Design and Analysis of Distributed Averaging with Quantized Communication

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***Abstract*— Consider a network whose nodes have some initial values, and it is desired to design an algorithm that builds on neighbor to neighbor interactions with the ultimate goal of convergence to the average of all initial node values or to some value close to that average. Such an algorithm is called generically "distributed averaging", and our goal in this paper is to study the performance of a subclass of distributed averaging algorithms where the information exchange between neighboring nodes (agents) is subject to deterministic uniform quantization. With such quantization, the precise average cannot be achieved (except in exceptional cases), but some value close to it, called quantized consensus. It is shown in this paper that in finite time, the algorithm will either cause all agents to reach a quantized consensus where the consensus value is the largest integer not greater than the average of their initial values, or will lead all variables to cycle in a small neighborhood around the average, depending on initial conditions. In the latter case, tight bounds for the size of the neighborhood are given, and it is further shown that the error can be made arbitrarily small by adjusting the algorithm's parameters in a distributed manner.**

I. INTRODUCTION

There has been considerable interest recently in developing algorithms for distributing information among members of interactive agents via local interactions (e.g., a group of sensors [1] or mobile autonomous agents [24]), especially for the scenarios where agents or sensors are constrained by limited sensing, computation, and communication capabilities. Notable among these are algorithms intended to cause such a group to reach a consensus in a distributed manner [17], [21]. Consensus processes play an important role in many other problems such as Google's PageRank [16], clock synchronization [27], and formation control [14].

One particular type of consensus process, distributed averaging, has received much attention lately [4], [10], [12], [23], [29]. Most existing algorithms for precise distributed averaging require that agents are able to send and receive real values with infinite precision. However, a realistic network can only allow messages with limited length to be transmitted between agents due to constraints on the capacity of communication links. With such a constraint, when a real value is sent from an agent to its neighbors, this value will be truncated and only a quantized version will be received by the neighbors. With such quantization, the precise average cannot be achieved (except in particular cases), but some value close to it can be achieved, called quantized consensus (the formal definition is given in Section IV). A number of papers have studied this quantized consensus problem and various *probabilistic* quantization strategies have been proposed to cause all the agents in a network to reach a quantized consensus with probability one (or at least with high probability) [2], [3], [13], [18]–[20]. Notwithstanding this, the problem of how to design and analyze consensus algorithms with *deterministic* quantization effects remains open [6], [15].

In this paper, we thoroughly analyze the performance of a class of deterministic distributed averaging algorithms in which the information exchange between neighboring agents is subject to certain types of uniform quantization. It is shown that in finite time, the algorithms will either cause all agents to reach a quantized consensus where the consensus value is the largest integer not greater than the average of their initial values, or lead all agents' variables to cycle in a small neighborhood around the average, depending on initial conditions. In the latter case, we give tight error bounds for the size of the neighborhood and it is further shown that the error can be made arbitrarily small by adjusting the algorithm's parameters in a distributed manner, at a cost of slower convergence.

A. Literature Review

Most of the related works for distributed averaging with quantized communication use either a deterministic algorithm (as our approach in this paper) or a probabilistic one.

There are only a few publications which study deterministic algorithms for quantized consensus. In [21] the distributed averaging problem with quantized communication is formulated as a feedback control design problem for coding/decoding schemes; the paper shows that with an appropriate scaling function and some carefully chosen control gain, the proposed protocol can solve the distributed averaging problem, but some spectral properties of the Laplacian matrix of the underlying fixed undirected graph have to be known in advance. More sophisticated coding/decoding schemes were proposed in [22] for time-varying undirected graphs and in [30] for time-varying directed graphs, all requiring carefully chosen parameters. Control performance of logarithmic quantizers was studied in [7] and recently a novel but complicated dynamic quantizer has been proposed in [28]. A biologically inspired algorithm was proposed in [9] which makes all agents reach some consensus with arbitrary precision, but at the cost of not preserving the desired

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average. Most closely related to the problem considered here is the work of [15] where a deterministic algorithm of the same form as in this paper has been only partially analyzed and the authors have approximated the system by a probabilistic model and left the design of the weights as an open problem.

Over the past decade quite a few probabilistic quantized consensus algorithms have been proposed. The probabilistic quantizer in [2] ensures almost sure consensus at a common but random quantization level for fixed (strongly connected) directed graphs; although the expectation of the consensus value equals the desired average, the deviation of the consensus value from the desired average is not tightly bounded. An alternative algorithm which gets around this limitation was proposed in [18] by adding dither to the agents' variables. The probabilistic algorithm in [3], called "interval consensus gossip", causes all n agents to reach a consensus in finite time almost surely on the interval in which the average lies, for time-varying (jointly connected) undirected graphs. Stochastic quantized gossip algorithms were introduced in [20], [31] and shown to work properly. The effects of quantized communication on the randomized gossip algorithm were analyzed in [8].

Another thread of research has studied quantized consensus with the additional constraint that the value at each node is an integer. The probabilistic algorithm in [19] causes all n agents reach quantized consensus almost surely for a fixed (connected) undirected graph; convergence time of the algorithm was studied in [13], with bounds on its expected value. In [5] a probabilistic algorithm was introduced to solve the quantized consensus problem for fixed (strongly connected) directed graphs using the idea of "surplus".

II. Distributed Averaging

Consider a group of $n > 1$ agents labeled 1 to n. Each agent i has control over a real-valued scalar quantity x_i called an *agreement variable* which the agent is able to update its value from time to time. Each agent can only communicate with its "neighbors". Neighbor relations are described as follows: agent j is a *neighbor* of agent i if $(i,j) \in \mathcal{E}$ is an edge in a given undirected n-vertex graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the vertex set and $\mathcal{E}$ is the edge set. We assume that the graph $\mathbb{G}$ is connected and does not change over time. Initially each agent i has a real number $x_i(0)$. Let $x_{ave}(k) = \frac{1}{n}\sum_{i\in\mathcal{V}} x_i(k)$ be the average of values of all agreement variables in the network at time k; we will refer to $x_{ave}(0)$ simply as x_{ave}. The purpose of the distributed averaging problem is to devise an algorithm for each agent which enables all n agents to asymptotically determine in a decentralized manner, the average of the initial values of their scalar variables, i.e., $\lim_{k\to\infty} x_i(k) = x_{ave}$.

A well studied approach to the problem is for each agent to use a linear iterative update rule of the form

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i} w_{ij}x_j(k), \quad \forall i \in \mathcal{V}, \qquad (1)$$

where k is a discrete time index, $\mathcal{N}_i$ is the set of neighbors of agent i and the w_{ij} are real-valued weights to be designed. Eq. (1) can be written in a matrix form as $\mathbf{x}(k+1) = W\mathbf{x}(k)$, where $\mathbf{x}(k)$ is the state vector of agreement values whose ith element equals $x_i(k)$, and W is the weight matrix whose ijth entry equals w_{ij}. A necessary and sufficient condition for the convergence of Eq. (1) to the desired average for any initial values $\mathbf{x}(0)$ is that each row sum and each column sum of W is equal to 1 and all eigenvalues of W, with the exception of a single eigenvalue of value 1, have magnitude strictly less than unity [29]. A well-known choice of weights satisfying this condition is called Metropolis algorithm where the nonzero entries of the weight matrix are given by $w_{ij} = \frac{1}{\max\{d_i,d_j\}+1}$, $\forall (i,j) \in \mathcal{E}$ and $w_{ii} = 1 - \sum_{j\in\mathcal{N}_i} w_{ij}$, $\forall i \in \mathcal{V}$, where $d_i = |\mathcal{N}_i|$ is the number of neighbors of agent i, or equivalently, the degree of vertex i in $\mathbb{G}$.

III. Quantized Communication

In a network where links have constraints on the capacity and have limited bandwidth (e.g., digital communication networks), messages cannot have infinite length. However, the distributed averaging algorithm requires sending real (infinite precision) values through these communication links. Therefore, with digital transmission, the messages transmitted between neighboring agents will have to be truncated. If the communication bandwidth was limited, the more the truncation of agents' values, the higher would be the deviation of agent's value from the desired average consensus x_{ave}.

To model the effect of quantized communication, we assume that the links perform a quantization effect on the values transmitted between agents. We still assume that each agent i can have infinite bandwidth to store its latest value $x_i(k)$ and perform computations. However, whenever agent i sends its value $x_i(k)$ through the communication network, its neighbors will receive a quantized value of $x_i(k)$. A *quantizer* is a function $\mathcal{Q} : \mathbb{R} \to \mathbb{Z}$ that maps a real value to an integer. In this paper we will study the performance of the distributed averaging algorithm due to deterministic quantization which entails two quantizers: a *truncation quantizer* $\mathcal{Q}_t(x)$ which truncates the decimal part of a real number and keeps the integer part, and a *rounding quantizer* $\mathcal{Q}_r(x)$ which rounds a real number to its nearest integer. These quantizers are defined as follows [8], [25]:

$$\mathcal{Q}_t(x) = \lfloor x \rfloor, \text{ and } \mathcal{Q}_r(x) = \begin{cases} \lfloor x \rfloor & \text{if } x - \lfloor x \rfloor < 1/2 \\ \lceil x \rceil & \text{if } x - \lfloor x \rfloor \geq 1/2 \end{cases}.$$

These map $\mathbb{R}$ into $\mathbb{Z}$ and have quantization jumps of size 1. Note that quantizers having a generic real positive quantization step ϵ can be simply recovered by a suitable scaling: $\mathcal{Q}^{(\epsilon)}(x) = \epsilon\mathcal{Q}(x/\epsilon)$ [8]. Thus the results in this paper cover these generic quantizers as well.

IV. Problem Formulation

Suppose that all n agents adhere to the same update rule of Eq. (1). Then with a quantizer $\mathcal{Q}(x)$, the network equation would be

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i} w_{ij}\mathcal{Q}(x_j(k)), \quad \forall i \in \mathcal{V}. \quad (2)$$

Simple examples show that this algorithm can cause the system to shift far away from the initial average x_{ave} [11].

Since agents know exactly the effect of the quantizer, for the agents not to lose any information caused by quantization, at each iteration k each agent i can send out the quantized value $\mathcal{Q}(x_i(k))$ (instead of sending $x_i(k)$) and store in a *local* scalar $c_i(k)$ the difference between the real value $x_i(k)$ and its quantized version, i.e.,

$$c_i(k) = x_i(k) - \mathcal{Q}(x_i(k)).$$

Then, the iteration update of agent i can be modified as

$$x_i(k+1) = w_{ii}\mathcal{Q}(x_i(k)) + \sum_{j\in\mathcal{N}_i} w_{ij}\mathcal{Q}(x_j(k)) + c_i(k). \quad (3)$$

A major difference between this equation and (2) is that here no information is lost; i.e., the total average is conserved in the network, as we will show shortly after. The state equation of the system defined by (3) is

$$\mathbf{x}(k+1) = W\mathcal{Q}(\mathbf{x}(k)) + \mathbf{x}(k) - \mathcal{Q}(\mathbf{x}(k)). \quad (4)$$

For any W where each column sums to 1 (i.e., $\mathbf{1}^T W = \mathbf{1}^T$ where $\mathbf{1}$ is the vector of all ones), the total sum of all n agreement variables does not change over time, i.e., if agents followed the protocol of equation (4), then

$$\begin{aligned}\mathbf{1}^T\mathbf{x}(k+1) &= \mathbf{1}^T(W\mathcal{Q}(\mathbf{x}(k)) + \mathbf{1}^T\mathbf{x}(k) - \mathbf{1}^T\mathcal{Q}(\mathbf{x}(k)) \\ &= \mathbf{1}^T\mathcal{Q}(\mathbf{x}(k)) + \mathbf{1}^T\mathbf{x}(k) - \mathbf{1}^T\mathcal{Q}(\mathbf{x}(k)) \\ &= \mathbf{1}^T\mathbf{x}(k) = \mathbf{1}^T\mathbf{x}(0) = nx_{ave}. \end{aligned} \quad (5)$$

Thus the average is also conserved (i.e., $x_{ave}(k) = x_{ave}, \quad \forall k$). Equation (4) would be our model of distributed averaging with deterministic quantized communication where the quantizer can take the form of the truncation $\mathcal{Q}_t$ or the rounding one $\mathcal{Q}_r$. It is worth noting that the two quantizers can be related by the following equation:

$$\mathcal{Q}_r(x) = \mathcal{Q}_t(x + 1/2).$$

Given a model with the rounding quantizer $\mathcal{Q}_r$ in (4), by taking $\mathbf{y}(k) = \mathbf{x}(k) + \frac{1}{2}\mathbf{1}$, the system evolves as:

$$\begin{aligned}\mathbf{y}(k+1) &= \mathbf{y}(k) + W\mathcal{Q}_t(\mathbf{y}(k)) - \mathcal{Q}_t(\mathbf{y}(k)) \\ \mathbf{y}(0) &= \mathbf{x}(0) + \frac{1}{2}\mathbf{1}.\end{aligned}$$

Therefore, by the analyzing the above system which has a truncation quantizer $\mathcal{Q}_t$, we can deduce the performance of $\mathbf{x}(k)$ that satisfies equation (4) with a rounding quantizer $\mathcal{Q}_r$ because they are related by a simple translation equation ($\mathbf{y}(k) = \mathbf{x}(k) + \frac{1}{2}\mathbf{1}$). Therefore the effects of the two quantizers are essentially the same. With this nontrivial observation in mind, we focus on the analysis of the truncation quantizer only in the rest of this paper. The results can then easily be extended to the case of the rounding quantizer.

In the sequel we will completely characterize the behavior of system (4) and its convergence properties. But first, we have the following definition:

Definition 1. *A network of n agents reaches a finite-time quantized consensus if there is an iteration k_0 such that*

$$\mathcal{Q}(x_i(k)) = \mathcal{Q}(x_j(k)), \ \forall i,j \in \mathcal{V}, \ \forall k \geq k_0.$$

V. Design and Analysis of the System

In this section, we carry out the analysis of the proposed quantized consensus system. As mentioned earlier, we only consider the truncation quantizer $\mathcal{Q}_t$ in (4). Then the system equation can be written as:

$$\mathbf{x}(k+1) = W\lfloor\mathbf{x}(k)\rfloor + \mathbf{x}(k) - \lfloor\mathbf{x}(k)\rfloor. \quad (6)$$

This can be written in a distributed way for every $i \in \mathcal{V}$ as follows:

$$\begin{aligned}x_i(k+1) &= x_i(k) + \sum_{j\in\mathcal{N}_i} w_{ji}\left(\lfloor x_j(k)\rfloor - \lfloor x_i(k)\rfloor\right), \quad (7)\\ &= x_i(k) + \sum_{j\in\mathcal{N}_i} w_{ji}L_{ji}(k), \quad (8)\end{aligned}$$

where $L_{ji}(k) \triangleq \lfloor x_j(k)\rfloor - \lfloor x_i(k)\rfloor = -L_{ij}(k)$. The nonlinearity of the system due to quantization complicates the analysis, and traditional stability analysis of linear systems (such as ergodicity, products of stochastic matrices, etc.) cannot be applied here as the system might not even converge.

The system behavior depends of course on the design of the weight matrix. In distributed averaging, it is important to consider weights that can be chosen locally and guarantee desired convergence properties. We impose the following assumption on W which can be satisfied in a distributed manner.

Assumption 1: The weight matrix W in our design has the following properties:

- W is a symmetric doubly stochastic matrix: $w_{ij} = w_{ji} \geq 0 \ \forall i,j \in \mathcal{V}$, and $\sum_i w_{ij} = \sum_j w_{ij} = 1$,
- Dominant diagonal entries of W: $w_{ii} > 1/2$ for all $i \in \mathcal{V}$,
- Network communication constraint: if $(i,j) \notin \mathcal{E}$, then $w_{ij} = 0$,
- For any link $(i,j) \in \mathcal{E}$ we have $w_{ij} \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of *positive* rational numbers.

These are also sufficient conditions for the linear system (1) to converge. The choice of weights being rational numbers is not restrictive because any practical implementation would satisfy this property intrinsically (we use it here to prove convergence results).

We now state the main result of this paper which will be proved in the following subsections V-A,V-B, and V-C.

Main Convergence Result 1. *Consider the quantized system (6). Suppose that Assumption 1 holds. Then for any initial value $\mathbf{x}(0)$, there is a finite time iteration where either*

1) *the system reaches quantized consensus, or*
2) *the nodes' values cycle in a small neighborhood around the average, where the neighborhood can be*

made arbitrarily small by a decentralized design of the weights (having trade-off with the speed of convergence).

A. Cyclic States

We study in this subsection the convergence properties of the system equation (6) under Assumption 1. Let us first show that due to quantized communication, the states of the agents lie in a discrete set. Since $w_{ij} \in \mathbb{Q}^+$ for any link (i,j), we can write $w_{ij} = \frac{a_{ij}}{b_{ij}}$ where a_{ij} and b_{ij} are co-prime positive integers. Suppose that B_i is the Least Common Multiple (LCM) of the integers $\{b_{ij}; (i,j) \in \mathcal{E}, j \in \mathcal{N}_i\}$. Let $c_i(k) = x_i(k) - \lfloor x_i(k) \rfloor$; then we have $c_i(k) \in [0,1)$. Moreover,

$$\begin{aligned} c_i(k+1) &= x_i(k+1) - \lfloor x_i(k+1) \rfloor \\ &= x_i(k) + \sum_{j \in \mathcal{N}_i} w_{ij} \times L_{ji}(k) - \lfloor x_i(k+1) \rfloor \\ &= \lfloor x_i(k) \rfloor + c_i(k) + \sum_{j \in \mathcal{N}_i} \frac{a_{ij}}{b_{ij}} L_{ji}(k) - \lfloor x_i(k+1) \rfloor \\ &= c_i(k) + \frac{Z(k)}{B_i}, \end{aligned} \tag{9}$$

where $Z(k) \in \mathbb{Z}$ is an integer. Then, with a simple recursion, we can see that for any iteration k we have:

$$c_i(k) = c_i(0) + \frac{\tilde{Z}(k)}{B_i}, \tag{10}$$

where $\tilde{Z}(k) \in \mathbb{Z}$. Since $c_i(k) \in [0,1)$, this equation shows that the states of the nodes are quantized, and the decimal part can have maximum B_i quantization levels. We now give the following definition,

Definition 2. *The quantized system* (6) *enters a cycle in finite time if there exists a positive integer P and a finite time k_0 such that $\mathbf{x}(k+P) = \mathbf{x}(k) \quad \forall k \geq k_0$. We call P the cycle period.*

Proposition 1. *Suppose that Assumption 1 holds. Then, the quantized system* (6)*, starting from any initial value* $\mathbf{x}(0)$*, enters a cycle in finite time.*

Proof. Let $m(k)$ and $M(k)$ be defined as follows:

$$m(k) \triangleq \min_{i \in \mathcal{V}} \lfloor x_i(k) \rfloor, \;\; M(k) \triangleq \max_{i \in \mathcal{V}} \lfloor x_i(k) \rfloor. \tag{11}$$

Notice that for any k, we have

$$\begin{aligned} x_i(k+1) &= x_i(k) + \sum_{j \in \mathcal{N}_i} w_{ji} L_{ji}(k) \\ &\leq c_i(k) + \lfloor x_i(k) \rfloor + (\sum_{j \in \mathcal{N}_i} w_{ji}) \left(M(k) - \lfloor x_i(k) \rfloor\right) \\ &\leq c_i(k) + M(k), \end{aligned}$$

from which it follows that $\lfloor x_i(k+1) \rfloor \leq M(k)$, and hence $M(k+1) \leq M(k)$. By a simple recursion we can see that the maximum cannot increase, $M(k) \leq M(0)$. Similarly, we have $m(k) \geq m(0)$. As a consequence, $\lfloor x_i(k) \rfloor \in \{m(0), m(0)+1, \ldots, M(0)-1, M(0)\}$ is a finite set. Moreover, from equation (10), $c_i(k)$ belongs to a finite set that can have at most B_i elements. Since $x_i(k) = \lfloor x_i(k) \rfloor + c_i(k)$, and each of the elements in the sum belongs to a finite set, $x_i(k)$ belongs to a finite set. But from equation (6), we have $\mathbf{x}(k+1) = f(\mathbf{x}(k))$ where the function $f(.)$ is a deterministic function of the input state at iteration k, so the system is a deterministic finite state automata. States of deterministic automata enter a cycle in finite time [26], and therefore the system is cyclic. □

B. Convergence Analysis

In this subsection, we will study the stability of the above system using a Lyapunov function. Equation (10) implies that there exist three fixed strictly positive constants $\gamma_1, \gamma_2, \gamma_3 > 0$, independent of time and only dependent on initial values and the network structure, which satisfy the following:

- For any i and any iteration k such that $c_i(k) > \left(\sum_{j \in \mathcal{N}_i} w_{ij}\right)$, we have: $c_i(k) - \sum_{j \in \mathcal{N}_i} w_{ij} \geq \gamma_1 > 0$,
- For any i and any iteration k such that $\bar{c}_i(k) > \left(\sum_{j \in \mathcal{N}_i} w_{ij}\right)$, we have: $\bar{c}_i(k) - \sum_{j \in \mathcal{N}_i} w_{ij} \geq \gamma_2 > 0$,
- For any i and any iteration k, we have: $\bar{c}_i(k) \geq \gamma_3 > 0$,

where $\bar{c}_i(k) = 1 - c_i(k)$. Let $m(k)$ and $M(k)$ be defined as in (11). Let us define the following set:

$$S_k = \{\mathbf{y} \in \mathbb{R}^n, |y_i - m(k) - 1| \leq \alpha_i\},$$

where $\alpha_i = 1 - w_{ii} + \gamma$, $\gamma = \min\{\frac{\gamma_1}{2}, \frac{\gamma_2}{2}, \gamma_3\}$. The set S_k depends on the iteration k because the value m does. Since according to the system (6), $m(k)$ cannot decrease and $M(k)$ cannot increase as indicated earlier, then S_k can only belong to one of the $M(0) - m(0)$ possible compact sets at each iteration k. Furthermore, if S_k changes to a different compact set due to an increase in m, it cannot go back to the old one as m cannot decrease.

Let us define the following candidate Lyapunov function:

$$\begin{aligned} V(k) = d(\mathbf{x}(k), S_k) &= \min_{\mathbf{y} \in S_k} ||\mathbf{y} - \mathbf{x}(k)||_1 \\ &= \min_{\mathbf{y} \in S_k} \sum_{i \in \mathcal{V}} |y_i - x_i(k)| \end{aligned} \tag{12}$$

By minimizing along each component of $\mathbf{y}$ independently, we get $V(k) = \sum_i \max\{|x_i(k) - m(k) - 1| - \alpha_i, 0\}$. Let us determine the change in the proposed candidate Lyapunov function. In order to understand the evolution of $\nabla V_k = V(k+1) - V(k)$, we group the nodes depending on their values at iteration k into 6 sets, $X_1(k)$, $X_2(k)$, $X_3(k)$, $X_4(k)$, $X_5(k)$, and $X_6(k)$:

- Node $i \in X_1(k)$ if $m(k) \leq x_i(k) < m(k) + 1 - \alpha_i$,
- Node $i \in X_2(k)$ if $m(k)+1-\alpha_i \leq x_i(k) < m(k)+1$,
- Node $i \in X_3(k)$ if $m(k)+1 \leq x_i(k) \leq m(k)+1+\alpha_i$,
- Node $i \in X_4(k)$ if $m(k)+1+\alpha_i < x_i(k) < m(k)+2$,
- Node $i \in X_5(k)$ if $m(k)+2 \leq x_i(k) < m(k)+2+\alpha_i$,
- Node $i \in X_6(k)$ if $m(k) + 2 + \alpha_i \leq x_i(k)$.

For simplicity we will drop the index k in the notation of the sets and $m(k)$ when there is no ambiguity.

Any increase in $V(k)$ is due to nodes changing to a higher set. However, any node changing its set to a higher one, should have neighbors in the higher sets that cause $V(k)$ to decrease by at least the same amount. The following lemma makes this argument formal.

Lemma 1. *Consider the quantized system* (6)*. Suppose that Assumption 1 holds. If $m(k+1) = m(k)$, we have $\nabla V_k \leq 0$.*

Proof. The proof of this lemma can be found in [11] and is omitted here due to space constraints. □

Lemma 1 implies that $V(k)$ is non-increasing with time. To show that $V(k)$ is eventually decreasing, we need some notation. Let $R(k_0) = \min\{k - k_0; k \geq k_0, \nabla V_k \leq -\beta\}$ where $\beta > 0$ is a positive constant. We will show that if there exists at least one node in $\{X_1, X_2, X_3\}$ at k_0 and $m(k) = m(k_0)$ for $k \leq R(k_0)$, then we can have a fixed upper bound on $R(k_0)$. In fact, $R(k_0)$ corresponds to the first time after k_0 when $V(k)$ strictly decreases, and an upper bound on this time interval is given by the following lemma:

Lemma 2. *If $\{X_4, X_5, X_6\} \neq \phi$ at an iteration k_0, and $m(k) = m(k_0)$ for $k \geq k_0$, then $R(k_0) \leq n\left(1 + \frac{1}{2\delta}\right)^{n-1}$, where $\delta = \min_{(i,j)\in\mathcal{E}} w_{ij}$.*

Proof. The proof of this lemma can be found in [11] and is omitted here due to space constraints. □

We also need the following lemma.

Lemma 3. *Suppose that Assumption 1 holds. For the quantized system* (6)*, at any time k_0, there is a finite time $k_1 \geq k_0$ such that for $k \geq k_1$, either $\{X_4, X_5, X_6\} = \phi$ or $m(k) > m(k_0)$.*

Proof. We prove this result by contradiction. Suppose that $\{X_4, X_5, X_6\} \neq \phi$ and $m(k) = m(k_0)$ for $k \geq k_0$. Therefore we can apply Lemma 2 to show that there is a finite time $R(k_0)$ for $\nabla V_k \leq -\beta$, otherwise $\nabla V_k \leq 0$. For $k > k_0 + n\left(\frac{V(k_0)}{\beta} + 1\right)\left(\frac{1}{2\delta} + 1\right)^{n-1}$, $V(k)$ has decreased at least $\left(\frac{V(k_0)}{\beta} + 1\right)$ times; then

$$V(k) \leq V(k_0) - \beta \times \left(\frac{V(k_0)}{\beta} + 1\right) \leq -\beta < 0,$$

which is a contradiction since $V(k) \geq 0$ is a Lyapunov function. As a result, there exists an iteration k_1 satisfying $k_1 \leq k_0 + n\left(\frac{V(k_0)}{\beta} + 1\right)\left(\frac{1}{2\delta} + 1\right)^{n-1}$ such that for $k \geq k_1$, either $\{X_4, X_5, X_6\} = \phi$ or $m(k) > m(k_0)$. □

We are now in position to prove the following proposition.

Proposition 2. *Consider the quantized system* (6)*. Suppose that Assumption 1 holds, and let $\alpha = \max_i \alpha_i$. Then for any initial value $\mathbf{x}(0)$, there is a finite time iteration where either*

- *the values of nodes are cycling in a small neighborhood around the average such that :*

$$\begin{cases} |x_i(k) - x_j(k)| \leq \alpha_i + \alpha_j \text{ for all } i, j \in \mathcal{V} \\ |x_i(k) - x_{ave}| \leq 2\alpha \text{ for all } i \in \mathcal{V}, \end{cases} \tag{13}$$

- *or the quantized values have reached consensus, i.e.*

$$\begin{cases} \lfloor x_i(k) \rfloor = \lfloor x_j(k) \rfloor \text{ for all } i, j \in \mathcal{V} \\ |x_i(k) - x_{ave}| < 1 \text{ for all } i \in \mathcal{V}. \end{cases} \tag{14}$$

Proof. The value $m(k)$ cannot increase more than $M(0) - m(0)$ number of times because $M(k)$ is non-increasing. Therefore, applying Lemma 3 for $M(0)-m(0)$ times, we see that $\{X_4, X_5, X_6\} = \phi$ in a finite number of iterations (say it is T). For $k \geq T$, if we look at the system $\mathbf{y}(k) = -\mathbf{x}(k)$, we see that $\mathbf{y}(k)$ satisfies equation (6). It is not difficult to check that by a similar argument for $\mathbf{y}(k)$, either X_1 or X_3 can be nonempty, but not both. Therefore, the two possibilities in Proposition 2 are just a consequence of the following two possible cases.

Case $\{X_1, X_4, X_5, X_6\} = \phi$: Here all nodes are in $\{X_2, X_3\}$ and by the definition of the sets we have $|x_i(k) - x_j(k)| \leq \alpha_i + \alpha_j$ for all $i, j \in \mathcal{V}$, so nodes are cycling (due to Proposition 1) around $m+1$. Moreover, since the average is conserved from Eq. (5), we have:

$$\begin{aligned} |x_i(k) - x_{ave}| &= |x_i(k) - x_{ave}(k)| \\ &\leq |\max_i x_i(k) - \min_i x_i(k)| \leq 2\alpha. \end{aligned}$$

Case $\{X_3, X_4, X_5, X_6\} = \phi$: Here all nodes are in $\{X_1, X_2\}$ and by the definition of the sets we have reached quantized consensus. Since for any i and j we have $c_i(k), c_j(k) \in [0, 1)$, then $|x_i(k) - x_j(k)| < 1$ and as in the previous case due to Eq. (5), we have $|x_i(k) - x_{ave}| < 1$. □

Corollary 1. *Consider the quantized system* (6)*. Suppose that Assumption 1 holds. If the initial values $\mathbf{x}(0)$ satisfy,*

$$\alpha \leq x_{ave} - \lfloor x_{ave} \rfloor \leq 1 - \alpha, \tag{15}$$

where $\alpha = \max_i \alpha_i$, then the network reaches quantized consensus.

Proof. If the system was cyclic, then for any node $i \in \mathcal{V}$, we have $i \in \{X_2, X_3\}$, so $x_i(k) \in [m+1-\alpha_i, m+1+\alpha_i]$. This implies that $x_{ave}(k) \in [m+1-\alpha_i, m+1+\alpha_i]$, but since the average is conserved (from equation (5)), it also implies that $x_{ave} \in [m+1-\alpha_i, m+1+\alpha_i]$. From the latter condition, since $\alpha = \max_i \alpha_i$, then if $\alpha \leq x_{ave} - \lfloor x_{ave} \rfloor \leq 1 - \alpha$, the system cannot be cyclic, and by Proposition 2, it must reach quantized consensus. □

C. Design of weights with arbitrarily small error

If the system has reached quantized consensus, the values of the agents' agreement variables become stationary and the deviation of these values from the average is no larger than 1. In the case when the system does not reach quantized consensus but becomes cyclic, Proposition 2 shows that the deviation of nodes' values from the average is upper bounded by 2α where $\alpha = \max_i \alpha_i$. It is possible to make the deviation arbitrarily small by adjusting the weights in a distributed manner. Toward that end, we propose the following modified Metropolis weights:

$$\begin{aligned} w_{ij} &= \frac{1}{C\left(\max\{d_i, d_j\} + 1\right)}, \quad \forall (i,j) \in \mathcal{E} \\ w_{ii} &= 1 - \sum_{j\in\mathcal{N}_i} w_{ij}, \quad \forall i \in \mathcal{V} \end{aligned}$$

where C is any rational constant such that $C \geq 2$. It can be easily checked that the proposed weights satisfy

Assumption 1. Moreover, the choice of C can be used to control the error. Notice that for any $i \in \mathcal{V}$, we have $w_{ii} > 1 - \frac{1}{C} \geq 1 - \frac{1}{C} + \gamma$, so $\alpha \leq \frac{1}{C}$. This shows that given an arbitrary level of precision known to all the agents, the agents can choose the weights with large enough C in a distributed manner, so that the neighborhood of the cycle will be close to the average with the given precision. Notice that if $x_{ave} \neq \lfloor x_{ave} \rfloor$, then for α small enough, the system cannot be cyclic and only quantized consensus can be reached (Corollary 1). In other words, for systems starting with different initial values, having a smaller α leads more of these systems to converge to quantized consensus (and of course if they cycled, they will cycle in a smaller neighborhood as well due to Proposition 2).

It is worth mentioning that this arbitrarily small neighborhood weight design has a trade-off with the speed of convergence of quantized consensus protocol (small error weight design leads to slower convergence).

VI. Conclusion

In this paper, we have studied the performance of distributed averaging protocols subject to deterministic communication quantization. We have shown that quantization due to links can force quantization on the state. Depending on initial conditions, the system either converges in finite time to a quantized consensus, or the nodes' values are entering into a cyclic behavior oscillating around the average. For more discussions, simulations, and fully detailed proofs we refer the reader to our technical report [11], all of which could not be included here due to space constraints.

For future work, it will be interesting to quantify the length of a period of the cycle when the system enters a cycle. We also plan to extend our results to the cases when the neighbor graph may change over time and the agents may update their variables in an asynchronous manner. Although cyclic behavior of the system will generally not occur for the above cases, the quantized consensus can still be achieved. The analysis tools presented in this paper are promising for these more complicated and challenging cases.

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