Advanced Logic

http://www-sop.inria.fr/members/Martin.Avanzini/teaching/2022/AL/

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MASTER INFORMATIQUE
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Last Lecture

- an alternating finite automata (AFA) is a tuple $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$ where all components are identical to an NFA except that

$$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$$

- AFAs are more concise but otherwise equi-expressive to NFAs

Theorem

For every AFA $\mathcal{A}$ there exist a DFA $\mathcal{B}$ with $O(2^{2^{|\mathcal{A}|}})$ states such that $L(\mathcal{A}) = L(\mathcal{B})$.

Corollary

AFAs recognize REG.

Today's Lecture

First Order-Logic Recap

- structures, formulas and satisfiability

Monadic Second-Order Logic

1. weak monadic second-order (WMSO) logic
2. Regularity and WMSO definability
3. Decision problems

First-Order Logic Recap

First-Order Logic

- let $\mathcal{V} = \{x, y, \dots\}$ be a set of variables
- let $\mathcal{R} = \{P, Q, \dots\}$ and $\mathcal{F} = \{f, g, \dots\}$ be a vocabulary of predicate/function symbols
- predicate and function symbols are equipped with an arity $\mathsf{ar} : \mathcal{R} \cup \mathcal{F} \to \mathbb{N}$
- first-order terms and formulas over $\mathcal{V}$, $\mathcal{R}$ and $\mathcal{F}$ are given by the following grammar:

$$
\begin{array}{rll}
s, t ::= x \mid f(t_1, \dots, t_{\mathsf{ar}(f)}) & & \textit{(terms)} \\
\phi, \psi ::= \top \mid \bot & & \textit{(atomic truth values)} \\
\mid P(t_1, \dots, t_{\mathsf{ar}(P)}) \mid s = t & & \textit{(predicates and equality)} \\
\mid \phi \vee \psi \mid \neg\phi & & \textit{(Boolean connectives)} \\
\mid \exists x.\phi & & \textit{(existential quantification)}
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- further connectives definable:

$$\phi \to \psi \triangleq \neg\phi \vee \psi \quad s \neq t \triangleq \neg(s = t) \quad \phi \wedge \psi \triangleq \neg(\neg\phi \vee \neg\psi) \quad \forall x.\phi \triangleq \neg(\exists x.\neg\phi) \quad \ldots$$

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- to avoid parenthesis, we fix precedence $\neg \;>\; \wedge, \vee \;>\; \exists, \forall$

Free Variables, Open and Closed Formulas

- a quantifier $\exists x.\phi$ binds the variable x within ϕ
- variables not bound are called free
- the set of variables free in ϕ is denoted by $\mathsf{fv}(\phi)$

$$\mathsf{fv}(E(x,y)) = \{x,y\} \qquad \mathsf{fv}(\exists y.E(x,y)) = \{x\} \qquad \mathsf{fv}(\forall x.\exists y.E(x,y)) = \varnothing$$

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- otherwise they are called open
- we consider formulas equal up to renaming of bound variables
  - $\exists y.E(x,y)$ is equal to $\exists z.E(x,z)$ but **neither** to $\exists y.E(x,z)$ nor $\exists y.E(z,y)$

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  - a non-empty domain D; and
  - an interpretation $\mathcal{I}(P) \subseteq D^{\mathrm{ar}(P)}$ for each predicate $P \in \mathcal{R}$
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- sentences describes properties of structures, consider e.g., $\forall x. \exists y. E(x, y)$:
  - on directed graphs, with E interpreted as "edge": every node has a successor
  - on natural numbers, with E interpreted as "$<$": for every number there is a strictly bigger one

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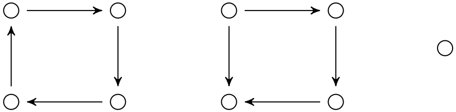
  - on natural numbers, with E interpreted as "$<$": for every number there is a strictly bigger one
- if a formula ϕ holds true in a model $\mathcal{M}$, we write

$$\mathcal{M} \models \phi$$

and say $\mathcal{M}$ models ϕ, or that ϕ is satisfiable with $\mathcal{M}$

Examples

1. consider the formula $\phi = \forall x.\exists y.E(x, y)$ and E interpreted by ...



G_1 G_2 G_3

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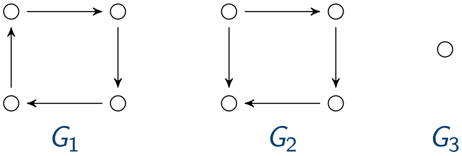

   – we have $G_1 \models \varphi$, $G_2 \not\models \varphi$ and $G_3 \not\models \varphi$

2. consider the formula $\exists x_1, x_2, x_3. \big(x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1\big)$

   – the formula is satisfiable by all models with three objects in the domain

Consequence, Equivalence and Validity

- a sentence ϕ is a consequence of sentences $\Phi = \psi_1; \ldots; \psi_n$, in notation

$$\Phi \models \phi$$

  if all models satisfying all $\psi_i \in \Phi$ also satisfy ϕ

  - $\forall x.P(x) \rightarrow Q(x); \exists x.P(x) \models \exists x.Q(x)$

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- a formula ϕ is valid if it is satisfiable for all models, in notation

$$\models \phi$$

  – this is to say that $\neg\phi$ is unsatisfiable
  – the formula $\forall x.x = x$ is trivially valid

Satisfiability, Formally

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- together with a model, we can now interpret open terms t in its domain D

$$\mathcal{I}_\alpha(x) \triangleq \alpha(x) \qquad \mathcal{I}_\alpha(f(t_1, \ldots, t_n)) \triangleq \mathcal{I}(f)(\mathcal{I}_\alpha(t_1), \ldots, \mathcal{I}_\alpha(t_n))$$

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- for a sentence ϕ, we can now define $\mathcal{M} \vDash \phi$ formally as $\mathcal{M}; \varnothing \vDash \phi$ where

$$\mathcal{M}; \alpha \vDash \top \qquad \mathcal{M}; \alpha \nvDash \bot$$

$$\mathcal{M}; \alpha \vDash P(t_1, \ldots, t_n) \quad :\Longleftrightarrow \quad (\mathcal{I}_\alpha(t_1), \ldots, \mathcal{I}_\alpha(t_n)) \in \mathcal{I}(P)$$

$$\mathcal{M}; \alpha \vDash s = t \quad :\Longleftrightarrow \quad \mathcal{I}_\alpha(s) = \mathcal{I}_\alpha(t)$$

$$\mathcal{M}; \alpha \vDash \phi \vee \psi \quad :\Longleftrightarrow \quad \mathcal{M}; \alpha \vDash \phi \text{ or } \mathcal{M}; \alpha \vDash \psi$$

$$\mathcal{M}; \alpha \vDash \neg\phi \quad :\Longleftrightarrow \quad \mathcal{M}; \alpha \nvDash \phi$$

$$\mathcal{M}; \alpha \vDash \exists x.\phi \quad :\Longleftrightarrow \quad \mathcal{M}; \alpha[x \mapsto d] \vDash \phi \text{ for some } d \in D$$

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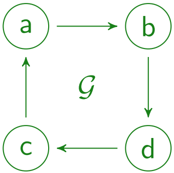
Example



$$\begin{aligned}
\mathcal{G} \models \exists x.\exists y.E(x, y) &\Longleftrightarrow \mathcal{G}; \varnothing \models \exists x.\exists y.E(x, y) \\
&\Longleftarrow \mathcal{G}; x \mapsto \mathrm{a} \models \exists y.E(x, y) \\
&\Longleftarrow \mathcal{G}; x \mapsto \mathrm{a}; y \mapsto \mathrm{b} \models E(x, y) \\
&\Longleftrightarrow (\mathrm{a}, \mathrm{b}) \in \mathcal{I}(E)
\end{aligned}$$

Monadic Second-Order Logic

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Second Order-Logic

- in first-order logic, quantification confined to elements of the domain
- in second-order logic, quantification is permitted on relations
  - $\forall x. \exists X. \forall y. X(x, y) \leftrightarrow x = y$

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- monadic second-order logic (MSO) confines second-order quantification to monadic predicates
  - monadic: $\forall x. \exists Y. \forall y. Y(y) \leftrightarrow x = y$
  - non-monadic: $\forall x. \exists X. \forall y. X(x, y) \leftrightarrow x = y$

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  - non-monadic: $\forall x. \exists X. \forall y. X(x, y) \leftrightarrow x = y$
- quantification over sets, but not over arbitrary predicates
  - on graphs: quantification over nodes but not edges

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- A theory T is complete if for any sentence ϕ we have $\phi \in T$ or $\neg\phi \in T$.
  - a complete theory speaks about all formulas
- for a class of structures $\mathcal{C}$, the theory of $\mathcal{C}$ is the set of sentences which are valid on all $\mathcal{M} \in \mathcal{C}$

Examples

1. The theory of Presburger Arithmetic, i.e., the theory of natural numbers with addition only is decidable
   - $\forall n.\exists m.(n = m + m) \vee (n = m + m + 1)$
   - Presburger Arithmetic admits a quantifier elimination procedure

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Theorem (Büchi)

The theory of monadic second-order logic over $(\mathbb{N}, <)$ *is decidable*

Theorem (Rabin)

The theory of monadic second-order logic over trees is decidable

A First Step Towards Rabin's and Büchi's Result

consider only models over $\mathbb{N}$, ordered by $<$

Theorem (Büchi-Elgot-Trakhtenbrot)

The theory of weak monadic second-order logic over $(\mathbb{N}, <)$ is decidable

quantification over finite sets

Weak Monadic Second-Order Logic

Weak Monadic Second-Order Logic (WMSO)

- let $\mathcal{V}_1 = \{x, y, \dots\}$ be a set of first-order variables (ranging over $\mathbb{N}$)
- let $\mathcal{V}_2 = \{X, Y, \dots\}$ be monadic second-order variables (ranging over finite sets of $\mathbb{N}$)
- $\mathcal{R} = \{<\}$ and $\mathcal{F} = \varnothing$ is fixed, with $\mathsf{ar}(<) = 2$
- the set of WMSO formulas over $\mathcal{V}_1, \mathcal{V}_2$ is given by the following grammar:

$$\phi, \psi ::= \top \;\big|\; \bot \;\big|\; x < y \;\big|\; X(x) \;\big|\; \phi \vee \psi \mid \neg\phi \;\big|\; \exists x.\phi \;\big|\; \exists X.\phi$$

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- further definable connectives / formulas

$\forall X.\phi \triangleq \neg(\exists X.\neg\phi)$ $\quad x = 0 \triangleq \neg(\exists y.y < x)$ $\quad x \leq y \triangleq \neg(y < x)$ $\quad x = y$ $\quad X(y + c)$ $\quad$ (exercise)

Weak Monadic Second-Order Logic (WMSO)

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- weak: second-order variables refer to finite sets
  - $X(y)$ means informally $y \in X$ where X is finite set over $\mathbb{N}$
  - $\models \exists X.\forall x.X(x) \to \exists y.x < y \wedge X(y)$ $\qquad \alpha(X) = \varnothing$
  - $\not\models \exists X.(\forall x.x = 0 \to X(x)) \wedge (\forall x.X(x) \to \exists y.x < y \wedge X(y))$

Satisfiability

- ⋆ since the model $(\mathbb{N}, \{<\})$ is fixed, the valuation of a formula depends only on an assignment α
- ⋆ α maps first-order variables $x \in \mathcal{V}_1$ to $\mathbb{N}$, and second-order variables $X \in \mathcal{V}_2$ to finite subsets of $\mathbb{N}$

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- satisfiability relation takes the form $\alpha \models \phi$ and is inductively defined as expected:

$$\alpha \models \top \qquad \alpha \not\models \bot$$

$$
\begin{aligned}
\alpha \models x < y \quad &:\Longleftrightarrow \quad \alpha(x) < \alpha(y) \\
\alpha \models X(x) \quad &:\Longleftrightarrow \quad \alpha(x) \in \alpha(X) \\
\alpha \models \phi \vee \psi \quad &:\Longleftrightarrow \quad \alpha \models \phi \text{ or } \alpha \models \psi \\
\alpha \models \neg\phi \quad &:\Longleftrightarrow \quad \alpha \not\models \phi \\
\alpha \models \exists x.\phi \quad &:\Longleftrightarrow \quad \alpha[x \mapsto n] \models \phi \text{ for some } n \in \mathbb{N} \\
\alpha \models \exists X.\phi \quad &:\Longleftrightarrow \quad \alpha[x \mapsto M] \models \phi \text{ for some \textit{finite} } M \subset \mathbb{N}
\end{aligned}
$$

Connections to Formal Languages

- to encode words $w \in \Sigma^*$ over alphabet Σ we use to kinds of variables
  - first-order variables $x \in \mathcal{V}_1$ refer to positions within w
  - for each letter $\mathtt{a} \in \Sigma$, second-order variables $P_\mathtt{a} \in \mathcal{V}_2$ indicate the positions of $\mathtt{a}$ in w

w	a b b a
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Examples

- $\underline{\mathtt{ab}} \models \exists x. P_{\mathtt{a}}(x)$
- $\underline{\mathtt{ab}} \not\models \exists x. P_{\mathtt{c}}(x)$
- $\underline{\mathtt{ab}} \not\models \exists x. \exists y. x < y \wedge P_{\mathtt{b}}(x) \wedge P_{\mathtt{a}}(y)$
- $\underline{\mathtt{ab}} \not\models \exists X. \forall x. (X(x) \rightarrow P_{\mathtt{b}}(x)) \wedge \exists y. y = 0 \wedge X(y)$

Language of a WMSO Formula

- for alphabet Σ and WMSO formula ϕ s.t. $\mathrm{fv}(\phi) \subseteq \{P_{\mathrm{a}} \mid \mathrm{a} \in \Sigma\}$, we let

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$\exists X.\forall x.(X(x) \rightarrow P_b(x)) \wedge \exists y.y = 0 \wedge X(y)$	$\{bw \mid w \in \Sigma^*\}$

Regularity and WMSO Definability

Büchi-Elgot-Trakhtenbrot

Theorem

Let $L \subseteq \Sigma^*$ be a language. The following are equivalent:

- ⋆ L *is regular*
- ⋆ L *is recognizable by a finite automata*
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From Automatons to Formulas

Encoding for given $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$

- ⋆ first-order variables $m, n, \ldots$ refer to positions in input words w
- ⋆ for $\mathtt{a} \in \Sigma$: second-order variables $P_\mathtt{a}$ encode words: as before
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example run	$p \xrightarrow{\mathtt{a}} q \xrightarrow{\mathtt{b}} p \xrightarrow{\mathtt{b}} r$
$P_{\mathtt{a}}$	$\{ 0 \}$
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X_p	$\{ (-1)\ 1 \}$
X_q	$\{ 0 \}$
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- ultimately, $\phi_{\mathcal{A}} \triangleq \exists X_{q_1}.\ldots.\exists X_{q_n}.\psi_{\mathcal{A}}$ with $\psi_{\mathcal{A}}$ saying that X_{q_i} encode an accepting run of $\mathcal{A}$ on input word described by $P_{\mathtt{a}}$.

Linking Run-Variables

for all word lengths *len*, we define:

- ⋆ $\psi_{setup} \triangleq \forall m. m < len \rightarrow (\bigvee_{q \in Q} X_q(m)) \wedge (\bigwedge_{p \neq q} \neg(X_q(m) \wedge X_p(m)))$
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- $\phi_{accept} \triangleq (len = 0 \wedge \ulcorner q_I \in F \urcorner) \vee \exists m. len = m + 1 \wedge \bigvee_{q \in F} (X_q(m))$
  - encoded transition of word $\mathrm{a}_0 \ldots \mathrm{a}_m$ of length $m + 1$ lands in a final state

$$\phi_{\mathcal{A}} \triangleq \exists X_{q_1}. \cdots \exists X_{q_n}.$$

$$\forall len. \underbrace{\left(\bigwedge_{\mathrm{a} \in \Sigma} \neg P_{\mathrm{a}}(len) \wedge \forall m. \bigwedge_{\mathrm{a} \in \Sigma} P_{\mathrm{a}}(m) \to m \leq len \right)}_{len \text{ gives length of input}} \to \psi_{setup} \wedge \psi_{initial} \wedge \psi_{run} \wedge \psi_{accept}$$

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From Formulas to Regular Languages

Encoding for given ϕ over $\mathcal{V}_2 = \{X_1, \ldots, X_m\}$ and $\mathcal{V}_1 = \{y_{m+1}, \ldots, y_{m+n}\}$

- ⋆ the alphabet Σ_ϕ is given by $m + n$ bit-vectors, i.e., $\Sigma_\phi \triangleq \{0, 1\}^{n+m}$

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- ⋆ word $w \in \Sigma_\phi^*$ can then be seen as a bit-matrix, encoding a valuation α:
  - rows $1 \le i \le m$ encode valuations of $X_i \in \mathcal{V}_2$: 1 at column $1 \le j \le |w| \iff j \in \alpha(X_i)$
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v	$\alpha(v)$
X_1	$\{0, 2\}$
X_2	$\{1, 3, 4\}$
y_3	3
y_4	0

$\equiv$

$w[0]$	$w[1]$	$w[2]$	$w[3]$	$w[4]$
$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

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- for a valuation α for ϕ, let us write $\underline{\alpha} \in \Sigma_\phi^*$ for its encoding

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let us denote by $\hat{L}(\phi) \subseteq \Sigma_\phi^*$ the language of coded valuations making ϕ true:

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Homomorphisms

Consider $h : \Sigma \to \Gamma^*$ and extend it to words w by replacing each letter $\mathtt{a}$ in w by $h(w)$:

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Lemma (Closure of $REG(\Sigma)$ under homomorphism)

The set of regular languages is closed under (inverse) applications of homomorphisms.

Example

For $1 \leq i \leq k$, let $del_{i,k} : \{0,1\}^k \to \{0,1\}^{k-1}$ delete the i-th entry of its argument, e.g.,

$$del_{1,3}\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \triangleq \begin{pmatrix} b \\ c \end{pmatrix} \qquad del_{2,3}\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \triangleq \begin{pmatrix} a \\ c \end{pmatrix} \qquad del_{3,3}\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \triangleq \begin{pmatrix} a \\ b \end{pmatrix}$$

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- Attention: One has to be slightly more careful with codings.

$$\phi \rightsquigarrow \begin{matrix} X \\ Y \end{matrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\cdots\begin{pmatrix} a_n \\ b_n \end{pmatrix}\begin{pmatrix} a_{n+1} \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \exists X.\phi \rightsquigarrow (\, b_1 \,)\cdots(\, b_n \,)(\, 1 \,)(\, 0 \,)$$

The Main Lemma (Continued)

Lemma

For any WMSO formula ϕ, $\hat{\mathrm{L}}(\phi)$ is regular

Proof Outline.

- ⋆ $\phi = \psi_1 \vee \psi_2$:
  - by induction hypothesis, $L_1 \triangleq \hat{\mathrm{L}}(\psi_1)$ and $L_2 \triangleq \hat{\mathrm{L}}(\psi_2)$ are regular
  - L_1 and L_2 speak about assignments to variables in ψ_1 and ψ_2
  - inverse applications of $del_{i,*}$ extends these codings to valuations over $\mathrm{fv}(\psi_1 \vee \psi_2)$
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- $\phi = \exists X_i.\psi$ or $\phi = \exists y_j.\psi$: from induction hypothesis, using homomorphism $del_{i,*}$ to drop the rows referring to X_i or y_j; taking care of trailing zero-vectors (see previous slide)

Büchi-Elgot-Trakhtenbrot

Theorem

Let $L \subseteq \Sigma^*$ be a language. The following are equivalent:

- ⋆ *L is regular*
- ⋆ *L is recognizable by a finite automata*
- ⋆ *L is WMSO definable*

Proof Outline.

- ⋆ (1) $\Leftrightarrow$ (2) Kleene's Theorem.
- ⋆ (2) $\Rightarrow$ (3) Given an Automata $\mathcal{A}$, we define a WMSO formula $\phi_{\mathcal{A}}$ s.t. $\mathrm{L}(\mathcal{A}) = \mathrm{L}(\phi_{\mathcal{A}})$
- ⋆ (3) $\Rightarrow$ (1) Given a WMSO formula ϕ, define a regular Language L_ϕ s.t. $\mathrm{L}(\phi) = L_\phi$
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  - as the former is regular and $REG(\Sigma)$ closed under homomorphisms, the direction follows

Decision Problems

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- Given: WMSO formula ϕ
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Theorem

Satisfiability and Validity are decidable for WMSO.

Proof Outline.

through the construction of corresponding DFAs, checking emptiness

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- ⋆ Emptiness for an DFA $\mathcal{A}_\phi$ is in PTIME (in the number $|\mathcal{A}_\phi|$ of nodes of $\mathcal{A}_\phi$)
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Satisfiability and validity are in DTIME$(2^c_{O(n)})$, *where* 2^c_k *is a tower of exponentials* $2^{2^{\cdot^{\cdot^{\cdot^{2^c}}}}}$ *of height* k.

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Theorem (Completeness)

Any language L *decidable in time* DTIME$(2^c_{O(n)})$ *can be reduced (within polynomial time) to the satisfiability of formulas* ϕ_w $(w \in L)$ *of size polynomial in* $|w|$.

WMSO and Alternating Finite Automata

- ⋆ What if we translate WMSO formulas to AFAs?
  - for basic formulas $x < y$ and $X(y)$, the construction is as seen previously
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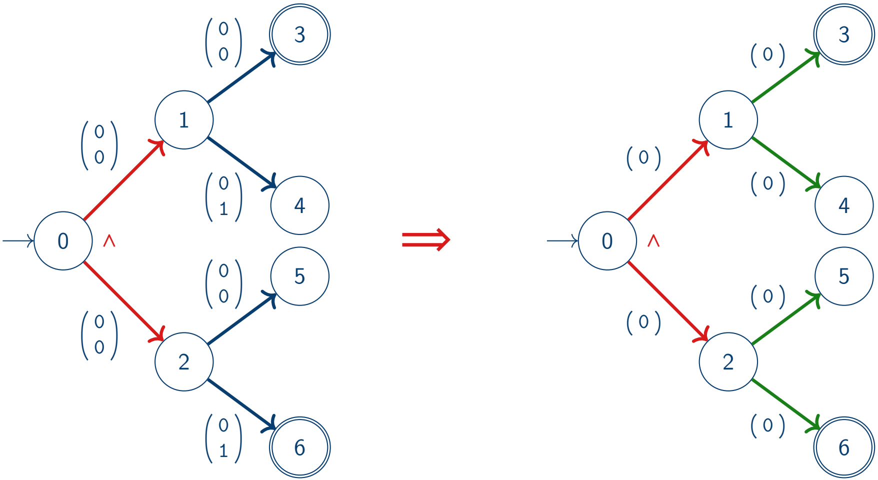
- this suggests resulting automaton is linear in size of formula

$\Longrightarrow$ WMSO model-checking in exponential time, contradicting the lower-bound result!

Projections and AFAs



$L(\mathcal{A}) = \varnothing$

$L(\mathrm{del}_{2,2}(\mathcal{A})) = \{00\}$

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Problem:
We do not have a polytime algorithm for homorphism applications on AFAs