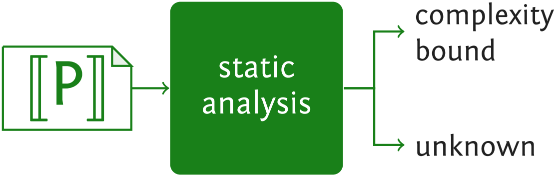
Probabilistic Term Rewriting and the Interpretation Method

Martin Avanzini and Ugo Dal Lago and Akihisa Yamada

Automated Complexity Analysis



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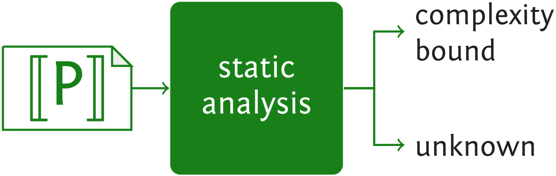
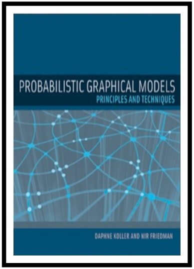


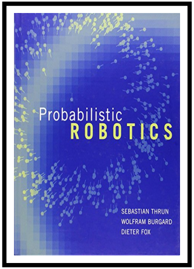
- **APROVE** (`http://aprove.informatik.rwth-aachen.de`)
- **TcT** (`http://cl-informatik.uibk.ac.at/software/tct`)
- **RaML** (`http://raml.co`)
- ...

Probabilistic Computation is Becoming Pervasive

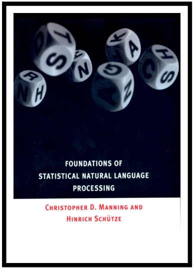
ARTIFICIAL INTELLIGENCE



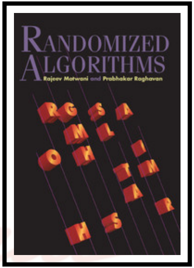
ROBOTICS



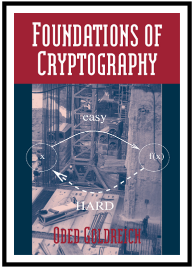
NATURAL LANGUAGE PROCESSING



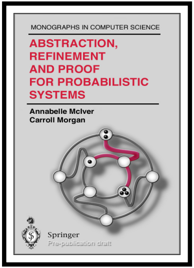
ALGORITHMS



CRYPTOGRAPHY



PROGRAM VERIFICATION



Starting Point

Proving Positive Almost-Sure Termination

Olivier Bournez, Florent Garnier

LORIA/INRIA, 615 Rue du Jardin Botanique
54602 Villers lès Nancy Cedex, France

Abstract In order to extend the modeling capabilities of rewriting systems, it is rather natural to consider that the firing of rules can be subject to some probabilistic laws. Considering rewrite rules subject to probabilities leads to numerous questions about the underlying notions and results.
We focus here on the problem of termination of a set of probabilistic rewrite rules. A probabilistic rewrite system is said almost surely terminating if the probability that a derivation leads to a normal form is one. Such a system is said positively almost surely terminating if furthermore the mean length of a derivation is finite. We provide several results and techniques in order to prove positive almost sure termination of a given set of probabilistic rewrite rules. All these techniques subsume classical ones for non-probabilistic systems.

1 Introduction

Since 30 years, term rewriting has shown to be a very powerful tool in several

O. Bournez and F. Garnier. *"Proving Positive Almost-Sure Termination"*. In *Proc. of 16th RTA*, pp. 323–337, 2005.

Probabilistic Abstract Reduction Systems

Definition (PARS – Bournez & Garnier, RTA'05)

Probabilistic abstract reduction system is tuple $\mathcal{A} = (A, \to)$ s.t.:

- A is countable set of objects
- $\to \subseteq A \times \mathsf{Dist}(A)$ maps elements from A to (discrete) probability distributions over A

Intuitions:

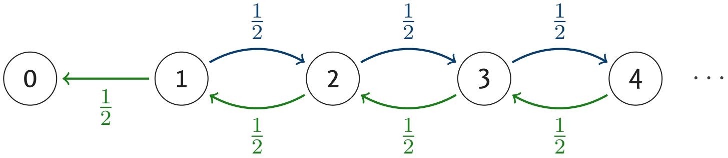
1. probabilistic choice:
   if $a \to \{p_i : b_i\}_i$ then a reduces to b_i with probability p_i;
2. non-deterministic choice:
   if $a \to d_1$ and $a \to d_2$ ($d_1 \neq d_2$), reduct is chosen from d_1 or d_2.

Example: Random walk

PARS $\mathcal{W} = (\mathbb{N}, \to)$ where

$$n \to \{\tfrac{1}{2} : n+1; \tfrac{1}{2} : n-1\} \quad \text{(for all } n > 0\text{)},$$

defines simple random walk on $\mathbb{N}$:



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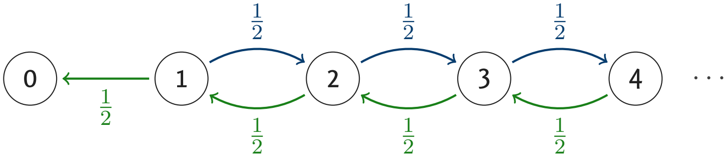

Some properties of interest:

- almost-sure termination

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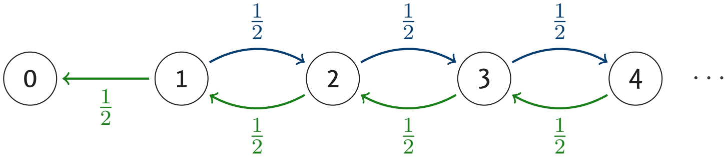

Some properties of interest:

- almost-sure termination
- *positive* almost-sure termination

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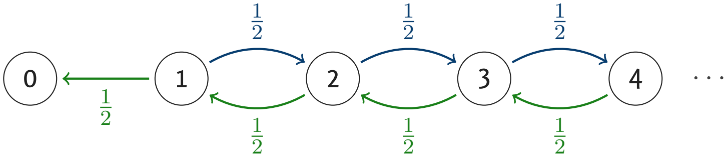

Some properties of interest:

- almost-sure termination
- *positive* almost-sure termination
- expected runtime

Dynamics: As Stochastic Processes

- Bournez and Garnier model reductions as stochastic processes

$$\mathbf{X} = X_0, X_1, X_2, X_3, \ldots$$

  - nth random variable X_n gives the nth reduct (or $\perp$)

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- random variable $T_{\mathbf{X}}$ over $\mathbb{N} \cup \{\infty\}$ measures reduction length
- expected reduction length defined as expectation of $T_{\mathbf{X}}$:

$$\mathbb{E}(T_{\mathbf{X}}) \triangleq \sum_{n=1}^{\infty} n \cdot \mathbb{P}(T_{\mathbf{X}} = n) \qquad (= \sum_{n \geq 1} \mathbb{P}(T_{\mathbf{X}} \geq n))$$

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Pros

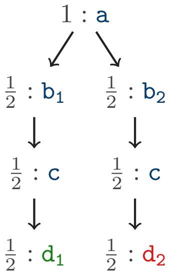
- solid theoretical foundation

Cons

- definition of whole machinery involved

Mixing Nondeterministic and Probabilistic Choice

$\mathtt{a} \rightarrow \{\frac{1}{2} : \mathtt{b}_1,\ \frac{1}{2} : \mathtt{b}_2\} \qquad \mathtt{b}_1 \rightarrow \mathtt{c} \qquad \mathtt{b}_2 \rightarrow \mathtt{c} \qquad \mathtt{c} \rightarrow \mathtt{d}_1 \qquad \mathtt{c} \rightarrow \mathtt{d}_2$



Trajectories

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$1 : \texttt{a}$

$\swarrow \qquad \searrow$

$\frac{1}{2} : \texttt{b}_1 \qquad \frac{1}{2} : \texttt{b}_2$

$\downarrow \qquad \downarrow$

$\frac{1}{2} : \texttt{c} \qquad \frac{1}{2} : \texttt{c}$

$\downarrow \qquad \downarrow$

$\frac{1}{2} : \texttt{d}_1 \qquad \frac{1}{2} : \texttt{d}_2$

Trajectories

$X_0 = \{1 : \texttt{a}\}$

$X_1 = \{\frac{1}{2} : \texttt{b}_1, \frac{1}{2} : \texttt{b}_2\}$

$X_2 = \{1 : \texttt{c}\}$

$X_3 = \{\frac{1}{2} : \texttt{d}_1, \frac{1}{2} : \texttt{d}_2\}$

Stochastic Process

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Stochastic Process

parameterised by strategy ϕ:

$\dots \quad \phi(\mathtt{ab_1c}) = (\mathtt{c} \to \mathtt{d}_1) \qquad \phi(\mathtt{ab_2c}) = (\mathtt{c} \to \mathtt{d}_2) \quad \dots$

Mixing Nondeterministic and Probabilistic Choice

$\mathtt{a} \to \{\frac{1}{2} : \mathtt{b}_1,\ \frac{1}{2} : \mathtt{b}_2\}$ $\quad \mathtt{b}_1 \to \mathtt{c}$ $\quad \mathtt{b}_2 \to \mathtt{c}$ $\quad \mathtt{c} \to \mathtt{d}_1$ $\quad \mathtt{c} \to \mathtt{d}_2$

$$
\begin{array}{ccc}
 & 1 : \mathtt{a} & \\
\swarrow & & \searrow \\
\frac{1}{2} : \mathtt{b}_1 & & \frac{1}{2} : \mathtt{b}_2 \\
\downarrow & & \downarrow \\
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\end{array}
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Trajectories

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Stochastic Process

$$\{\!\{1 : \mathtt{a}\}\!\} \rightsquigarrow \{\!\{\tfrac{1}{2} : \mathtt{b}_1, \tfrac{1}{2} : \mathtt{b}_2\}\!\} \rightsquigarrow \{\!\{\tfrac{1}{2} : \mathtt{c}, \tfrac{1}{2} : \mathtt{c}\}\!\} \rightsquigarrow \{\!\{\tfrac{1}{2} : \mathtt{d}_1, \tfrac{1}{2} : \mathtt{d}_2\}\!\}$$

Multidistribution Reduction

parameterised by strategy ϕ:

$\ldots \quad \phi(\mathtt{ab}_1\mathtt{c}) = (\mathtt{c} \to \mathtt{d}_1) \qquad \phi(\mathtt{ab}_2\mathtt{c}) = (\mathtt{c} \to \mathtt{d}_2) \quad \ldots$

Dynamics: As Multidistribution Reductions

- a (finite) multidistribution over A is a (finite) *multiset*

$$\mu = \{\!\{p_1 : a_1, \ldots, p_n : a_n\}\!\} ,$$

where $0 \leq p_i \leq 1$, $a_i \in A$ and

$$|\mu| \triangleq \sum_{i=1}^{n} p_i \leq 1 .$$

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- for PARS $\mathcal{A} = (A, \to)$, reduction relation $\rightsquigarrow_{\mathcal{A}}$ is defined s.t.:

$$\{\!\{ p_1 : a_1, \ldots, p_n : a_n \}\!\} \rightsquigarrow_{\mathcal{A}} p_1 \cdot \nu_1 \uplus \cdots \uplus p_n \cdot \nu_n ,$$

where either
  - $\nu_i = d_i$ for some d_i with $a_i \to d_i$, or
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- expected derivation length of reduction $M = \mu_0 \rightsquigarrow_{\mathcal{A}} \mu_1 \rightsquigarrow_{\mathcal{A}} \dots$:

$$\mathrm{edl}(M) \triangleq \sum_{n \geq 1} |\mu_n| .$$

Reductions vs Stochastic Processes

Theorem (A., Dal Lago & Yamada, FLOPS'18)

Stochastic sequence $\mathbf{X} = \{X_n\}_{n \in \mathbb{N}}$ *and probabilistic reductions* $M = \mu_0 \rightsquigarrow \mu_1 \rightsquigarrow \ldots$ *are in one-to-one correspondence with:*

$$\mathbb{P}(X_n = a) = |\mu_n|_a \qquad \text{(for all } n \in \mathbb{N} \text{ and } a \in A\text{)},$$

where $|\mu|_a \triangleq \sum_{(p:a) \in \mu} p$ *denotes the total probability of* a *in* μ*.*

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Corollary

$$\mathrm{edl}(M) = \mathbb{E}(T_{\mathbf{X}}) .$$

Probabilistic Ranking Functions

Definition (Bournez & Garnier, RTA'05)

Function $[\![\cdot]\!] : A \to \mathbb{R}_{\geq 0}$ is (Lyapunov) ranking function for PARS $\mathcal{A} = (A, \to)$ if for some $\epsilon > 0$,

$$a \to d \implies [\![a]\!] >_\epsilon \mathbb{E}([\![d]\!]) \ ,$$

where $\mathbb{E}(\{p_i \cdot x_i\}_i) \triangleq \sum_i p_i \cdot x_i$ and $x >_\epsilon y$ if $x \geq y + \epsilon$.

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Example

Consider the *biased* random walk $\mathcal{W}_{\frac{1}{3}} = (\mathbb{N}, \to)$ where

$$n \to \{1/3 \colon n+1; 2/3 \colon n-1\} \quad \text{(for all } n > 0\text{)},$$

define $[\![n]\!] \triangleq n$ and take $\epsilon = \frac{1}{3}$. Then for all $n > 0$,

$$[\![n]\!] >_\epsilon (1/3) \cdot [\![n+1]\!] + (2/3) \cdot [\![n-1]\!] \ .$$

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Theorem (Bournez & Garnier, RTA'05)

1. *Soundness: If $[\![\cdot]\!]$ is a ranking function for $\mathcal{A}$, then $\mathcal{A}$ is PAST.*
2. *Completeness: If $\mathcal{A}$ is finitely branching and PAST, then there exists a ranking function $[\![\cdot]\!]$ for $\mathcal{A}$.*

Probabilistic Ranking Functions, revisited

Theorem (A., Dal Lago & Yamada, FLOPS'18)

Let $[\![\cdot]\!]$ *be a ranking function for* $\mathcal{A}$.

1. *Soundness: If* $[\![\cdot]\!]$ *is a ranking function for* $\mathcal{A}$*, then* $\mathrm{edh}_{\mathcal{A}}(a) \leq [\![a]\!] \cdot \frac{1}{\epsilon}$ *for all* $a \in A$.
2. *Completeness: If* $\mathrm{edh}_{\mathcal{A}}(a) \in \mathbb{N}$ *for all* $a \in A$*, then* $[\![a]\!] \triangleq \mathrm{edh}_{\mathcal{A}}(a)$ *is a ranking function for* $\mathcal{A}$.

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Example

Consider

$$\mathtt{a_n} \to \{\tfrac{1}{2} : \mathtt{a_{n+1}}; \tfrac{1}{2} : 0\} \qquad \mathtt{a_n} \to 2^n \cdot n \qquad n + 1 \to n \qquad (\text{for all } n \in \mathbb{N})\ .$$

- this PARS is PAST, i.e., $\mathrm{edl}(M) \in \mathbb{N}$ for every reduction sequence M;
- $\mathrm{edh}(\mathtt{a_0}) > \frac{1}{2^n} \cdot (2^n \cdot n) = n$ for all $n \in \mathbb{N}$, i.e., is not bounded.

Probabilistic Term Rewrite Systems

- probabilistic TRS $\mathcal{R}$ is finite set of probabilistic rewrite rules $l \to d$

$$\mathtt{rand_L}(xs) \to \{\tfrac{1}{2} : xs, \tfrac{1}{2} : \mathtt{rand_N}(0) :: \mathtt{rand_L}(xs)\}$$
$$\mathtt{rand_N}(n) \to \{\tfrac{1}{2} : n, \tfrac{1}{2} : \mathtt{rand_N}(\mathtt{succ}(n))\}$$

- rewrite relation $\rightsquigarrow_{\mathcal{R}}$ defined in terms of underlying PARS $\hat{\mathcal{R}}$

Interpretation Method for Runtime Analysis

Definition (Hirokawa & Moser, IJCAR'08)

Monotone algebra $(\llbracket\cdot\rrbracket, \succ)$ on domain X consists of:

- interpretations $\llbracket \mathtt{f} \rrbracket : X^k \to X$ satisfying:
  - monotonicity: $x \succ y \implies \llbracket \mathtt{f} \rrbracket(\dots, x, \dots) \succ \llbracket \mathtt{f} \rrbracket(\dots, y, \dots)$;
- order $\succ \subseteq X \times X$ satisfying:
  - collapsibility: $x \succ y \implies \mathsf{G}(x) >_\epsilon \mathsf{G}(y)$ for some $\mathsf{G} : X \to \mathbb{R}_{\geq 0}$;

orients TRS $\mathcal{R}$ if

$$l \to r \in \mathcal{R} \implies \llbracket l \rrbracket\alpha \succ \llbracket r \rrbracket\alpha \qquad \text{for all assignments } \alpha : V \to X.$$

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Theorem (Hirokawa & Moser, IJCAR'08)

Suppose $(\llbracket\cdot\rrbracket, \succ)$ orients the TRS $\mathcal{R}$. Then

$$\mathrm{dh}_{\mathcal{R}}(t) \leq \mathsf{G}(\llbracket t \rrbracket) \cdot \frac{1}{\epsilon}\ .$$

Interpretation Method for Runtime Analysis

Definition (A., Dal Lago & Yamada, FLOPS'18)

Barycentric, monotone algebra $([\![\cdot]\!], \mathbb{E}, \succ)$ on domain X consists of:

- barycentric operation $\mathbb{E} : \mathsf{Dist}(X) \to X$
- interpretations $[\![\mathtt{f}]\!] : X^k \to X$ satisfying:
  - monotonicity: $x \succ y \implies [\![\mathtt{f}]\!](\ldots, x, \ldots) \succ [\![\mathtt{f}]\!](\ldots, y, \ldots)$;
  - concavity: $[\![\mathtt{f}]\!](\ldots, \mathbb{E}(\{p_i : x_i\}_i), \ldots) \succeq \mathbb{E}(\{p_i : [\![\mathtt{f}]\!](\ldots, x_i, \ldots)\}_i)$.
- order $\succ \subseteq X \times X$ satisfying:
  - collapsibility: $x \succ y \implies \mathsf{G}(x) >_\epsilon \mathsf{G}(y)$ for some $\mathsf{G} : X \to \mathbb{R}_{\geq 0}$;

orients TRS $\mathcal{R}$ if

$$l \to d \in \mathcal{R} \implies [\![l]\!]\alpha \succ \mathbb{E}([\![d]\!]\alpha) \quad \text{for all assignments } \alpha : V \to X.$$

Theorem (A., Dal Lago & Yamada, FLOPS'18)

Suppose $([\![\cdot]\!], \mathbb{E}, \succ)$ *orients the PTRS* $\mathcal{R}$. *Then*

$$\mathsf{edh}_{\mathcal{R}}(t) \leq \mathsf{G}([\![t]\!]) \cdot \frac{1}{\epsilon}\,.$$

Instances of Barycentric Algebras

1. multi-linear polynomial interpretations $(\llbracket\cdot\rrbracket, \mathbb{E}, >_\epsilon)$ where

$$\llbracket \mathtt{f} \rrbracket(x_1, \ldots, x_n) = \sum_{V \subseteq \{x_1, \ldots, x_n\}} c_V \cdot \prod_{x_i \in V} x_i \qquad (c_V \in \mathbb{N}/\mathbb{Q}/\mathbb{R})$$

2. matrix interpretations $(\llbracket\cdot\rrbracket, \mathbb{E}, \gg_\epsilon)$ where

$$\llbracket \mathtt{f} \rrbracket(x_1, \ldots, x_n) = \sum_{i=1}^{n} C_i \cdot \vec{x}_i + \vec{c} \qquad (C_i \in \mathbb{N}^{m \times m}/\mathbb{Q}^{m \times m}/\mathbb{R}^{m \times m})$$

and

$$(\vec{x})^T \gg_\epsilon (\vec{y})^T :\Longleftrightarrow \vec{x}_1 >_\epsilon \vec{y}_1 \text{ and } (\vec{x})^T \geq (\vec{y})^T \,.$$

Conclusion

- simple notion of reduction for probabilistic ARSs / TRSs based on multidistributions
- recovered the completeness proof of Lyapunov ranking functions
- barycentric algebras for reasoning about expected runtimes of probabilistic TRSs
- implementation of polynomial & matrix interpretations (over $\mathbb{R}^n_{\geq 0}$) in NaTT