AN ALTERNATIVE OBSERVER FOR ZERO DEFICIENCY CHEMICAL NETWORKS

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Abstract: This paper provides an explicit observer for chemical reaction networks of the Feinberg-Horn-Jackson zero deficiency type. This observer is obtained by a construction different from the one employed in a previous paper. The new observer exhibits slower performance, but this is traded for more robustness to additive disturbances, expressed through an ISS property.

Keywords: observers, chemical reaction systems, detectability

1. INTRODUCTION

This paper represents a continuation of the work started in (Chaves and Sontag, 2001) (see also (Sontag, 2001) for background material on stability), which dealt with the construction of observers for the class of systems which describe chemical reaction networks of the type introduced by Feinberg, Horn, and Jackson in (Feinberg, 1995; Horn and Jackson, 1972; Horn, 1974). The outputs of the network are a subset of state variables or, more generally, monomials in state variables (which could, in practice, be associated to measured reaction rates). The original motivation for the work reported in (Chaves and Sontag, 2001), as well as in the previous paper (Sontag, 2001), came from the study of certain biological models, specifically the kinetic proofreading system proposed by McKeithan in (McKeithan, 1995) for T-cell receptor signal transduction. Observers have been often mentioned as relevant for biotechnology (Bastin and Dochain, 1990).

In (Chaves and Sontag, 2001), we provided a necessary and sufficient theoretical condition for detectability, and went on to explicitly construct a full-state observer that is guaranteed to converge globally, under the hypothesis that the system is detectable.

In this paper, we construct an alternative observer, which works under the same necessary and sufficient conditions for detectability. An important feature of the new observer is that it is more "robust" to perturbations in measurements. This robustness can be quantified by an input-to-state stability (ISS) type of estimate, and we prove that the present observer is ISS with respect to a far wider class of external disturbances than the one previously given. Among other interesting features, the observer state never becomes negative, which is natural on physical grounds (since the state being observed is always nonnegative). This nonnegativity is enforced by a "barrier feedback" reminiscent of optimization techniques.

1.1 *Review of setup and previous result*

We study n-dimensional systems $\dot{x} = f(x)$ with outputs, where the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the "mass-action kinetics" form:

$$\dot{x} \;=\; \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}\, x_1^{b_{1j}} x_2^{b_{2j}} \ldots x_n^{b_{nj}} (b_i - b_j). \quad (1)$$

Here, $m \leq n$ and each b_j is a column vector in $\mathbb{R}^n$ and has entries $b_{1j}, b_{2j}, \ldots, b_{nj}$, which are

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nonnegative integers. It is assumed furthermore that $B := [b_1, b_2, \dots, b_m]$ has rank m and that none of its rows vanishes. The constants a_{ij} are all nonnegative, and the matrix $A = [a_{ij}]$ is assumed to be irreducible. (This last condition can be relaxed to a block-irreducibility one.) It is easy to show that $\mathbb{R}^n_{>0} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0 \text{ for all } i\}$ is a forward-invariant set for the system.

The output function $h : \mathbb{R}^n \to \mathbb{R}^p$ consists of a set of monomials on state variables, which allows consideration of both the case when the concentrations of some of the substances are measured as well as cases when some of the reaction rates are measured. That is, $h(x) = (x_1^{c_{11}} x_2^{c_{12}} \dots x_n^{c_{1n}}, \cdots, x_1^{c_{p1}} x_2^{c_{p2}} \dots x_n^{c_{pn}})'$ where $C = (c_{ij}) \in \mathbb{R}^{p \times n}$ is a matrix with nonnegative entries.

A *(positive) class* $\mathcal{C}$ is any intersection of $\mathbb{R}^n_{>0}$ with an affine manifold of the form $\xi + \mathcal{D}$, where $\xi \in \mathbb{R}^n_{>0}$ and $\mathcal{D} = \text{span}\{b_i - b_j : i, j = 1, \dots, m\}$. As in (Chaves and Sontag, 2001), we make the simplifying assumption that the systems considered have no boundary equilibrium in the closure of any class, i.e., if $x = (x_1, \dots, x_n) \in \overline{\mathbb{R}^n_{>0}}$ is so that some component x_i of x vanishes and $x - \bar{x} \in \mathcal{D}$, for some $\bar{x} \in \mathbb{R}^n_{>0}$, then $f(x) \neq 0$. (No reaction consistent with positive concentrations can be in equilibrium if one of the participating substances is at zero concentration.) We then have the following fact as a simple consequence of the theory developed by Feinberg et. al., see (Sontag, 2001): For each positive class $\mathcal{C}$ there exists a (unique) state $\bar{x} = \bar{x}_{\mathcal{C}} \in \mathbb{R}^n_{>0}$ which is a globally asymptotically stable point relative to $\mathcal{C}$, i.e., for each $x_0 \in \mathcal{C}$, the solution of $\dot{x} = f(x), \quad x(0) = x_0$ is defined for all $t \geq 0$, and $x(t) \to \bar{x}$ as $t \to \infty$, and for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $|\bar{x} - x_0| < \delta$, then $|\bar{x} - x(t)| < \varepsilon$ for all $t > 0$.

Detectability As in (Chaves and Sontag, 2001), we say that a system (1) is *detectable* (on $\mathbb{R}^n_{>0}$) if, for every two trajectories $x_1(\cdot)$ and $x_2(\cdot)$ in $\mathbb{R}^n_{>0}$,

$$h(x_1(t)) \equiv h(x_2(t)) \;\Rightarrow$$
$$|x_1(t) - x_2(t)| \to 0 \text{ as } t \to \infty.$$

We showed in (Chaves and Sontag, 2001) that detectability is equivalent to the algebraic condition

$$\mathcal{D} + \text{ im } C' = \mathbb{R}^n .$$

Note that $\dim\mathcal{D} = m - 1$ and that detectability implies $m - 1 + \text{rank } C' = n$. In the case the matrix C has full rank, then rank $C' = p$ and detectability implies $m - 1 + p = n$.

By a (full-state) *observer* for (1) we mean a system $\dot{z} = g(z, h(x))$ such that, for each $x(0), z(0)$ in $\mathbb{R}^n_{>0}$, the composite system has solutions defined for all $t > 0$, and $|z(t) - x(t)| \to 0$ as $t \to +\infty$ (This is a weak definition, on that only "attraction" and not stability is required; however, in our proofs we achieve a stability property as well, as will follow from "KL" estimates.) Detectability is necessary for the existence of an observer. Conversely, we showed in (Chaves and Sontag, 2001) that, if the system (1) is detectable, then the following system is an observer for it:

$$\dot{z} = f(z) + C'(h(x) - h(z)) . \tag{2}$$

This looks very much, of course, like a Kalman filter (or Luenberger observer), but notice that (a) our systems are nonlinear, and (b) the "gain" matrix is C' and not an arbitrary matrix L making $\dot{x} = f(x) - Lh(x)$ in some sense stable.

2. AN ALTERNATIVE OBSERVER

We now use logarithms of outputs as the input to the observer. Specifically, consider for each p the vector function: $\rho_p(x) = (\ln x_1, \dots, \ln x_p)'$. (We drop the subscript p when clear from the context.) Then we have:

Theorem 1. If (1) is detectable, the following system is an observer for it:

$$\dot{z} = f(z) + C'(\rho(h(x)) - \rho(h(z))) . \tag{3}$$

A convenient notation is: $H(x) := \rho(h(x))$. So, $H(x) = [H_1(x)\ H_2(x) \dots H_p(x)]' = C\rho(x)$ where $H_i(x) = c_{i1} \ln x_1 + \cdots + c_{in} \ln x_n$ (defined on $\mathbb{R}^n_{>0}$ only). The above observer can be written as:

$$\dot{z} = f(z) + C'(H(x) - H(z)) . \tag{4}$$

To prove this theorem, one shows that the system with inputs

$$\dot{z} = f^*(z, u) := f(z) + C'(u - H(z)), \tag{5}$$

satisfies an "input to state stability" (ISS) condition. The analysis of this system with inputs is interesting in its own right, since it provides a means of studying the behavior of the observer under bounded inputs, which may represent, for example, the effect of additive disturbances on observations. The ISS property obtained allows a precise characterization of the stability properties of the observer.

2.1 *An ISS property*

The definition of an input-to-state stable (ISS) system and several equivalent characterizations are given in (Sontag and Wang, 1995). Here, we adapt this notion to deal with constrained inputs and relative equilibria, as well as positive states (i.e., those states with all coordinates in the strictly positive half-line).

We will say a system $\dot{z} = f^*(z,u)$, with input-value set $\mathbb{U} \subset \mathbb{R}^p$, evolving on $\mathbb{R}^n_{>0}$, is $\mathbb{R}^n_{>0}$-*(forward) invariant* if, for each initial state $z(0) \in \mathbb{R}^n_{>0}$ and each $\mathbb{U}$-valued input $u(\cdot)$, the corresponding maximal solution of $\dot{z} = f^*(z,u)$, which is defined on an interval $J_{z(0),u} = [0, t_{\max})$, has values $z(t) \in \mathbb{R}^n_{>0}$ for all $t \in J_{z(0),u}$. The system is $\mathbb{R}^n_{>0}$-*(forward) complete* if it is $\mathbb{R}^n$-(forward) invariant and, for each $z(0) \in \mathbb{R}^n_{>0}$ and $\mathbb{U}$-valued input $u(\cdot)$, $J_{z(0),u} = [0, +\infty)$.

For the following definitions, fix points $\bar{x} \in \mathbb{R}^n_{>0}$ and $\bar{u} \in \mathbb{U}$, where $\mathbb{U}$ is a subset of $\mathbb{R}^p$. Let $\|\cdot\|$ denote the essential supremum norm.

Definition 2.1. A system $\dot{z} = f^*(z,u)$, evolving on $\mathbb{R}^n_{>0}$, is *input-to-state stable with input-value set* $\mathbb{U}$ (with respect to the point $\bar{x}$ and the input $\bar{u}$) if it is $\mathbb{R}^n_{>0}$-complete and there exist a function β of class $\mathcal{KL}$ and a function φ of class $\mathcal{K}_\infty$ such that, for each $\mathbb{U}$-valued input $u(\cdot)$, and each initial condition $z_0 \in \mathbb{R}^n_{>0}$, for all t it holds that

$$|z(t) - \bar{x}| \leq \beta(|z_0 - \bar{x}|, t) + \varphi(\|u - \bar{u}\|). \quad (6)$$

Definition 2.2. An *ISS-Lyapunov function with respect to the point $\bar{x}$ and input $\bar{u}$*, for the system $\dot{z} = f^*(z,u)$ with inputs in $\mathbb{U} \subseteq \mathbb{R}^p$, evolving on $\mathbb{R}^n_{>0}$, is a continuous function $V : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$, whose restriction to $\mathbb{R}^n_{>0}$ is continuously differentiable, which satisfies: (i) $V(z) \geq 0$ ($V(z) = 0 \Leftrightarrow z = \bar{x}$); (ii) V is proper in $\mathbb{R}^n_{\geq 0}$; (iii) There exist functions $\alpha, \gamma \in \mathcal{K}_\infty$ such that, for all $u \in \mathbb{U}$ and $z \in \mathbb{R}^n_{>0}$,

$$\nabla V(z)\ f^*(z,u) \leq -\alpha(|z - \bar{x}|) + \gamma(|u - \bar{u}|).$$

It is not difficult to check that, for each fixed positive equilibrium $\bar{x}$

$$V(x) = \sum_{i=1}^{n} x_i(\ln x_i - \ln \bar{x}_i) + \bar{x}_i - x_i$$

is an ISS-Lyapunov function with respect to the point $\bar{x}$ and input $\bar{u} = H(\bar{x})$, for the system (5).

Lemma 2.3. Consider an $\mathbb{R}^n_{>0}$-invariant system $\dot{z} = f^*(z,u)$, with input-value set $\mathbb{U}$. Suppose that the system is $\mathbb{R}^n_{>0}$-complete. Fix a state $\bar{x} \in \mathbb{R}^n_{>0}$ and input value $\bar{u} \in \mathbb{U}$. Suppose that that there is some ISS-Lyapunov funtion V with respect to $\bar{x}$ and $\bar{u}$. Then, the system is input-to-state stable with input-value set $\mathbb{U}$ (with respect to the same $\bar{x}$ and $\bar{u}$).

The proof of this Lemma is very similar to what is done in the case of the usual definition of an ISS system (as in (Sontag and Wang, 1995)). One can also show that

Proposition 2.4. Assume system (1), with output h, to be detectable. Then the system (5) is $\mathbb{R}^n_{>0}$-invariant with input-value set $\mathbb{R}^p$. Furthermore, pick any $\bar{x} \in E_+$ and let κ be a positive constant. Define $\mathbb{U}_\kappa$ to be the subset of $\mathbb{R}^p$ consisting of vectors u which satisfy $|u| \leq \kappa$. Then, the system (5) is $\mathbb{R}^n_{>0}$-complete with input-value set $\mathbb{U}_\kappa$.

Since system (1) is globally asymptotically stable relative to each positive class, the trajectory of system (1) for each initial condition will stay in a compact set of $\mathbb{R}^n_{>0}$. So let κ be such that $|H(x(t))| \leq \kappa$ for all t, and let $\mathbb{U}_\kappa$ be the corresponding set of controls. The proof of Theorem 1 now follows immediately from Proposition 2.4 and Lemma 2.3, upon substitution by the specific input $u(t) = H(x(t))$ obtained from observations.

2.2 *Robustness*

The system $\dot{z} = f^*(z,u)$ corresponding to our first observer (Chaves and Sontag, 2001) also exhibits an ISS stability property, but only with respect to a more restrictive set of inputs, namely, u has to be nonnegative as well as uniformly bounded. In fact, if it happens that $u(t)$ becomes less than an (arbitrary) amount $-\varepsilon$ on some interval of time (in the case of a systematic negative disturbance or some negative offset in a control), then solutions may blow up in finite time. This problem arises even on McKeithan's kinetic proofreading example discussed in that paper. This example is obtained by letting $n = 3, m = 2$, $b_1 = (1,1,0)'$, $b_2 = (0,0,1)'$. For simplicity, take rate constants equal to 1. Since $\mathcal{D} = \text{span}\,\{(1,1,-1)'\}$, the following output $h(x) = (x_1^2, x_2^2)'$ garantees detectability of (1).

Consider an input $u(t) = (u_1(t), u_2(t))'$ with $2(u_1(t) + u_2(t)) < -\varepsilon$ for some $\varepsilon > 0$ and all t. For the McKeithan example, the system with inputs (similar to (5)) suggested by our previous observer (2), is then of the form

$$\begin{aligned} \dot{x}_1 &= -x_1x_2 + x_3 + 2\,(u_1 - x_1^2) \\ \dot{x}_2 &= -x_1x_2 + x_3 \\ \dot{x}_3 &= +x_1x_2 - x_3 + 2\,(u_2 - x_3^2). \end{aligned}$$

Now look at the variable $x_1 + x_3$:

$$\begin{aligned} \dot{x}_1 + \dot{x}_3 &= 2(u_1 + u_2) - (x_1^2 + x_3^2) \\ &\leq -\varepsilon - \frac{1}{2}(x_1 + x_3)^2. \end{aligned}$$

Viewing this as a differential inequality, we get that $(x_1+x_3)(t) \leq w(t)$ where $w(t)$ is the solution to $\dot{w} = -\varepsilon - \frac{1}{2}w^2,\ w(0) = (x_1+x_3)(0)$, i.e.,

$$(x_1 + x_3)(t) \leq -\sqrt{\varepsilon}\tan\left(\frac{\sqrt{\varepsilon}}{2}t - a_0\right)$$

where $a_0 = \arctan\left(\frac{(x_1+x_3)(0)}{\sqrt{\varepsilon}}\right) > 0$. Therefore, for a suitably chosen initial condition, $w(t)$ blows up to $-\infty$ as t tends to $\frac{\sqrt{\varepsilon}}{2}(a_0 + \frac{\pi}{2})$. Hence, since $(x_1 + x_3)(t) \leq w(t)$, $x_1 + x_3$ also has to blow up to $-\infty$ in finite time, and this can happen even if all states started positive.

In contrast, model (5), allows each coordinate of u to have negative values, for an arbitrarily large interval of time, and only requires that $|u(t)| < \kappa$ for all t, for some (fixed but arbitratry) positive constant, κ.

3. SOME SIMULATIONS

We carried out some simulations to explore the responses of the observers in the case of noise in the output measurements. We also compare the performance of our observers with that of an extended Kalman filter.

As a working example we choose $\dot{x} = f(x)$ to be the network proposed by McKeithan described above, and took the output to be $h(x) = (x_1x_2^2, x_1x_3)'$. The reaction rate constants were taken to be: $a_{12} = 1, a_{21} = 3$.

In one simulation, white noise was added to the outputs, so that the equation for the alternative observer becomes

$$\dot{z} = f(z) + C'(\rho(h(x(t)) + n(t)) - \rho(h(z)))$$

and $n(t)$ is an $\mathbb{R}^2$-valued vector white noise. Since $n(t)$ is a bounded function, we know from Proposition 2.4 and Lemma 2.3 that this system has a solution for all t (in the case of our first observer, solutions also exist for all t provided that $h(x(t)) + n(t)$ is nonnegative). Thus we chose $x(0)$ so that $x_i(t) \geq 2$ and $|n(t)| < 2$ for all t and all i.

In Figure 1(a) we can see that the alternative observer, although slower in the convergence, has smoothed out the noise effect, whereas the main observer exhibits small magnitude perturbations.

In the chemical reactor literature, observers are typically constructed using extended Kalman filters (EKF). This approach is not guaranteed to work for nonlinear systems, but it is often the case that they perform adequately in specific examples, and hence their practical success.

Using an EKF standard construction (Bastin and Dochain, 1990) to obtain an observer for the same example as before, Figure 1 (b) illustrates that, even for very simple examples, these standard techniqes can fail in a major way. The extended Kalman filter is seen to converge locally, but for many other initial conditions, away from $\bar{x}$, its solution diverges, while our observer is, as predicted by the theory, convergent to the right estimate.

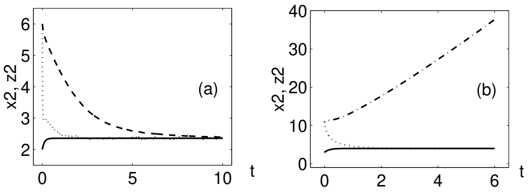

Fig. 1. (a) The trajectories of the system (solid line) and observers (2) (dotted line) and (3) (dashed line), in the presence of observation noise; (b) The extended Kalman filter (dash-dotted line) diverges for many initial conditions. Here $x(0) = (2, 3, 3)'$, and $z(0) = (1, 11, 11)'$ for both observers.

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