Advanced Logic

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Last Lecture

- an alternating finite automata (AFA) is a tuple $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$ where all components are identical to an NFA except that

$$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$$

- AFAs are more concise but otherwise equi-expressive to NFAs

Theorem

For every AFA $\mathcal{A}$ there exist a DFA $\mathcal{B}$ with $O(2^{2^{|\mathcal{A}|}})$ states such that $L(\mathcal{A}) = L(\mathcal{B})$.

Corollary

AFAs recognize REG.

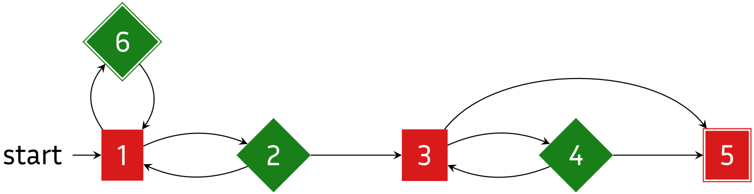
Today's Lecture

- ⋆ a short excursion to game theory
- ⋆ word-recognition in AFAs through games

Short Excursion to Game Theory

Reachability Games

- ⋆ a reachability game is played by two players, players ◆ and ■
- ⋆ the game is played on a graph which determines the current player and her possible moves



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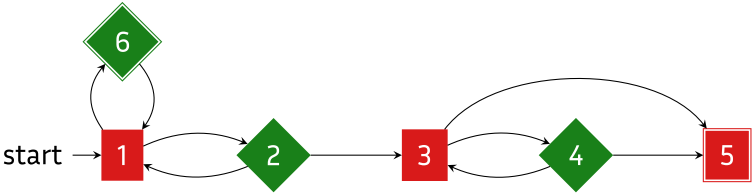

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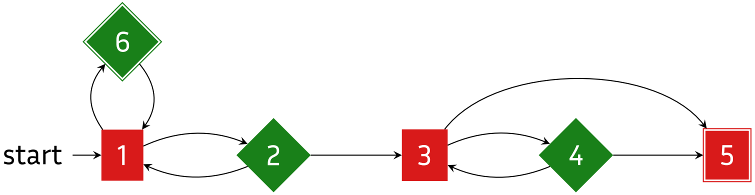

Objectives

- ⋆ player ◆: reach a certain positions (among possibly many)
- ⋆ player ■: prevent player ◆ from winning

Main Question: has player ◆ a winning strategy

Definitions

- an arena is a tuple $\mathcal{G} = (V, V_{\blacklozenge}, V_{\blacksquare}, E, v_I, Z)$ such that
  - $V = V_{\blacklozenge} \uplus V_{\blacksquare}$ are the playing positions
  - E are the possible moves
  - v_I is the initial position
  - $Z \subseteq V$ is the set of goal positions for player $V_{\blacklozenge}$

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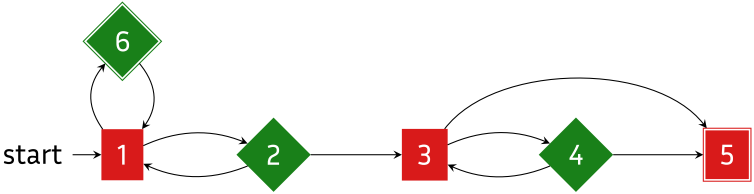
  - $Z \subseteq V$ is the set of goal positions for player $V_{\blacklozenge}$
- a match π on $\mathcal{G}$ is a (possible infinite) maximal path within (V, E) starting from v_I
- player ◆ wins a match if it passes through a position in F, or the path ends in a node $V_{\blacksquare}$ (player ■ got stuck)
- otherwise player ■ wins

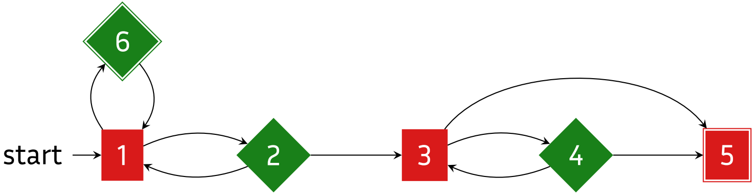
Example



the above depicts the arena $(V, V_\blacklozenge, V_\blacksquare, E, v_I, Z)$ where

- ⋆ $V_\blacklozenge = \{2, 4, 6\}$ and $V_\blacksquare = \{1, 3, 5\}$
- ⋆ $v_I = 1$ and $Z = \{5, 6\}$ (employing automata notation)

Example



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Example Matches

- ⋆ the path π_1:  is a match won by ◆

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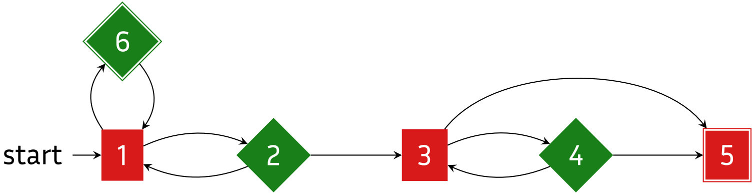

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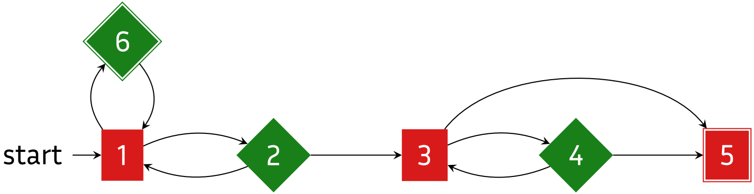

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- ★ in this arena, ◆ can always win if played properly $\Longrightarrow$ ◆ has a winning strategy

Strategies

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- formally, a strategy on an arena $\mathcal{G} = (V, V_{\blacklozenge}, V_{\blacksquare}, E, v_I, Z)$ for player $P \in \{\blacklozenge, \blacksquare\}$ is a function

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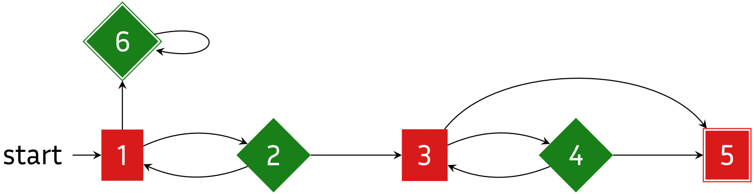
Theorem

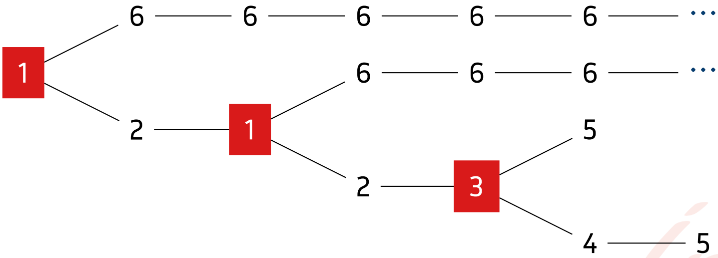
For every arena $\mathcal{G}$, either $\blacklozenge$ or $\blacksquare$ has a winning strategy.

- instance of a more general theorem due to Donald A. Martin (1982)

Strategies Seen as Trees



An arena



A strategy for player ◆; this is a winning strategy

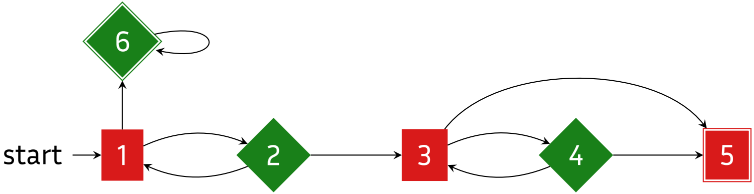
Memoryless Strategies

- a strategy σ is memoryless (or positional) if it depends only on the current players position

$$\sigma(\pi_1 \cdot v) = \sigma(\pi_2 \cdot v)$$

- a memoryless strategy for player $P \in \{\blacklozenge, \blacksquare\}$ can be seen as a function $\sigma : V_P \to V$

Example



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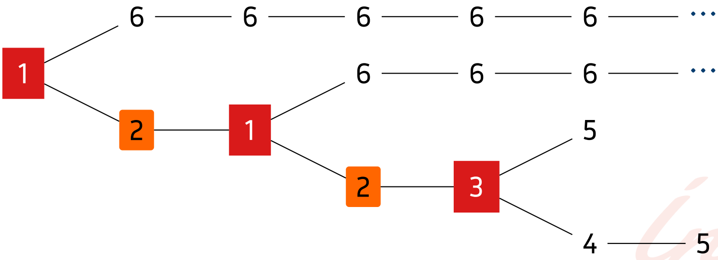
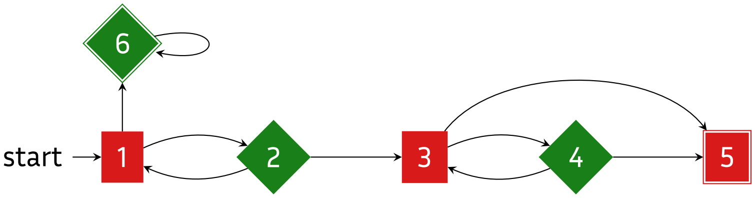


A strategy for ◆ that is not memoryless

Example



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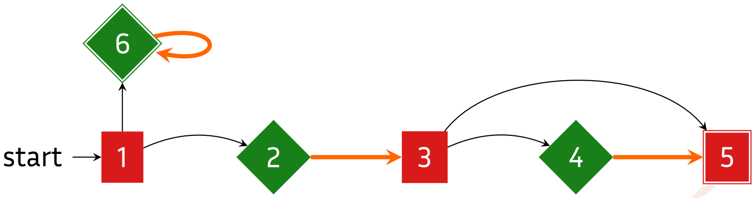

A memoryless strategy for ◆

Positional Winning Strategies

Theorem

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Proof Outline.

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- ⋆ Let W be the set of positions v such that the game with initial position v admits a positional winning strategy for player ◆
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  - $v \in V_{◆}$: Then no successors of v is in W
    - otherwise, a winning strategy for player ◆ on v could be defined by stepping to such a position, contradicting $v \notin W$
  - $v \in V_{■}$: Then at least one successor w of v is not in W
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    - otherwise, independent of the next move of player ■, player ◆ would win hence $v \in W$
  - moving for all $v \in V_{\blacksquare}$ to a successor $w \in V_{\blacksquare}$ yields a winning strategy for ■

Effective Computation of Winner

- For a set of positions W, denote by $\mathrm{Pre}_{\blacklozenge}(W)$ the set of positions v such that:
  - if $v \in V_{\blacklozenge}$, there is a successor of v in W
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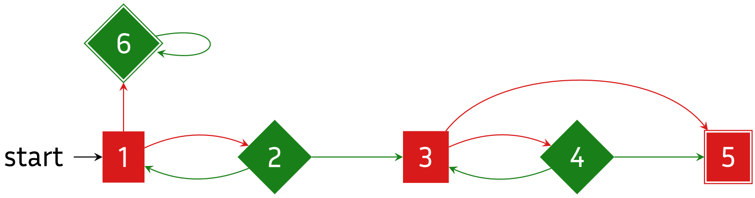
- player $\blacklozenge$ has a winning strategy iff v_I is within this fixed-point

Example



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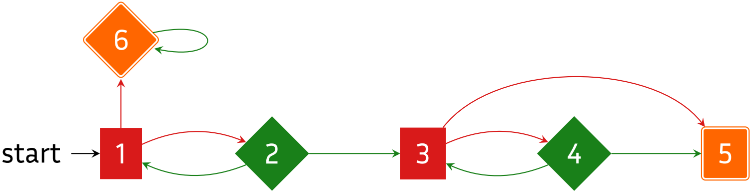

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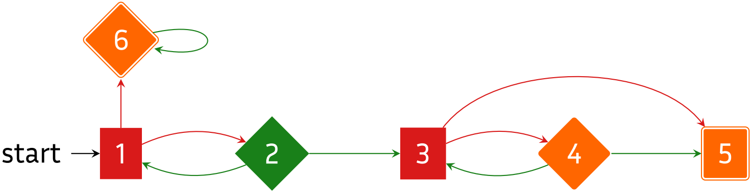
6	6

Example



$v \in V_{\blacklozenge}$	σ
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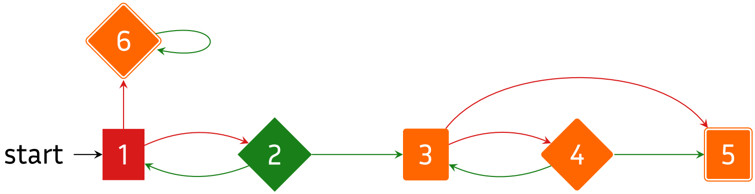
4	5

Example



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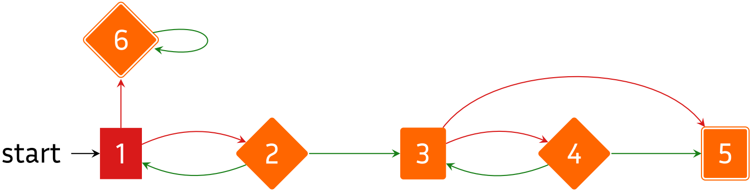
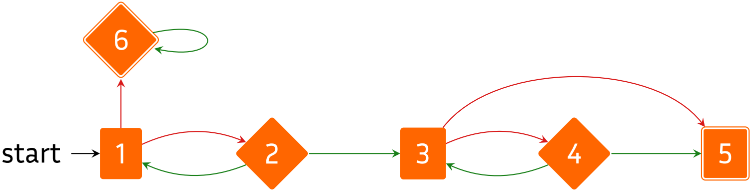


$v \in V_{\blacklozenge}$	σ
6	6
4	5
2	3

Example



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◆ has winning strategy σ

Optimal Complexity

Theorem

For an arena $\mathcal{G} = (V, V_{\blacklozenge}, V_{\blacksquare}, E, v_I, Z)$*, the set of winning positions can be computed in time* $O(|V| + |E|)$.

Optimal Complexity

Theorem

For an arena $\mathcal{G} = (V, V_{\blacklozenge}, V_{\blacksquare}, E, v_I, Z)$, the set of winning positions can be computed in time $O(|V| + |E|)$.

Note: to arrive at this optimal bound, we proceed as before and iteratively compute the set of winner positions W for player $\blacklozenge$, but:

- ⋆ we associate each node in $v \in V_{\blacksquare}$ with a counter k_V
- ⋆ k_v indicates the number of successors outside of W, initially it is simply the out-degree of v
- ⋆ once $k_v = 0$, v is added to W

Word-Recognition in AFAs through games

AFAs and Reachability Games

consider a word $w = \mathrm{a}_0 \ldots \mathrm{a}_{n-1} \in \Sigma^*$ and AFA $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$

the question $w \in L(\mathcal{A})$ be conceived as a reachability game $\mathcal{G}_{\mathcal{A},w} \triangleq (V, V_{\blacklozenge}, V_{\blacksquare}, E, v_I, Z)$

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  - being in a current state q, player ◆ picks a model M which should satisfy $\delta(q, \mathtt{q}_\mathtt{i})$

$$V_{◆} \triangleq Q \times \{0, \ldots, n-1\} \qquad (q, i) \rightarrow (M, i+1) :\Longleftrightarrow M \models \delta(q, a_i) \text{ and } i < n$$

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  - having received a model M, player ■ picks a state $q \in M$ contradicting this fact

$$V_\blacksquare \triangleq 2^Q \times \{1, \ldots, n\} \qquad (M, i) \to (q, i) :\Longleftrightarrow q \in M$$

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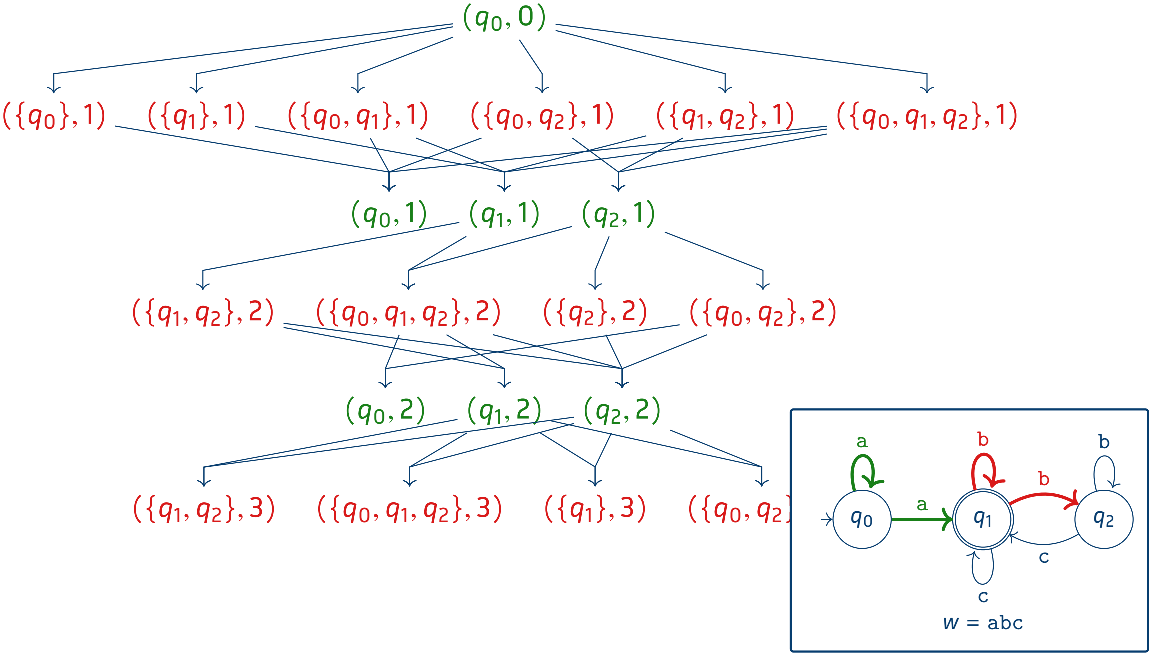
- ⋆ the game starts at $(q_I, 0)$, the goal for player ◆ is to reach (M, n) with M final

  $$v_I \triangleq (q_I, 0) \qquad F \triangleq \{(M, n) \mid M \subseteq F\}$$

Example



$(q_0, 0)$
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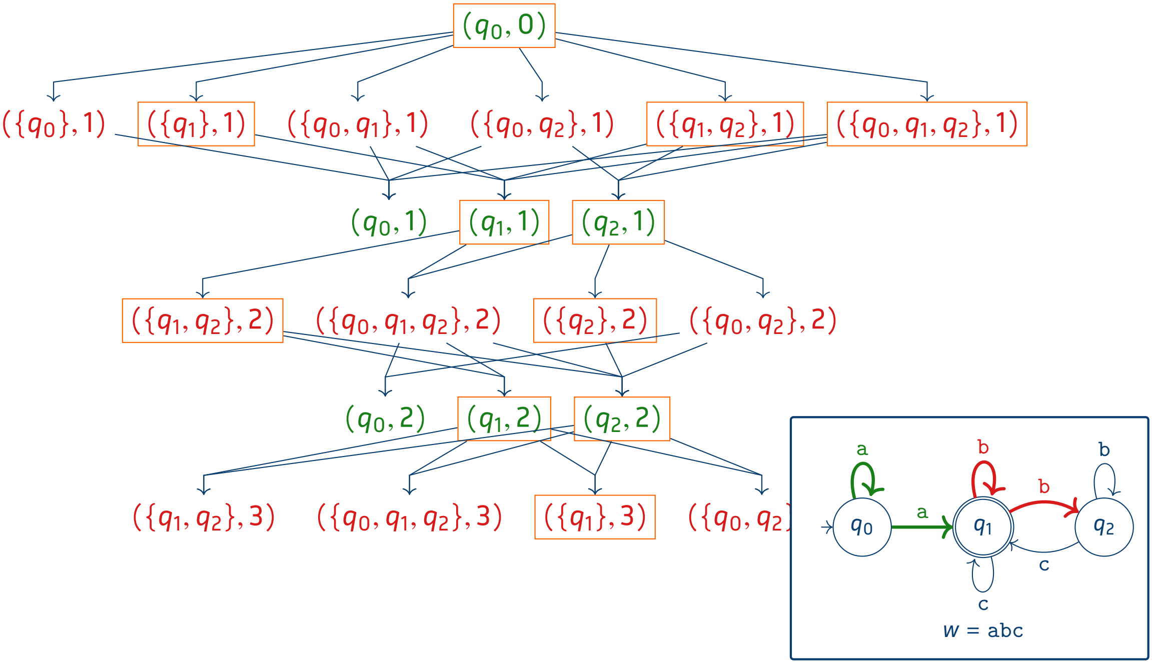
$(q_1, 2)$
$(q_2, 2)$
$(\{q_1, q_2\}, 3)$
$(\{q_0, q_1, q_2\}, 3)$
$(\{q_1\}, 3)$
a
b
b
b
a
q_0
q_1
q_2
c
c
w = abc

Example


$(q_0, 0)$
$(\{q_0\}, 1)$
$(\{q_1\}, 1)$
$(\{q_0, q_1\}, 1)$
$(\{q_0, q_2\}, 1)$
$(\{q_1, q_2\}, 1)$
$(\{q_0, q_1, q_2\}, 1)$
$(q_0, 1)$
$(q_1, 1)$
$(q_2, 1)$
$(\{q_1, q_2\}, 2)$
$(\{q_0, q_1, q_2\}, 2)$
$(\{q_2\}, 2)$
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$(\{q_1\}, 3)$
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a
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AFAs and Reachability Games (II)

Theorem

$w \in \mathrm{L}(\mathcal{A})$ *if and only if player* ◆ *has a winning strategy in* $\mathcal{G}_{\mathcal{A},w}$.

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Remarks

- translating an AFA to DFA takes $O(2^{2^n})$ space
- it is more efficient to resolve the game instead
- however, it may be more efficient to construct the DFA on the fly, avoiding the state-space explosion to some extend

Programming Project (II)

write a solver for a reachability game, computing the set of winning positions for player ◆

- ⋆ the algorithm should obey the optimal complexity bound

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- ⋆ send solutions including instructions to `martin.avanzini@inria.fr`
- ⋆ **deadline** Friday 14/05 08:00, exercise will be discussed in lecture 7