Advanced Logic

http://www-sop.inria.fr/members/Martin.Avanzini/teaching/2022/AL/

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MASTER INFORMATIQUE
UNIVERSITÉ CÔTE D'AZUR

2nd Semester M1, 2022

Last Lecture

INFORMATION AND COMPUTATION 75, 87–106 (1987)

Learning Regular Sets from Queries and Counterexamples*

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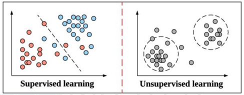
The problem of identifying an unknown regular set from examples of its members and nonmembers is addressed. It is assumed that the regular set is presented by a *minimally adequate Teacher*, which can answer membership queries about the set and can also test a conjecture and indicate whether it is equal to the unknown set and provide a counterexample if not. (A counterexample is a string in the symmetric difference of the correct set and the conjectured set.) A learning algorithm L^* is described that correctly learns any regular set from any minimally adequate Teacher in time polynomial in the number of states of the minimum dfa for the set and the maximum length of any counterexample provided by the Teacher. It is shown that in a stochastic setting the ability of the Teacher to test conjectures may be replaced by a random sampling oracle, $EX(\ )$. A polynomial-time learning algorithm is shown for a particular problem of context-free language identification.

Dana Angluin



Today's Lecture

- ⋆ infinite words
- ⋆ regular languages over infinite words
- ⋆ Büchi automata
- ⋆ Monadic Second-Order Logic on Infinite Words

Infinite Words

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  - note $|w|_a$ may be infinite
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- the left-concatenation of $u \in \Sigma^*$ and $v \in \Sigma^\omega$, is denoted by $u \cdot v \in \Sigma^\omega$

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- The complement of $V \subseteq \Sigma^\omega$ is given by $\overline{V} \triangleq \Sigma^\omega \setminus V$
- the ω-iteration of $U \subseteq \Sigma^*$ is given by

$$U^\omega \triangleq \{w_0 \cdot w_1 \cdot w_2 \cdot \cdots \mid w_i \in U \text{ and } w_i \neq \epsilon \text{ for all } i \in \mathbb{N}\}$$

Generalising the Theory of Regular Languages to Infinite Words

Recall...

For a language $L \in \Sigma^*$, the following are equivalent:

1. L is regular
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Outlook...

For a language $L \in \Sigma^\omega$, the following are equivalent:

1. L is ω-regular
   - defined next
2. L is recognized by a Büchi Automaton
   - a finite automaton with a suitable acceptance condition for infinite words
3. L is defined through a MSO formula
   - we drop the requirement on finite models present in wMSO

Regular Languages over Infinite Words

ω-Regular Languages

- a language $L \subseteq \Sigma^{\omega}$ is ω-regular (or simply regular) if

$$L = \bigcup_{0 \leq i \leq n} U_i \cdot V_i^{\omega}$$

  for regular languages U_i, V_i $(0 \leq i \leq n)$

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Proof Outline.

- Union is obvious
- concerning left-concatenation $U \cdot L$ where L is as above

$$U \cdot L = U \cdot \Big(\bigcup_{0 \leq i \leq n} U_i \cdot V_i^\omega \Big) = \bigcup_{0 \leq i \leq n} U \cdot (U_i \cdot V_i^\omega) = \bigcup_{0 \leq i \leq n} (U \cdot U_i) \cdot V_i^\omega$$

Examples

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Büchi Automata

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- A non-deterministic (deterministic) Büchi Automaton $\mathcal{A}$, short NBA (DBA), is a tuple $(Q, \Sigma, q_I, \delta, F)$ identical to an NFA (DFA)
- a run on $w = \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\ldots$ is an infinite sequence

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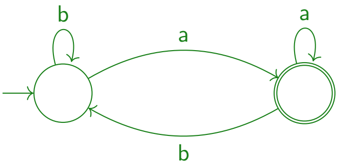
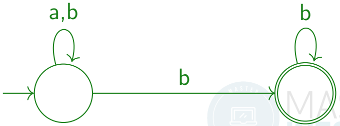


$L(\mathcal{A}_1) = \{w \in \Sigma^\omega \mid |w|_\mathtt{a} = \infty\}$



$L(\mathcal{A}_2) = \{w \in \Sigma^\omega \mid |w|_\mathtt{a} \neq \infty\}$

Non-Determinisation

Theorem

There are NBAs without equivalent DBA.

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Proof Outline.

- ⋆ the NBA $\mathcal{A}_2$ with $L(\mathcal{A}_2) = \{w \in \Sigma^\omega \mid |w|_a \neq \infty\}$
- ⋆ it can be shown that $L(\mathcal{A}_2)$ is not recognized by a DBA (exercise)

Closure Properties on NBAs

Theorem

For recognisable $U \in \Sigma^*$ and $V, W \in \Sigma^\omega$ the following are recognisable:

1. *union* $V \cup W$
2. *intersection* $V \cap W$
3. *left-concatenation* $U \cdot V$
4. ω*-iteration* U^ω
5. *complement* $\overline{V}$

Proof Outline.

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- ⋆ (2) Similar to NFA case. For Büchi condition, keep additional counter mod 2

$$\rho : \begin{pmatrix}\bigcirc\\ \bigcirc\\ 0\end{pmatrix} \xrightarrow{\mathtt{a}_1} \cdots \begin{pmatrix}\circledcirc\\ \bigcirc\\ 1\end{pmatrix} \xrightarrow{\mathtt{a}_{i_1}} \cdots \underbrace{\begin{pmatrix}\bigcirc\\ \circledcirc\\ 2\end{pmatrix}}_{\triangleq\text{final}} \xrightarrow{\mathtt{a}_{i_2}} \begin{pmatrix}\bigcirc\\ \bigcirc\\ 0\end{pmatrix} \xrightarrow{\mathtt{a}_{i_2+1}} \cdots$$

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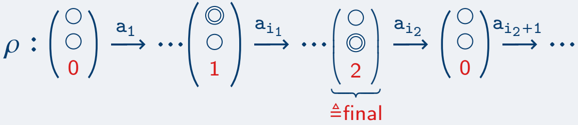

- ⋆ (4) exercise
- ⋆ (5) non-trivial, see next

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$L \in \omega REG(\Sigma)$ *if and only if* $L = \mathsf{L}(\mathcal{A})$ *for some NBA* $\mathcal{A}$

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$$w \in \mathrm{L}(\mathcal{A}) \Longleftrightarrow \exists q \in F.\ w = u \cdot v^{\omega} \text{ for some } u \in L_{q_I,q} \text{ and } v \in L_{q,q}^{\omega}$$

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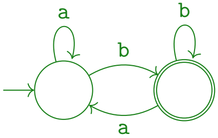
  - hence

$$\mathrm{L}(\mathcal{A}) = \bigcup_{q \in F} L_{q_I,q} \cdot L_{q,q}^{\omega} \in \omega REG(\Sigma)$$

Complementation of NBA (I)

even for DBAs, unlike for NFAs, complementation is non-trivial



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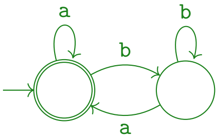
$\ni$ ababa···



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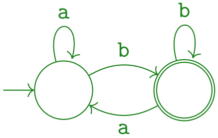

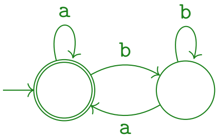

Idea

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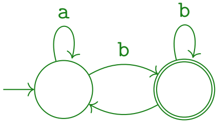

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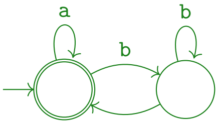

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$$\overline{L(\mathcal{A})} \overset{(ii)}{=} \Big(\bigcup_{U,V\in P} U \cdot V^\omega\Big) \setminus L(\mathcal{A}) \overset{(i)}{=} \bigcup_{\substack{U,\, V \in P \\ U \cdot V^\omega \cap L(\mathcal{A}) = \varnothing}} U \cdot V^\omega$$

Complementation of NBAs (II)

- ⋆ define $p \xrightarrow{w}_{\text{fin}} q :\Longleftrightarrow p \xrightarrow{u} q_f \xrightarrow{v} q$ for some $q_f \in F$ and $u \cdot v = w$

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Proof Outline.

Every class $[w]_\sim$ is described through two sets of state-pairs (at most $O(2^{2n^2})$ many)

Complementation of NBAs (III)

Lemma

1. *For any two* $U, V \in \Sigma^*/\sim$, *either (i)* $U \cdot V^\omega \subseteq L(\mathcal{A})$ *or (ii)* $U \cdot V^\omega \subseteq \overline{L(\mathcal{A})}$.
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Theorem

For any NBA $\mathcal{A}$, *there is an NBA* $\mathcal{B}$ *such that* $\mathsf{L}(\mathcal{B}) = \overline{\mathsf{L}(\mathcal{A})}$.

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Proof Outline.

- ⋆ the auxiliary lemmas yield that

$$\overline{\mathrm{L}(\mathcal{A})} = \bigcup \{U \cdot V^\omega \mid U, V \in \Sigma^*/\sim, U \cdot V^\omega \cap \mathrm{L}(\mathcal{A}) = \varnothing\}$$

- ⋆ as $U, V \in \Sigma^*/\sim$ is regular, $\overline{\mathrm{L}(\mathcal{A})}$ language is regular, and thus described by an NBA

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Notes

- ⋆ the above equation directly yield a recipe for building $\mathcal{B}$
- ⋆ the size of the constructed NBA is proportional to the cardinality of $\Sigma^*/\sim$ $(O(2^{2n^2}))$

Monadic Second-Order Logic on Infinite Words

MSO on Infinite Words

- the set of MSO formulas over $\mathcal{V}_1, \mathcal{V}_2$ coincides with that of weak MSO formulas:

$$\phi, \psi ::= \top \mid \bot \mid x < y \mid X(x) \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\phi \mid \exists X.\phi$$

- the satisfiability relation $\alpha \models \phi$ is defined equivalently, but allows infinite valuations of second order variables

$$\alpha \models \exists X.\phi \quad :\Longleftrightarrow \quad \alpha[x \mapsto M] \models \phi \text{ for some } M \subseteq \mathbb{N}$$

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Example

$$\exists X.\forall y.X(y) \leftrightarrow X(y+2)$$

- not satisfiable in WMSO
- valid in MSO

MSO Decidability

- consider MSO formula ϕ over $\mathcal{V}_2 = \{X_1, \ldots, X_m\}$ and $\mathcal{V}_1 = \{y_{m+1}, \ldots, y_{m+n}\}$
- as in the case of WMSO, the alphabet Σ_ϕ is given by $m + n$ bit-vectors, i.e., $\Sigma_\phi \triangleq \{0, 1\}^{n+m}$
- MSO assignment α can be coded as infinite words $\underline{\alpha} \in \Sigma_\phi^\omega$
  - $n \in \alpha(X_i)$ iff the i-th entry in n-th letter of $\underline{\alpha}$ is 1
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the language $\hat{\mathsf{L}}(\phi) \subseteq \Sigma_\phi^\omega$ of coded valuations making ϕ true is given by:

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Theorem

For every MSO formula ϕ there exists an NBA $\mathcal{A}_\phi$ s.t. $\hat{\mathsf{L}}(\phi) = \mathsf{L}(\mathcal{A}_\phi)$.

Proof Outline.

construction analoguous to the case of WMSO