Closing the Gap Between Runtime Complexity and Polytime Computability

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RTA '10

Simple "Functional Program"

① $\mathsf{d}(\mathsf{c}) = 0$

② $\mathsf{d}(x \times y) = \mathsf{d}(x) \times y + x \times \mathsf{d}(y)$

③ $\mathsf{d}(x + y) = \mathsf{d}(x) + \mathsf{d}(y)$

④ $\mathsf{d}(x - y) = \mathsf{d}(x) - \mathsf{d}(y)$

data Exp = Zero	0
| Const	c
| Times Exp Exp	$e_1 \times e_2$
| Plus Exp Exp	$e_1 + e_2$
| Minus Exp Exp	$e_1 - e_2$

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Computation

$$\mathsf{d}(\mathsf{c} \times \mathsf{c})$$

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$$\mathsf{d}(\mathsf{c} \times \mathsf{c}) = 0 \times \mathsf{c} + \mathsf{c} \times 0$$

Simple "Functional Program" $\approx$ Term Rewrite System (TRS)

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Computation $\approx$ Rewriting

$$\mathsf{d}(\mathsf{c} \times \mathsf{c}) \to^{!}_{\mathcal{R}} 0 \times \mathsf{c} + \mathsf{c} \times 0$$

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Runtime Complexity

“number of reduction steps as function in the size of the initial terms”

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Runtime Complexity

"number of reduction steps as function in the size of the initial terms"

- initial terms are argument normalised

let $\mathcal{C}$ collect all constructor symbols and let $\mathcal{V}\mathsf{al}$ abbreviate $\mathcal{T}(\mathcal{C}, \mathcal{V})$

Definition (Computation)

Let $\mathcal{R}$ denote a confluent and terminating TRS.
$\mathcal{R}$ computes a function $f \colon \mathcal{V}\mathsf{al}^k \to \mathcal{V}\mathsf{al}$ if $\exists$ function symbol f

$$f(v_1, \ldots, v_k) = w \qquad \Longleftrightarrow \qquad \mathsf{f}(v_1, \ldots, v_k) \to^!_{\mathcal{R}} w$$

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$$(v_1, \ldots, v_k, w) \in R \qquad \Longleftrightarrow \qquad \mathsf{f}(v_1, \ldots, v_k) \to^!_{\mathcal{R}} w \text{ and } w \text{ accepting}$$

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$$\mathrm{rc}_{\mathcal{R}}(n) = \max\{ \mathrm{dl}(t, \to_{\mathcal{R}}) \mid |t| \leqslant n \qquad \}$$

where $\mathrm{dl}(t, \to_{\mathcal{R}}) = \max\{\ell \mid \exists (t_1, \dots, t_\ell).\ t \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} t_\ell\}$

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- derivational complexity of above TRS is at least exponential
- runtime complexity of above TRS is linear

```
$ tct -a rc -p dif.trs
  YES(?,O(n^1))

 'Weak Dependency Pairs'
  -----------------------
  Answer:           YES(?,O(n^1))
  Input Problem:    runtime-complexity with respect to
    Rules:
       { d(c) -> 0()
       , d(*(x, y)) -> +(*(y, d(x)), *(x, d(y)))
       , d(+(x, y)) -> +(d(x), d(y))
       , d(-(x, y)) -> -(d(x), d(y))}
  Proof Details:
   ...
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Our Question

what can we infer about the computational complexity from this proof?

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  Proof Details:
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```

Our Question

In particular, does it certify polytime computability of the functions defined?

Main Result

Yes

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runtime complexity is a reasonable cost model for rewriting

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1. runtime complexity naturally expresses the cost of computation

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1. runtime complexity naturally expresses the cost of computation
2. polynomially related to actual cost of an implementation

   on a Turing machine

Main Result

Theorem

For any term s with $\mathrm{dl}_{\mathcal{R}}(s) \leqslant \ell$ $\qquad \ell = \Omega(|s|)$

1. *some normal-form is computable in deterministic time* $\mathrm{O}(\log(\ell)^3 \ell^7)$

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$$\begin{tikzcd}
 & & t_7 & t_6 \\
 & \cdot \arrow[ur] \arrow[r] \arrow[dr] & \cdot \arrow[ur] \arrow[r] \arrow[dr] & t_5 \\
s \arrow[ur] \arrow[dr] & & t_3 & t_4 \\
 & \cdot \arrow[r] \arrow[dr] & t_2 & \\
 & & t_1 &
\end{tikzcd}$$

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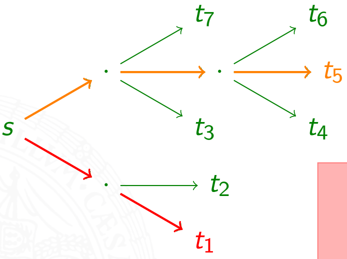

computed NF depends on employed strategy

Main Result

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For any term s with $\mathrm{dl}_{\mathcal{R}}(s) \leqslant \ell$ $\qquad \ell = \Omega(|s|)$

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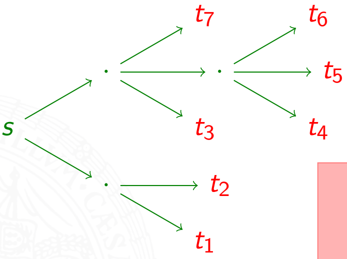
2. *any normal-form is computable in nondeterministic time* $\mathrm{O}(\log(\ell)^2 \ell^5)$



choice of redex nondeterministically

Main Result

Difficulty

a single rewrite step may copy arbitrarily large terms

- terms may grow exponential in the length of derivations

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$$\mathsf{d}(\mathsf{c}) \to^{!}_{\mathcal{R}} 0$$

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$$\mathsf{d}(\mathsf{c}) \to^{!}_{\mathcal{R}} 0$$
$$\mathsf{d}(\mathsf{c} \times \mathsf{c}) \to^{!}_{\mathcal{R}} 0 \times \mathsf{c} + \mathsf{c} \times 0$$

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$$\begin{aligned} \mathsf{d}(\mathsf{c}) &\to^!_{\mathcal{R}} 0 \\ \mathsf{d}(\mathsf{c} \times \mathsf{c}) &\to^!_{\mathcal{R}} 0 \times \mathsf{c} + \mathsf{c} \times 0 \\ \mathsf{d}((\mathsf{c} \times \mathsf{c}) \times \mathsf{c}) &\to^!_{\mathcal{R}} (0 \times \mathsf{c} + \mathsf{c} \times 0) \times \mathsf{c} + (\mathsf{c} \times \mathsf{c}) \times 0 \end{aligned}$$

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$$\begin{aligned}
\mathsf{d}(\mathsf{c}) &\to^{!}_{\mathcal{R}} 0 \\
\mathsf{d}(\mathsf{c} \times \mathsf{c}) &\to^{!}_{\mathcal{R}} 0 \times \mathsf{c} + \mathsf{c} \times 0 \\
\mathsf{d}((\mathsf{c} \times \mathsf{c}) \times \mathsf{c}) &\to^{!}_{\mathcal{R}} (0 \times \mathsf{c} + \mathsf{c} \times 0) \times \mathsf{c} + (\mathsf{c} \times \mathsf{c}) \times 0 \\
\mathsf{d}((\mathsf{c} \times \mathsf{c}) \times (\mathsf{c} \times \mathsf{c})) &\to^{!}_{\mathcal{R}} ((0 \times \mathsf{c} + \mathsf{c} \times 0) \times \mathsf{c} + (\mathsf{c} \times \mathsf{c}) \times 0) \times (\mathsf{c} \times \mathsf{c}) \\
&\quad + (\mathsf{c} \times \mathsf{c}) \times ((0 \times \mathsf{c} + \mathsf{c} \times 0) \times \mathsf{c} + (\mathsf{c} \times \mathsf{c}) \times 0)
\end{aligned}$$

$$
\begin{tikzcd}
f(s_1,\dots,s_n) \arrow[r, "\ell", "\mathcal{R}"'] \arrow[d] & t & \text{term rewriting} \\
\langle f(s_1,\dots,s_n)\rangle \arrow[r, "\leqslant p(\ell)", "\mathsf{M}"'] & \langle t\rangle \arrow[u] & \text{Turing machine}
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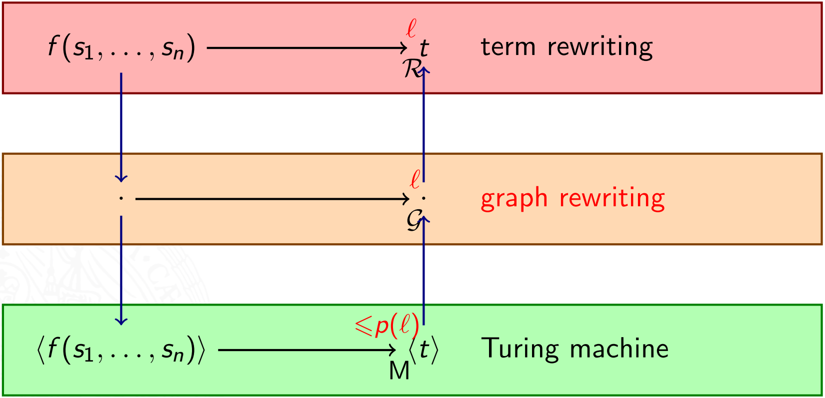
$$

Main Result

Proof Outline



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Adequacy of Graph Rewriting – "Complexity Version"

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\cdot \arrow[r, "\ell", "\mathcal{G}"'] \arrow[d] & \cdot \arrow[u] & \text{graph rewriting} \\
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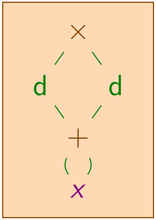
Graph Rewriting in a Nutshell

❶ term rewriting on graphs

Example

term $t = \mathsf{d}(x + x) \times \mathsf{d}(x + x)$ represented by



- same variable represented by unique node

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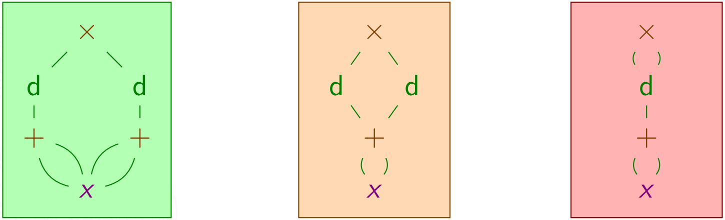

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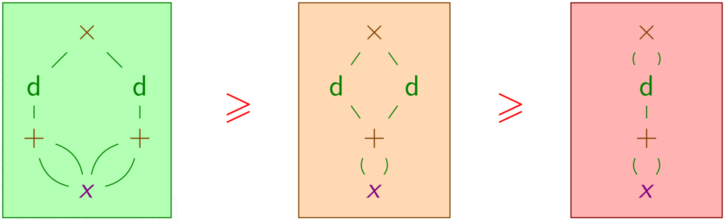

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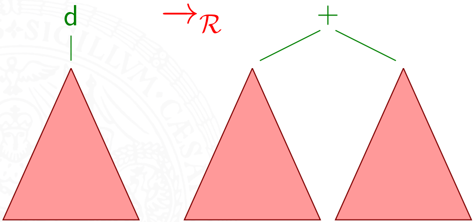
Graph Rewriting in a Nutshell

1. term rewriting on graphs
2. copying $\rightsquigarrow$ sharing

Example

Term Rewriting



$$\boxed{\mathsf{d}(x) \to x + x}$$

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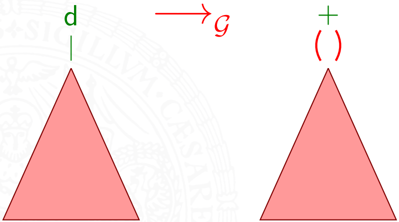

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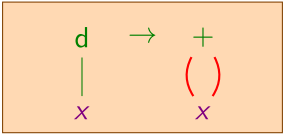
$$\Downarrow$$

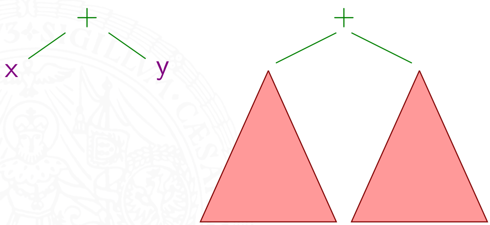


Graph Rewriting in a Nutshell

1. term rewriting on graphs
2. copying ⇝ sharing
3. structural equality ⇝ "pointer equality"

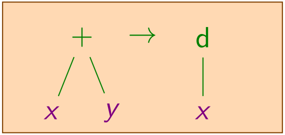
Example

Matching



$$x + y \to \mathsf{d}(x)$$

$$\Downarrow$$



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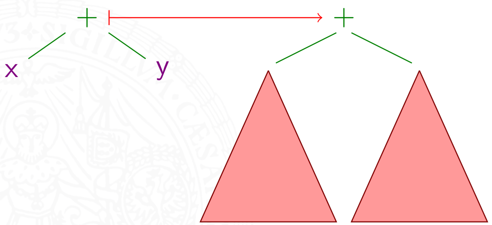

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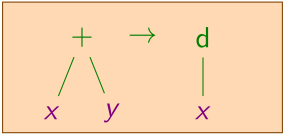

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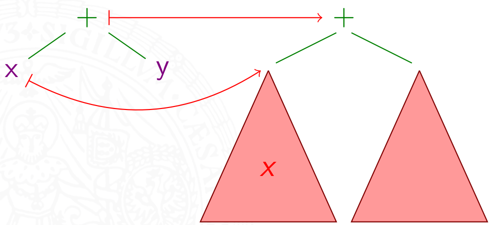

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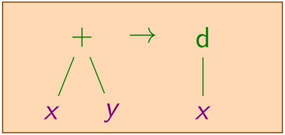

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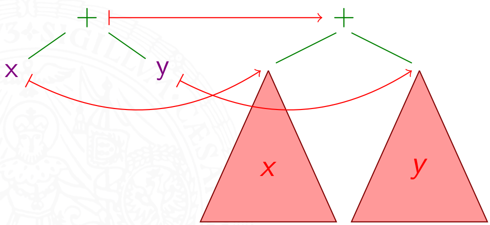

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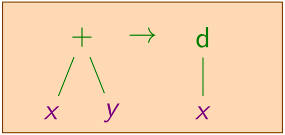
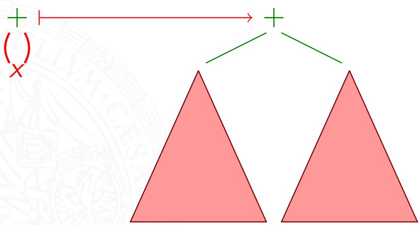


Graph Rewriting in a Nutshell

1. term rewriting on graphs
2. copying $\rightsquigarrow$ sharing
3. structural equality $\rightsquigarrow$ "pointer equality"

Example

Matching



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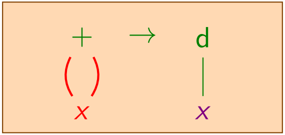

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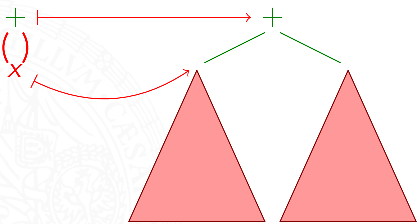

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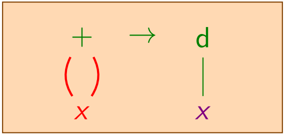

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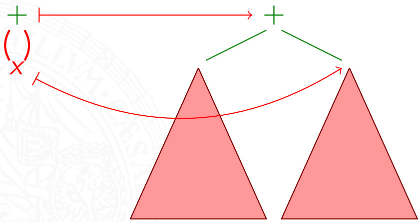

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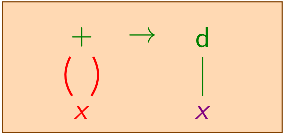

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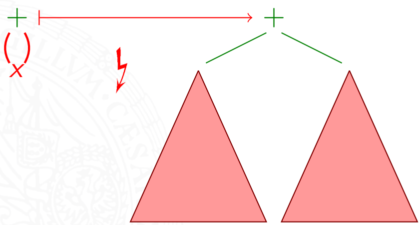

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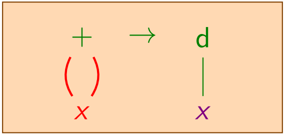

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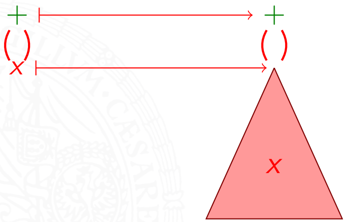

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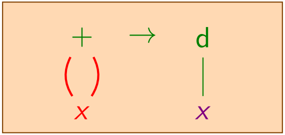

Adequacy of Graph Rewriting

Problems

$$\boxed{x + x \to \mathsf{d}(x)}$$

$$\big((0+0)+(0+0)\big) \times \big((0+0)+(0+0)\big)$$

$$\to_{\mathcal{R}} \big((0+0)+(0+0)\big) \times \big((0+0)+\mathsf{d}(0)\big)$$

$$x + x \to \mathsf{d}(x)$$

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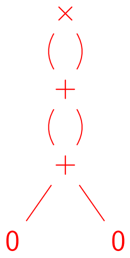

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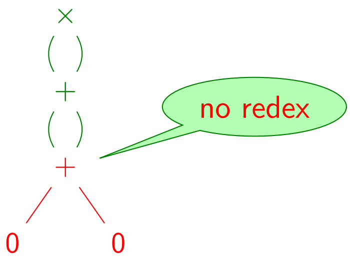

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Problem ①

below redex

maximal sharing

required

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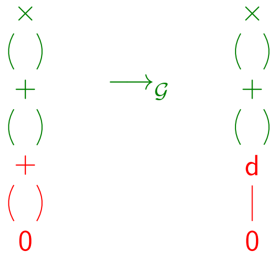


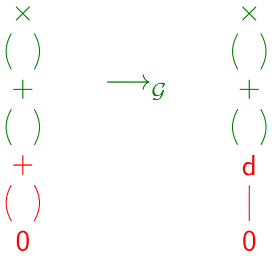
$$x + x \to \mathsf{d}(x)$$

$$((0+0)+(0+0)) \times ((0+0)+(0+0))$$

$$\to_{\mathcal{R}} ((0+0)+(0+0)) \times ((0+0)+\mathsf{d}(0))$$

$$\to^{3}_{\mathcal{R}} (\mathsf{d}(0)+\mathsf{d}(0)) \times (\mathsf{d}(0)+\mathsf{d}(0))$$

Problem ①
below redex
maximal sharing
required



Problem ②
both arguments
of $+\backslash\times$ rewritten

Adequacy of Graph Rewriting

Theorem

suppose S *is a term graph such that*

1. *node corresponding to* p *is unshared*
2. *subgraph* $S \upharpoonright p$ *is maximally shared*

Then

$$S \longrightarrow_{\mathcal{G},p} T \qquad \iff \qquad \mathsf{term}(S) \to_{\mathcal{R},p} \mathsf{term}(T)$$

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Idea

- extend rewrite relation $\longrightarrow_{\mathcal{G}}$ with folding and unfolding steps that recover condition ❶ and ❷

$$S \leqslant \cdot \geqslant \cdot \longrightarrow_{\mathcal{G}} T \qquad \Longleftrightarrow \qquad \mathsf{term}(S) \rightarrow_{\mathcal{R}} \mathsf{term}(T)$$

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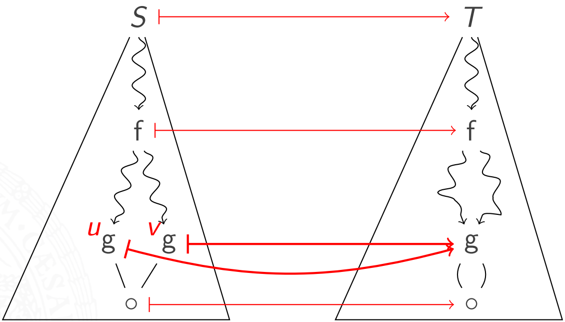
Observation

- unfolding may lead to exponential blowup

Restricted Folding and Unfolding

define for term graphs S, T

- $S \sqsupset^u_v T \;:\Longleftrightarrow\;$ "T obtained from S by identifying nodes u and v"



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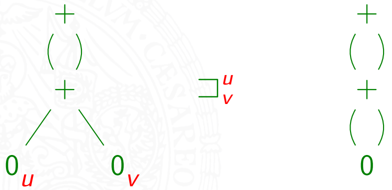


Restricted Folding and Unfolding

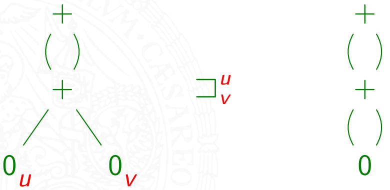
define for term graphs S, T

- $S \sqsupset^u_v T \;:\Longleftrightarrow\;$ "T obtained from S by identifying nodes u and v"

and position p

- $S \blacktriangleright_p T \;:\Longleftrightarrow\; S \sqsupset^u_v T$ for nodes $u, v \in S$ strictly below position p

Example



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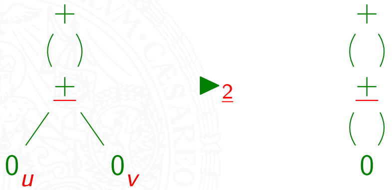

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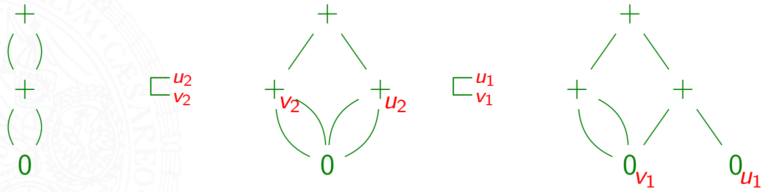

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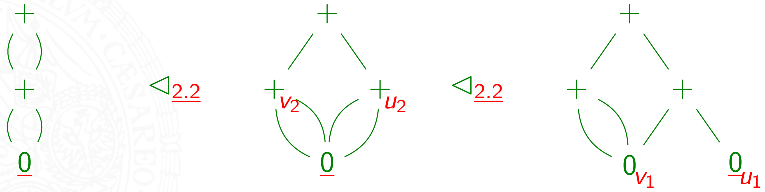

Adequacy Theorem

"Complexity Version"

Lemma

1. *if* S *is* $\blacktriangleright_p$*-minimal then* $S \upharpoonright p$ *is maximally sharing*
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$$S \lhd_p^! \cdot \blacktriangleright_p^! \cdot \longrightarrow_{\mathcal{G},p} T \qquad \Longleftrightarrow \qquad \mathrm{term}(S) \to_{\mathcal{R},p} \mathrm{term}(T)$$

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- set $\twoheadrightarrow_{\mathcal{G},p} := \lhd_p^! \cdot \blacktriangleright_p^! \cdot \longrightarrow_{\mathcal{G},p}$

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Lemma

If $S \twoheadrightarrow_{\mathcal{G}}^{\ell} T$ then $|T| \leqslant (\ell + 1) \cdot |S| + \ell^2 \cdot \Delta$ for fixed $\Delta \in \mathbb{N}$

- polynomial size growth in $|S|$ and length ℓ

Main Result Revisited

Theorem

For any term s with $\mathsf{dl}_{\mathcal{R}}(s) \leqslant \ell$ $\qquad\qquad \ell = \Omega(|s|)$

1. *some normal-form is computable in deterministic time* $\mathrm{O}(\log(\ell)^3 \ell^7)$
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Proof Idea.

$$s = s_0 \quad \to_{\mathcal{R}} \quad s_1 \quad \to_{\mathcal{R}} \quad \cdots \quad \to_{\mathcal{R}} \quad s_l$$

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$\square$

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Let $\mathcal{R}$ *be a terminating TRS with* $\mathsf{rc}_{\mathcal{R}}(n) \in \mathrm{O}(n^k)$ $\qquad k \geqslant 1$

1. *if* $\mathcal{R}$ *computes the function* f *then* $f \in \mathrm{FP}$

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1. *if $\mathcal{R}$ computes the function f then* $f \in \mathsf{FP}$
2. *if $\mathcal{R}$ computes the relation R then the function problem associated with R is in* FNP. $\qquad$ *given* $\vec{v} \in \mathcal{V}\mathsf{al}^k$, *find* $w \in \mathcal{V}\mathsf{al}$ *with* $(\vec{v}, w) \in R$

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FNP "is class of function problems associated with $\mathcal{L} \in \mathsf{NP}$"

$F_{\mathsf{SAT}} =$ given formula ϕ, find satisfying assignment α

Conclusion and Related Work

notion of runtime-complexity is a reasonable cost model for rewriting

1. cost of computation naturally expressed
2. polynomially related to actual cost on Turing machines

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2. polynomially related to actual cost on Turing machines

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Results
polynomially bounded innermost runtime complexity induces polytime computability on orthogonal constructor TRSs

Conclusion and Related Work

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U. Dal Lago and S. Martini
Derivational Complexity is an Invariant Cost Model.
In FOPARA, 2009.

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- AProVE — innermost runtime complexity
  http://aprove.informatik.rwth-aachen.de/
- CaT
  http://cl-informatik.uibk.ac.at/research/software/ttt2
- T_CT
  http://cl-informatik.uibk.ac.at/research/software/tct
- Matchbox/Poly — derivational complexity
  http://dfa.imn.htwk-leipzig.de/matchbox/poly