Advanced Logic

http://www-sop.inria.fr/members/Martin.Avanzini/teaching/2021/AL/

Martin Avanzini

Summer Semester 2021

Last Lecture

- a bottom-up tree automata (BUTA) $\mathcal{A}$ is a tuple (Q, Σ, δ, F) where

$$\delta_{\mathtt{f}} : Q^{\mathrm{ar}(\mathtt{f})} \to 2^Q$$

- a top-down tree automata (TDTA) $\mathcal{A}$ is a tuple (Q, Σ, q_I, δ) where

$$\delta_{\mathtt{f}} : Q \to 2^{Q^{\mathrm{ar}(\mathtt{f})}}$$

- an input tree is recognised if nodes can be re-labeled, bottom-up ending in a final state, or top-down from an initial state

Theorem

the set of languages recognized by BUTAs, deterministic BUTAs and TDTAs coincide

- There are languages recognised by TDTAs which are not recognised by deterministic TDTAs

Theorem

Emptyness is decidable in linear time; universality and equivalence are decidable (EXPTIME-complete)

Today's Lecture

- infinite words
- regular languages over infinite words
- Büchi automata
- Monadic Second-Order Logic on Infinite Words

Infinite Words

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  - in fact, $|w|_a = \infty$ holds for at least one $a \in \Sigma$
- the left-concatenation of $u \in \Sigma^*$ and $v \in \Sigma^\omega$, is denoted by $u \cdot v \in \Sigma^\omega$

Languages over Infinite Words

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- The complement of $V \subseteq \Sigma^\omega$ is given by $\overline{V} \triangleq \Sigma^\omega \setminus V$
- the ω-iteration of $U \subseteq \Sigma^*$ is given by

$$U^\omega \triangleq \{w_0 \cdot w_1 \cdot w_2 \cdot \cdots \mid w_i \in U \text{ and } w_i \neq \epsilon \text{ for all } i \in \mathbb{N}\}$$

Generalising the Theory of Regular Languages to Infinite Words

Recall...

For a language $L \in \Sigma^*$, the following are equivalent:

1. L is regular
2. L is recognized by an NFA
3. L is defined through a wMSO formula

Generalising the Theory of Regular Languages to Infinite Words

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Outlook...

For a language $L \in \Sigma^\omega$, the following are equivalent:

1. L is ω-regular
   - defined next
2. L is recognized by a Büchi Automaton
   - a finite automaton with a suitable acceptance condition for infinite words
3. L is defined through a MSO formula
   - we drop the requirement on finite models present in wMSO

Regular Languages over Infinite Words

ω-Regular Languages

- a language $L \subseteq \Sigma^\omega$ is ω-regular (or simply regular) if

$$L = \bigcup_{0 \leq i \leq n} U_i \cdot V_i^\omega$$

  for regular languages U_i, V_i $(0 \leq i \leq n)$

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Lemma

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Proof Outline.

- Union is obvious
- concerning left-concatenation $U \cdot L$ where L is as above

$$U \cdot L = U \cdot \Big(\bigcup_{0 \le i \le n} U_i \cdot V_i^\omega\Big) = \bigcup_{0 \le i \le n} U \cdot (U_i \cdot V_i^\omega) = \bigcup_{0 \le i \le n} (U \cdot U_i) \cdot V_i^\omega$$

Examples

Let $\Sigma = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

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- ★ $L_4 \triangleq \{w \mid |w|_a \neq \infty$ and $|w|_b = \infty\}$ is regular $L_4 = \Sigma^*(bc^*)^\omega$
- ★ $L_5 \triangleq \{w^\omega \mid w \in \Sigma^n\}$ is regular

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Büchi Automata

Büchi Automata

- A non-deterministic (deterministic) Büchi Automaton $\mathcal{A}$, short NBA (DBA), is a tuple $(Q, \Sigma, q_I, \delta, F)$ identical to an NFA (DFA)
- a run on $w = \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\ldots$ is an infinite sequence

$$\rho : \quad q_I = q_0 \xrightarrow{\mathtt{a}_1} q_1 \xrightarrow{\mathtt{a}_2} q_2 \xrightarrow{\mathtt{a}_n} \cdots$$

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- Büchi Condition: a run is accepting if $\mathsf{Inf}(\rho) \cap F \neq \varnothing$, where

$$\mathsf{Inf}(\rho) \triangleq \{q \in Q \mid |\rho|_q = \infty\}$$

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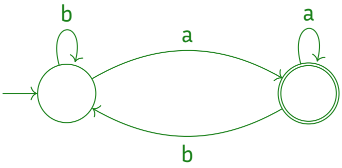
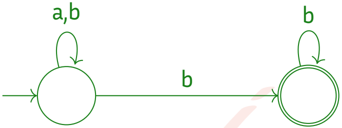
Example



$\mathrm{L}(\mathcal{A}_1) = \{w \in \Sigma^\omega \mid |w|_\mathtt{a} = \infty\}$



$\mathrm{L}(\mathcal{A}_2) = \{w \in \Sigma^\omega \mid |w|_\mathtt{a} \neq \infty\}$

Non-Determinisation

Theorem

There are NBAs without equivalent DBA.

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Proof Outline.

- the NBA $\mathcal{A}_2$ with $L(\mathcal{A}_2) = \{w \in \Sigma^\omega \mid |w|_a \neq \infty\}$
- it can be shown that $L(\mathcal{A}_2)$ is not recognized by a DBA (exercise)

Closure Properties on NBAs

Theorem

For recognisable $U \in \Sigma^*$ and $V, W \in \Sigma^\omega$ the following are recognisable:

1. *union* $V \cup W$
2. *intersection* $V \cap W$
3. *left-concatenation* $U \cdot V$
4. ω*-iteration* U^ω
5. *complement* $\overline{V}$

Proof Outline.

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- ★ (2) Similar to NFA case. For Büchi condition, keep additional counter mod 2

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- ⋆ (4) exercise
- ⋆ (5) non-trivial, see next

NBAs Characterise $\omega REG(\Sigma)$

Theorem

$L \in \omega REG(\Sigma)$ *if and only if* $L = \mathrm{L}(\mathcal{A})$ *for some NBA* $\mathcal{A}$

Proof Outline.

- ⋆ $\Longrightarrow$: consequence of closure properties

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Proof Outline.

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- $\Longleftarrow$:
  - for finite word $w = \mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n}$ define

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  - $w \in \mathrm{L}(\mathcal{A})$ if and only if a run on w traverses some $q \in F$ infinitely often

$$w \in \mathrm{L}(\mathcal{A}) \Longleftrightarrow \exists q \in F.\, w = u \cdot v^{\omega} \text{ for some } u \in L_{q_I,q} \text{ and } v \in L_{q,q}^{\omega}$$

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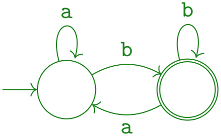
  - hence

$$\mathrm{L}(\mathcal{A}) = \bigcup_{q \in F} L_{q_I,q} \cdot L_{q,q}^{\omega} \in \omega REG(\Sigma)$$

Complementation of NBA (I)

even for DBAs, unlike for NFAs, complementation is non-trivial



$(a^* b)^\omega$
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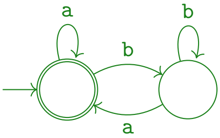

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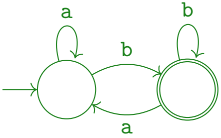

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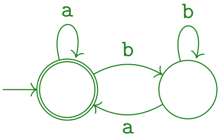

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Idea

- ⋆ find a finite partition P of Σ^* of regular languages such that

$$(i)\ \text{either } U \cdot V^\omega \subseteq L(\mathcal{A}) \text{ or } U \cdot V^\omega \subseteq \overline{L(\mathcal{A})} \text{ for } U, V \in P \qquad (ii)\ \Sigma^\omega = \bigcup_{U,V \in P} U \cdot V^\omega$$

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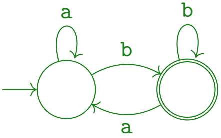

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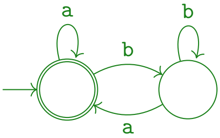

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- hence

$$\overline{L(\mathcal{A})} \overset{(ii)}{=} \Big(\bigcup_{U,V \in P} U \cdot V^\omega\Big) \setminus L(\mathcal{A}) \overset{(i)}{=} \bigcup_{\substack{U, V \in P \\ U \cdot V^\omega \cap L(\mathcal{A}) = \varnothing}} U \cdot V^\omega$$

Complementation of NBAs (II)

- define $p \xrightarrow{w}_{\text{fin}} q :\Longleftrightarrow p \xrightarrow{u} q_f \xrightarrow{v} q$ for some $q_f \in F$ and $u \cdot v = w$

Complementation of NBAs (II)

- define $p \xrightarrow{w}_{\text{fin}} q :\Longleftrightarrow p \xrightarrow{u} q_f \xrightarrow{v} q$ for some $q_f \in F$ and $u \cdot v = w$
- $u \sim v :\Longleftrightarrow \forall p.q \in Q.\ (p \xrightarrow{u} q \iff p \xrightarrow{v} q)$ and $(p \xrightarrow{u}_{\text{fin}} q \iff p \xrightarrow{v}_{\text{fin}} q)$ defines an equivalence on Σ^*
- if $u \sim v$ then u and v are "indistinguishable" by the considered NBA

Lemma

For every $w \in \Sigma^*$, $[w]_\sim$ is regular.

Complementation of NBAs (II)

- define $p \xrightarrow{w}_{\text{fin}} q :\Longleftrightarrow p \xrightarrow{u} q_f \xrightarrow{v} q$ for some $q_f \in F$ and $u \cdot v = w$
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- if $u \sim v$ then u and v are "indistinguishable" by the considered NBA

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Reformulating the definition, $[w]_\sim = \left(\bigcap_{p \xrightarrow{w} q} \{u \mid p \xrightarrow{u} q\}\right) \cap \left(\bigcap_{p \xrightarrow{w}_{\text{fin}} q} \{u \mid p \xrightarrow{u}_{\text{fin}} q\}\right)$

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Every class $[w]_\sim$ is described through two sets of state-pairs (at most $O(2^{2n^2})$ many)

Complementation of NBAs (III)

Lemma

1. *For any two* $U, V \in \Sigma^*/{\sim}$*, either (i)* $U \cdot V^\omega \subseteq \mathsf{L}(\mathcal{A})$ *or (ii)* $U \cdot V^\omega \subseteq \overline{\mathsf{L}(\mathcal{A})}$.
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- the auxiliary lemmas yield that

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Notes

- the above equation directly yield a recipe for building $\mathcal{B}$
- the size of the constructed NBA is proportional to the cardinality of $\Sigma^*/\sim$ ($O(2^{2n^2})$)

Monadic Second-Order Logic on Infinite Words

MSO on Infinite Words

- the set of MSO formulas over $\mathcal{V}_1, \mathcal{V}_2$ coincides with that of weak MSO formulas:

$$\phi, \psi ::= \top \mid \bot \mid x < y \mid X(x) \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\phi \mid \exists X.\phi$$

- the satisfiability relation $\alpha \models \phi$ is defined equivalently, but allows infinite valuations of second order variables

$$\alpha \models \exists X.\phi \quad :\Longleftrightarrow \quad \alpha[x \mapsto M] \models \phi \text{ for some } M \subseteq \mathbb{N}$$

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Example

$$\exists X.\forall y.X(y) \leftrightarrow X(y+2)$$

- not satisfiable in WMSO
- valid in MSO

MSO Decidability

- consider MSO formula ϕ over $\mathcal{V}_2 = \{X_1, \ldots, X_m\}$ and $\mathcal{V}_1 = \{y_{m+1}, \ldots, y_{m+n}\}$
- as in the case of WMSO, the alphabet Σ_ϕ is given by $m + n$ bit-vectors, i.e., $\Sigma_\phi \triangleq \{0, 1\}^{n+m}$
- MSO assignment α can be coded as infinite words $\underline{\alpha} \in \Sigma_\phi^\omega$
  - $n \in \alpha(X_i)$ iff the i-th entry in n-th letter of $\underline{\alpha}$ is 1
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Theorem

For every MSO formula ϕ there exists an NBA $\mathcal{A}_\phi$ s.t. $\hat{\mathsf{L}}(\phi) = \mathsf{L}(\mathcal{A}_\phi)$.

Proof Outline.

construction analoguous to the case of WMSO