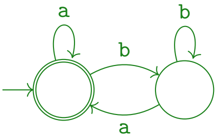
Complementation of NBA (I)

even for DBAs, unlike for DFAs, complementation is non-trivial



$(\mathtt{a}^*\mathtt{b})^\omega$

$\ni \mathtt{ababa}\cdots$



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Idea

- ⋆ find a finite partition P of Σ^* of regular languages such that

$$(i) \text{ either } U \cdot V^\omega \subseteq \mathrm{L}(\mathcal{A}) \text{ or } U \cdot V^\omega \subseteq \overline{\mathrm{L}(\mathcal{A})} \text{ for } U, V \in P \qquad (ii)\ \Sigma^\omega = \bigcup_{U,V \in P} U \cdot V^\omega$$

- ⋆ hence

$$\overline{\mathrm{L}(\mathcal{A})} \overset{(ii)}{=} \Big(\bigcup_{U,V \in P} U \cdot V^\omega \Big) \setminus \mathrm{L}(\mathcal{A}) \overset{(i)}{=} \bigcup_{\substack{U,\, V \in P \\ U \cdot V^\omega \cap \mathrm{L}(\mathcal{A}) = \emptyset}} U \cdot V^\omega$$

Complementation of NBAs (II)

- define $p \xrightarrow{w}_{\text{fin}} q :\Longleftrightarrow p \xrightarrow{u} q_f \xrightarrow{v} q$ for some $q_f \in F$ and $u \cdot v = w$

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- $u \sim v :\Longleftrightarrow \forall p.q \in Q.\ (p \xrightarrow{u} q \iff p \xrightarrow{v} q)$ and $(p \xrightarrow{u}_{\mathsf{fin}} q \iff p \xrightarrow{v}_{\mathsf{fin}} q)$ defines an equivalence on Σ^*
- if $u \sim v$ then u and v are "indistinguishable" by the considered NBA

Lemma

For every $w \in \Sigma^*$, $[w]_\sim$ is regular.

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Proof Outline.

Reformulating the definition, $[w]_\sim = \left(\bigcap_{p \xrightarrow{w} q} \{u \mid p \xrightarrow{u} q\}\right) \cap \left(\bigcap_{p \xrightarrow{w}_{\text{fin}} q} \{u \mid p \xrightarrow{u}_{\text{fin}} q\}\right)$

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Proof Outline.

Every class $[w]_\sim$ is described through two sets of state-pairs (at most $O(2^{2n^2})$ many)

Complementation of NBAs (III)

Lemma

1. *For any two* $U, V \in \Sigma^*/\sim$, *either (i)* $U \cdot V^\omega \subseteq \mathrm{L}(\mathcal{A})$ *or (ii)* $U \cdot V^\omega \subseteq \overline{\mathrm{L}(\mathcal{A})}$.
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- ⋆ the auxiliary lemmas yield that

$$\overline{\mathrm{L}(\mathcal{A})} = \bigcup \{U \cdot V^\omega \mid U, V \in \Sigma^*/\!\sim, U \cdot V^\omega \cap \mathrm{L}(\mathcal{A}) = \varnothing\}$$

- ⋆ as $U, V \in \Sigma^*/\!\sim$ is regular, $\overline{\mathrm{L}(\mathcal{A})}$ language is regular, and thus described by an NBA

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Notes

- the above equation directly yield a recipe for building $\mathcal{B}$
- the size of the constructed NBA is proportional to the cardinality of $\Sigma^*/\sim$ ($O(2^{2n^2})$)

Monadic Second-Order Logic on Infinite Words

MSO on Infinite Words

- the set of MSO formulas over $\mathcal{V}_1, \mathcal{V}_2$ coincides with that of weak MSO formulas:

$$\phi, \psi ::= \top \mid \bot \mid x < y \mid X(x) \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\phi \mid \exists X.\phi$$

- the satisfiability relation $\alpha \vDash \phi$ is defined equivalently, but allows infinite valuations of second order variables

$$\alpha \vDash \exists X.\phi \quad :\Longleftrightarrow \quad \alpha[x \mapsto M] \vDash \phi \text{ for some } M \subseteq \mathbb{N}$$

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Example

$$\exists X. \forall y. X(y) \leftrightarrow X(y+2)$$

- not satisfiable in WMSO
- valid in MSO

MSO Decidability

- consider MSO formula ϕ over $\mathcal{V}_2 = \{X_1, \ldots, X_m\}$ and $\mathcal{V}_1 = \{y_{m+1}, \ldots, y_{m+n}\}$
- as in the case of WMSO, the alphabet Σ_ϕ is given by $m + n$ bit-vectors, i.e., $\Sigma_\phi \triangleq \{0, 1\}^{n+m}$
- MSO assignment α can be coded as infinite words $\underline{\alpha} \in \Sigma_\phi^\omega$
  - $n \in \alpha(X_i)$ iff the i-th entry in n-th letter of $\underline{\alpha}$ is 1
  - $\alpha(y_j) = n$ iff the i-th entry in n-th letter of $\underline{\alpha}$ is 1

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Theorem

For every MSO formula ϕ there exists an NBA $\mathcal{A}_\phi$ s.t. $\hat{\mathrm{L}}(\phi) = \mathrm{L}(\mathcal{A}_\phi)$.

Proof Outline.

construction analoguous to the case of WMSO