Advanced Logic

http://www-sop.inria.fr/members/Martin.Avanzini/teaching/2021/AL/

Martin Avanzini

Summer Semester 2021

Last Lecture

Presburger Arithmetic refers to the first-order theory over $(\mathbb{N}, \{0, +, <\})$

$$s, t ::= 0 \mid x \mid s + t$$

$$\phi, \psi ::= \top \mid \bot \mid s = t \mid s < t \mid \phi \wedge \psi \mid \neg\psi \mid \exists x.\phi$$

Theorem

Satisfiability and Validity are decidable for Presburger Arithmetic.

Theorem

For any formula ϕ, the constructed DFA recognizing $\hat{L}(\phi)$ has size $O(2^{2^n})$.

- ⋆ this bound can be reached

Today's Lecture

- non-determinism
- alternative finite automata
- relationship with regular languages

Non-Determinism

Angelican vs Demonic Non-Determinism

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Example

- ⋆ NFAs are based on anglican non-determinism
- ⋆ worst-case complexity analysis assumes demonic non-determinism

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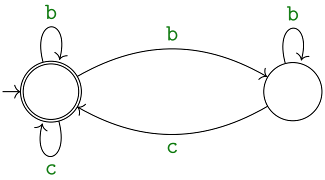

- ⋆ $L(\mathcal{A}) = (\mathtt{b} \cup \mathtt{c})^*$
- ⋆ $L^{-}(\mathcal{A}) = \epsilon \cup (\mathtt{b} \cup \mathtt{c})^* \cdot \mathtt{c}$

Duality of Non-Determinism

- recall that for each NFA $\mathcal{A}$, its dual $\overline{\mathcal{A}}$ is given by complementing final states
- in general, only when $\mathcal{A}$ is deterministic, then $\mathsf{L}(\overline{\mathcal{A}}) = \overline{\mathsf{L}(\mathcal{A})}$

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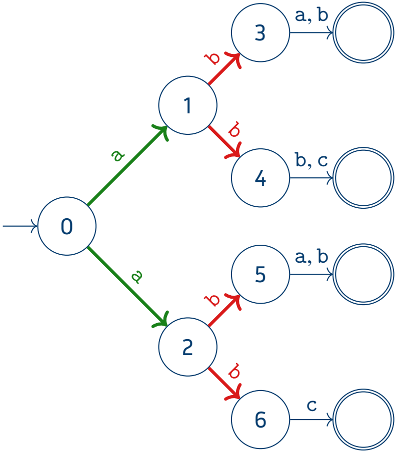
what happens if we leave regime internal to the automata?

Alternating Finite Automata

Alternating Finite Automata

- General Idea: mix Anglican an Demonic choice on the level of individual transitions
  - a player resolves Anglican choice
  - an oppenent resolves Demonic choice



$$\delta(0, a) = 1 \vee 2$$
$$\delta(1, b) = 3 \wedge 4$$
$$\delta(2, b) = 5 \wedge 6$$
$$\vdots$$

$$\begin{aligned} L(\mathcal{A}) &= a\big(b(a \cup b) \cap b(b \cup c)\big) \\ &\quad \cup a\big(b(a \cup b) \cap bc\big) \\ &= abb \cup \varnothing \\ &= abb \end{aligned}$$

Alternating Finite Automata, Formally

Positive Boolean Formulas

- let $A = \{a, b, \dots\}$ be a set of atoms
- the positive Boolean formulas $\mathbb{B}^+(A)$ over atoms A are given by the following grammar:

$$\phi, \psi ::= a \;\Big|\; \phi \wedge \psi \;\Big|\; \phi \vee \psi$$

  - such formulas are called positive because negation is missing

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- a set $M \subseteq A$ is a model of ϕ if $M \vDash \phi$ where

$$M \vDash a :\Longleftrightarrow a \in M \qquad M \vDash \phi \wedge \psi :\Longleftrightarrow M \vDash \phi \textbf{ and } M \vDash \psi \qquad M \vDash \phi \vee \psi :\Longleftrightarrow M \vDash \phi \textbf{ or } M \vDash \psi$$

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$$M \models a :\Longleftrightarrow a \in M \qquad M \models \phi \wedge \psi :\Longleftrightarrow M \models \phi \textbf{ and } M \models \psi \qquad M \models \phi \vee \psi :\Longleftrightarrow M \models \phi \textbf{ or } M \models \psi$$

Example

consider $\phi = a \wedge (b \vee c)$, then

$$\{a, b\} \models \phi \qquad \{a, c\} \models \phi \qquad \{a\} \not\models \phi \qquad \{b, c\} \not\models \phi$$

Alternating Finite Automata, Formally (II)

an alternating finite automata (AFA) is a tuple $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$ where all components are identical to an NFA except that

$$\delta : Q \times \Sigma \to \mathbb{B}^+(Q)$$

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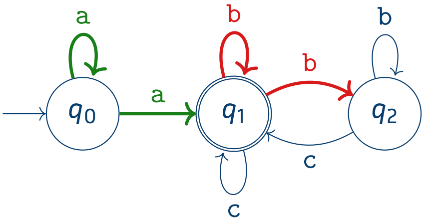

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Runs in an AFA

let $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$ be an AFA

- an execution for a word $w = \mathtt{a}_1 \ldots \mathtt{a}_n \in \Sigma^*$ is a tree T_w whose nodes are labeled by states Q s.t.:
  1. the root node of T_w is labeled by the initial state q_I
  2. for all nodes v on the ith layer ($i = 0, \ldots, n-1$) with successors $v_1, \ldots, v_k$ on layer $i+1$, labeled by $q_1, \ldots, q_k$, respectively:

$$\{q_1, \ldots, q_k\} \models \delta(q, \mathtt{a}_{\mathtt{i+1}})$$

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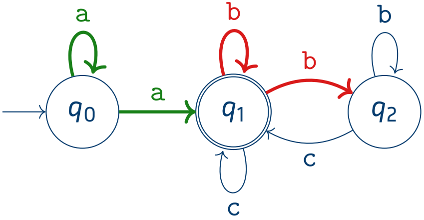
- ⋆ an execution is accepting if all leafs are labeled by final states
- ⋆ the language recognized by $\mathcal{A}$ is given by

$$L(\mathcal{A}) \triangleq \{w \mid \text{there exists an accepting execution } T_w \text{ for } w\}$$

Example of Accepting Execution for w = abbc



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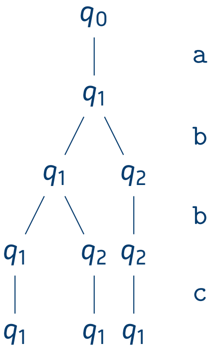

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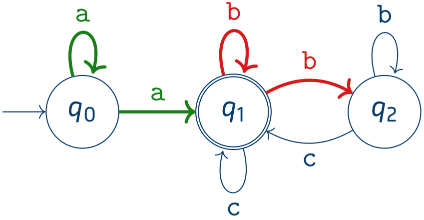

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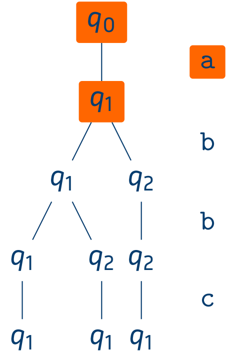

$\{q_1\} \models q_0 \vee q_1$

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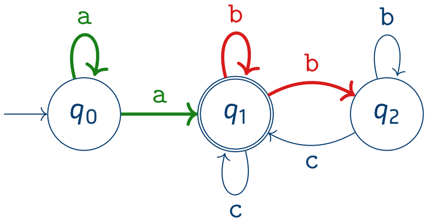

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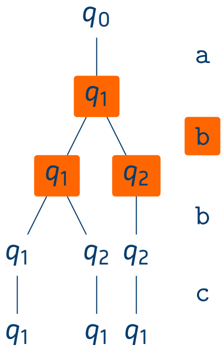

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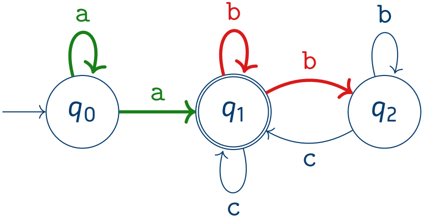

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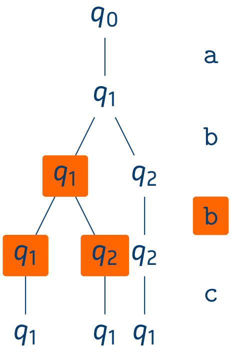

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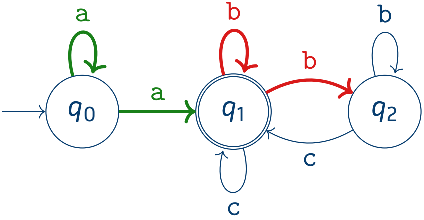

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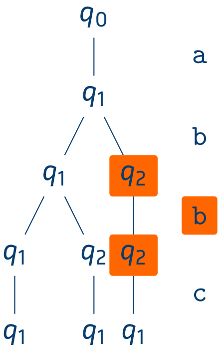
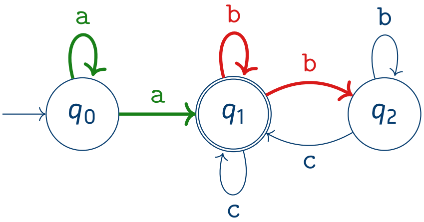


$\boxed{\{q_2\} \vDash q_2}$

Example of Accepting Execution for $w = \mathtt{abbc}$



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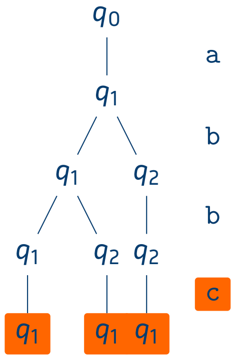

$\boxed{\{q_1\} \models q_1}$

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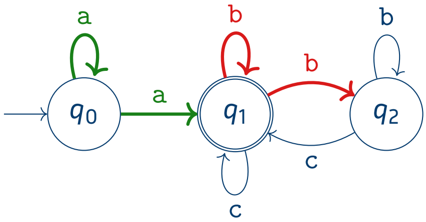

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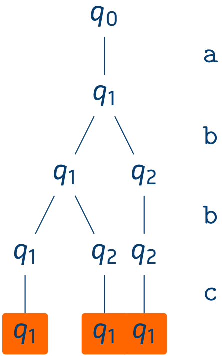

$\{q_1, q_1, q_1\} \subseteq F$

Extended Transition Function

the extended transition function

$$\hat{\delta} : \mathbb{B}^+(Q) \times \Sigma^* \to \mathbb{B}^+(Q)$$

is recursively defined by:

$$\hat{\delta}(q, \epsilon) \triangleq q \qquad \hat{\delta}(\phi \vee \psi, w) = \hat{\delta}(\phi, w) \vee \hat{\delta}(\psi, w)$$

$$\hat{\delta}(q, \mathtt{a} \cdot w) \triangleq \hat{\delta}(\delta(q, \mathtt{a}), w) \qquad \hat{\delta}(\phi \wedge \psi, w) = \hat{\delta}(\phi, w) \wedge \hat{\delta}(\psi, w)$$

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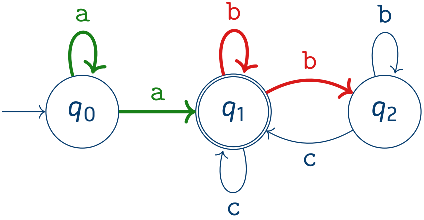
Lemma

$$\mathrm{L}(\mathcal{A}) = \{w \mid F \models \hat{\delta}(q_I, w)\}$$

Example of Accepting Execution for w = abbc (II)



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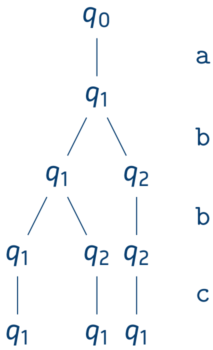

$$
\begin{aligned}
\hat{\delta}(q_0, \mathtt{abbc}) &= \hat{\delta}(q_0 \vee q_1, \mathtt{bbc}) \\
&= \hat{\delta}(q_0, \mathtt{bbc}) \vee \hat{\delta}(q_1, \mathtt{bbc}) \\
&= \hat{\delta}(q_\perp, \mathtt{bc}) \vee (\hat{\delta}(q_1, \mathtt{bc}) \wedge \hat{\delta}(q_2, \mathtt{bc})) \\
&= \hat{\delta}(q_\perp, \mathtt{c}) \vee (\hat{\delta}(q_1, \mathtt{c}) \wedge \hat{\delta}(q_2, \mathtt{c})) \\
&= \hat{\delta}(q_\perp, \epsilon) \vee \hat{\delta}(q_1, \epsilon) \\
&= q_\perp \vee q_1
\end{aligned}
$$

$$\{q_1\} \models q_\perp \vee q_1$$

Comparison to NFAs and DFAs

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  - every DFA is a NFA is an AFA

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- let $\mathcal{A}^{(m)}m = (Q^{(m)}, \{\mathrm{a}\}, \delta^{(m)}, q_I^{(m)}, F^{(m)})$ be an NFA such that $\mathrm{L}(\mathcal{A}_i) = \{w \mid |w| = 0 \bmod m\}$
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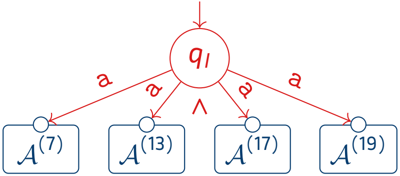
- consider the AFA $\mathcal{A}$ defined from $\mathcal{A}^{(m)}$ for primes $m = 7, 13, 17, 19$ by



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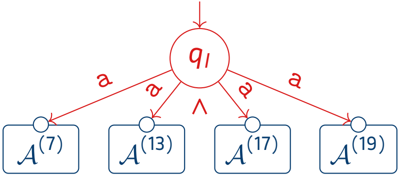

  - $\mathrm{L}(\mathcal{A}) = \{w \mid |w| = 1 \bmod 29393\}$ since $29393 = 7 \cdot 13 \cdot 17 \cdot 19$

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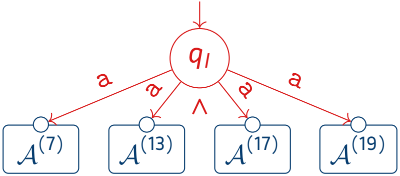

  - $\mathrm{L}(\mathcal{A}) = \{w \mid |w| = 1 \bmod 29393\}$ since $29393 = 7 \cdot 13 \cdot 17 \cdot 19$
  - AFA $\mathcal{A}$ has $57 = 1 + 7 + 13 + 17 + 19$, whereas a corresponding NFA needs 29393 states

Complementation

- recall: NFA-complementation may blow-up automata sizes by an exponential

Lemma

For every AFA $\mathcal{A}$ there exists an AFA $\overline{\mathcal{A}}$ of equal size such that $\mathsf{L}(\overline{\mathcal{A}}) = \overline{\mathsf{L}(\mathcal{A})}$

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Proof Outline.

- let $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$
- define the dual formula $\overline{\phi}$ of $\phi \in \mathbb{B}^+(Q)$ following De Morgans rules

$$\overline{q} \triangleq q \qquad \overline{\phi \vee \psi} \triangleq \overline{\phi} \wedge \overline{\psi} \qquad \overline{\phi \wedge \psi} \triangleq \overline{\phi} \vee \overline{\psi}$$

  - morally, $q \in Q$ re-used for their "negation"; we have (i) $M \vDash \phi$ iff $Q \backslash M \nvDash \overline{\phi}$

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- we now define $\overline{\mathcal{A}} \triangleq (Q, \Sigma, \overline{\delta}, q_I, Q \setminus F)$ where $\overline{\delta}(q, \mathrm{a}) \triangleq \overline{\delta(q, \mathrm{a})}$ for all $q \in Q$, $\mathrm{a} \in \Sigma$

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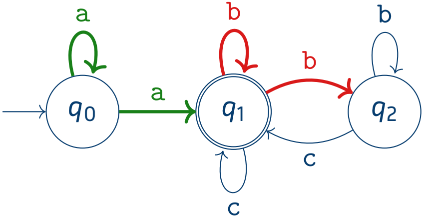
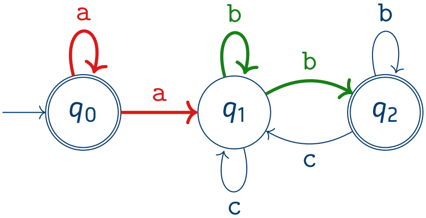
  - overall, we have

$$w \notin \mathsf{L}(\mathcal{A}) \overset{def.}{\iff} F \nvDash \hat{\delta}(q_I, w) \overset{(i)}{\iff} Q \backslash F \vDash \overline{\hat{\delta}(q_I, w)} \overset{(ii)}{\iff} Q \backslash F \vDash \hat{\overline{\delta}}(q_I, w) \overset{def.}{\iff} w \in \mathsf{L}(\overline{\mathcal{A}})$$

Example



$\Updownarrow$ complement



Relationship with Regular Languages

AFAs Recognize *REG*

Theorem

For every AFA $\mathcal{A}$ *there exist a DFA* $\mathcal{B}$ *with* $O(2^{2^{|\mathcal{A}|}})$ *states such that* $L(\mathcal{A}) = L(\mathcal{B})$.

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Idea:

- the states of $\mathcal{B}$ are formulas
- $\phi \xrightarrow{a} \psi$ in $\mathcal{B}$ if $\hat{\delta}(\phi, a) = \psi$
  - Example: $\delta(p, a) = q \wedge r$ and $\delta(q, a) = r \quad \Longrightarrow \quad p \vee q \xrightarrow{a} (q \wedge r) \vee r$
  - a run $q_I \xrightarrow{a_1} \cdots \xrightarrow{a_n} \phi$ thus models $\hat{\delta}(q_I, a_1 \ldots a_n) = \phi$
- the formula q_I is the initial state
- the formulas modeled by F are final

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- the formulas modeled by F are final
- to keep the construction finite, we'll identify equivalent formulas

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let $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$

Formally:

- the equivalence $\sim$ on $\mathbb{B}^+(Q)$ is given by $\phi \sim \psi$ if $\{M \mid M \models \phi\} = \{M \mid M \models \psi\}$
  - $q \sim q \vee q \sim q \wedge q$ but $q \not\sim p \vee q \not\sim p \wedge q$

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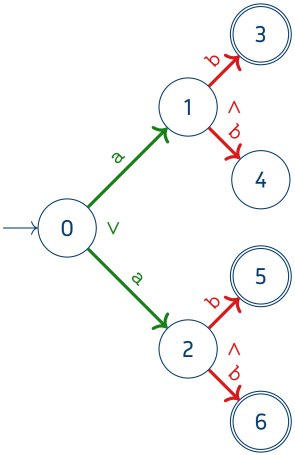
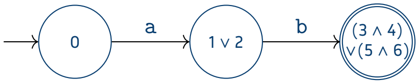
- ⋆ the set of all such equivalence classes $\mathbb{B}^+(Q)/\sim$ contains $\mathrm{O}(2^{2^{|Q|}})$ elements
- ⋆ $\mathcal{B} \triangleq (\mathbb{B}^+(Q)/\sim, \Sigma, q_I, \delta_\sim, \{[\phi]_\sim \mid F \vDash \phi\})$ where $\delta_\sim([\phi]_\sim, \mathrm{a}) \triangleq [\hat{\delta}(\phi, \mathrm{a})]_\sim$ recognises $\mathrm{L}(\mathcal{A})$

Example



the initial AFA



the translated DFA

From AFAs to NFA

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For every AFA $\mathcal{A}$ there exist a NFA $\mathcal{B}$ with $O(2^{|\mathcal{A}|})$ states such that $L(\mathcal{A}) = L(\mathcal{B})$.

Proof Outline.

- let $\mathcal{A} = (Q, \Sigma, q_I, \delta, F)$
- idea: rather then “recording” to be validated formulas as in the DFA construction, the corresponding NFA “records” valuations
  - the construction is simpler, at the expense of non-determinism

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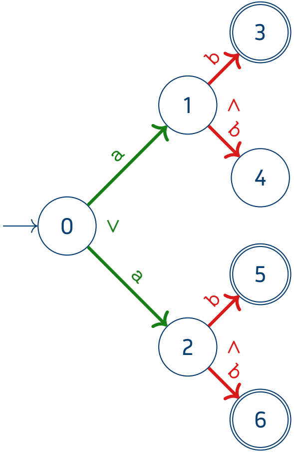
- idea: rather then "recording" to be validated formulas as in the DFA construction, the corresponding NFA "records" valuations
  - the construction is simpler, at the expense of non-determinism
- the NFA is given by $\mathcal{B} \triangleq (2^Q, \Sigma, \{q_I\}, \delta', \{M \mid M \subseteq F\})$ where

$$N \in \delta'(M, \mathrm{a}) \quad :\Longleftrightarrow \quad N \vDash \bigwedge_{q \in M} \delta(q, \mathrm{a})$$

Example (II)



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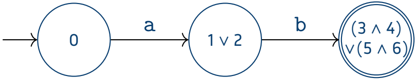
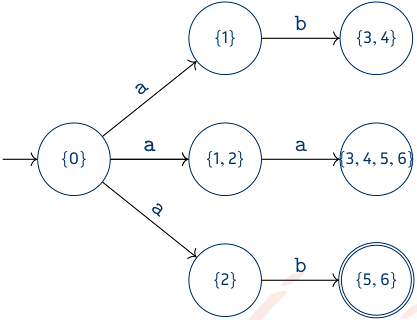


the translated DFA



the translated NFA

Discussion

- What if we translate wMSO formulas to AFAs?
  - for basic formulas $x < y$ and $X(y)$, the construction is as seen previously
  - Boolean connectives are reflected directly in the transition
  - Quantifier elimination through projection homomorphisms

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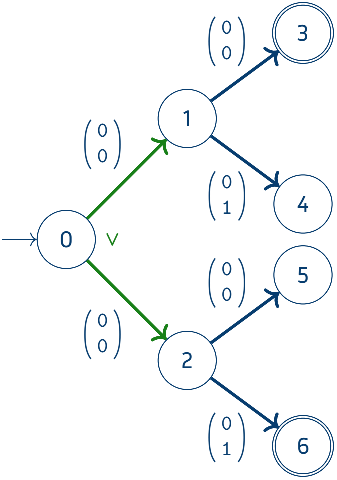
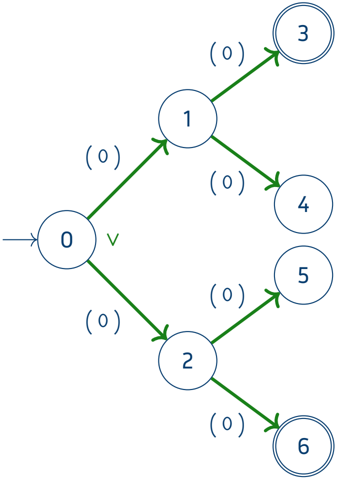
- ⋆ this suggests resulting automaton is linear in size of formula

$\Longrightarrow$ wMSO model-checking in exponential time, contradicting the lower-bound result!

Projections and AFAs



$\Longrightarrow$



$L(\mathcal{A}) = \varnothing$

$L(\mathrm{del}_{2,2}(\mathcal{A})) = \{00\}$

Discussion

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Problem:

We do not have a polytime algorithm for homorphism applications on AFAs