A Parameter–Robust Observer as an Application of ISS Techniques

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Abstract

Systems that model chemical networks are often defined through a set of parameters (the reaction rate constants) whose values may be determined with a small margin of error. These parameters will typically also appear in the construction of observers. The idea of robustness of observers with respect to the parameters is discussed. A definition of parameter-robustness is proposed and an explicit observer for zero-deficiency chemical networks is presented, which is robust in this sense.

1 Introduction

Consider the following model for chemical reaction networks of Feinberg-Horn-Jackson zero deficiency type ([3, 4, 5]), with mass-action kinetics:

$$\dot{x} = f_A(x) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_1^{b_{1j}} x_2^{b_{2j}} \ldots x_n^{b_{nj}} (b_i - b_j) \tag{1.1}$$

together with a set of measurements

$$y = h(x). \tag{1.2}$$

The vector $x \in \mathbb{R}^n$ represents the concentration of each species involved in the reactions and thus we will be interested only in those trajectories that evolve in the positive orthant $\mathbb{R}^n_{>0} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, \ \forall\, i\}$. The matrix $A = (a_{ij})$ is the matrix of the reaction rate constants, whose entries are all nonnegative (without loss of generality, we assume that its diagonal entries are zero), and which is assumed to be irreducible. The matrix $B = (b_1, \ldots, b_m)$, with $b_j = (b_{1j}, \ldots, b_{nj})$, is the matrix whose columns represent the several complexes involved in the reactions; all its entries are nonegative integers. It is assumed that B has full rank m, and that none of its rows vanishes completely.

The output map $h : \mathbb{R}^n \to \mathbb{R}^p$ is assumed to be of the form

$$h(x) = \begin{pmatrix} x_1^{c_{11}} x_2^{c_{12}} \cdots x_n^{c_{1n}} \\ \vdots \\ x_1^{c_{p1}} x_2^{c_{p2}} \ldots x_n^{c_{pn}} \end{pmatrix}, \tag{1.3}$$

where the matrix of exponents C is allowed to have entries either $c_{ij} = 0$ or $c_{ij} \geq 1$, so that $h(x)$ is a locally Lipschitz function.

In the paper [2], a necessary and sufficient condition for detectability of system (1.1), with outputs (1.2) of the form just described, was stated and proved. Under this condition, it was also shown that system (1.1) admits a *global observer*, of the Luenberger type

$$\dot{z} = f(z) + C'(h(x) - h(z)), \tag{1.4}$$

such that, $|z(t) - x(t)| \to 0$ for any initial conditions $z(0) \in \mathbb{R}^n_{\geq 0}$ of (1.4) and $x(0) \in \mathbb{R}^n_{>0}$ of (1.1).

We would like to investigate the performance of the observer (1.4) under small perturbations of the matrix A, that is, suppose that the observer is constructed, not with the "real" A, but instead with some "ideal" value A_0. Will the observer still produce a reasonable estimate of the state $x(t)$? Our goal is to establish that, provided the difference $\|A - A_0\|_{\mathrm{F}}$ is small, the difference $|z(t) - x(t)|$ will also be small, and to provide error estimates.

Throughout this paper we will consider the matrix B to be fixed and allow only A to vary. This is reasonable, since a change in B would mean a change in the complexes involved in the reaction network: this would be a more profound change in the system and a very different problem to analyse.

Important results for the system

An important object associated with this system is its stoichiometric space, given by

$$\mathcal{D} = \text{ span } \{b_i - b_j : i, j = 1, \ldots, m\}.$$

Define the *classes* of the system (1.1) by:

$$\mathcal{S} = (p + \mathcal{D}) \cap \mathbb{R}^n_{\geq 0} = \{p + d : d \in \mathcal{D}\} \cap \mathbb{R}^n_{\geq 0}, \quad p \in \mathbb{R}^n_{\geq 0}.$$

Note that the classes $\mathcal{S}$ depend on the matrix B but not on A, and that, given an initial condition $x_0 \in \mathbb{R}^n_{\geq 0}$, the system (1.1) will stay for all t in the class $\mathcal{S} = (x_0 + \mathcal{D}) \cap \mathbb{R}^n_{\geq 0}$. If $\mathcal{S} \cap \mathbb{R}^n_{>0} \neq \emptyset$, $\mathcal{S}$ is said to be a *positive class.*

Another important object associated with (1.1) is the set of equilibria:

$$E_A := \{x \in \mathbb{R}^n_{\geq 0} : f_A(x) = 0\}.$$

The set of strictly positive equilibria is denoted $E_{A,+} = E_A \cap \mathbb{R}^n_{>0}$. In the work of Feinberg and Horn & Jackson, it has been proved that in each positive class $\mathcal{S}$ there exists a unique positive equilibrium, $\{\bar{x}\} = \mathcal{S} \cap E_{A,+}$, and that $\bar{x}$ is asymptotically stable relative to $\mathcal{S}$. For this reason, it will be useful to identify the positive classes by their positive equilibrium, and thus give another representation of each positive class:

$$\mathcal{S}_{\bar{x}} = \{x \in \mathbb{R}^n_{\geq 0} : \langle v_i, x\rangle = \langle v_i, \bar{x}\rangle, i = 1, \ldots, n - m + 1\}$$

where $\{v_1, \ldots, v_{n-m+1}\}$ forms a basis of $\mathcal{D}^\perp$, since $\dim \mathcal{D}^\perp = n - \dim \mathcal{D} = n - m + 1$.

The set of elements of E_A which have at least one coordinate equal to zero (the *boundary equilibria*) will be denoted by E_0. It is interesting to note that the boundary equilibria do not depend on the matrix A, but only on the matrix B. Indeed, Lemma VI.2 in [7] gives the following characterization

$$x \in E_0 \;\Leftrightarrow\; x_1^{b_{1j}} x_2^{b_{2j}} \ldots x_n^{b_{nj}} = 0, \quad \forall\, j = 1, \ldots, m$$

The zero-deficiency chemical reaction networks are also described by Sontag in the paper [7] which summarizes the stability properties of system (1.1) and also gives further results in the context of control theory. In particular, it is shown that, if no boundary equilibria exist in a positive class (i.e., $S_{\bar{x}} \cap E_0 = \emptyset$, for $\bar{x} \in E_{A,+}$), then the unique positive equilibrium, $\bar{x}$, is in fact globally asymptotically stable relative to $\mathcal{S}_{\bar{x}}$. From now on, we will assume that (1.1) satisfies this *no boundary equilibria* assumption. (From a physical point of view, this assumption is reasonable since many known chemical reactions of this type don't have boundary equilibria in the positive classes.)

2 Parameter–robust observers

The concept of "perturbation to the ideal A_0" is next discussed. Given a chemical network, characterized by B and A_0, each nonzero entry of A_0 stands for an existing reaction between two of the complexes. In this paper, a perturbation of the matrix A_0 will mean a *perturbation of the nonzero entries of A_0 only.* Thus, we are not considering here any alterations to the *structure* of the existing network. Furthermore, we will also assume that any *perturbation is constant* along time, that is, we assume the true system may look like $\dot{x} = f_A(x)$, where A is constant, while the design of an observer is based on a *nominal system* $\dot{x} = f_{A_0}(x)$.

To formalize our concept of robustness, some notation is needed. Let

$$\mathcal{A}_{\geq 0} = \{A \in \mathbb{R}^{m \times m} : A \geq 0 \text{ and } (A + I)^k > 0 \text{ for some power } k\}.$$

Thus $\mathcal{A}_{\geq 0}$ is the set of irreducible $m \times m$ matrices whose entries are nonnegative. The inequality $A \geq 0$ (resp. $A > 0$), means that every entry of the matrix on the left hand side is nonegative (resp. positive). Let $\|A\|_F$ denote the matrix norm induced by the vector norm $|\cdot|$ (euclidean norm).

Each matrix $A \in \mathcal{A}_{\geq 0}$ characterizes a system of the form (1.1), so let $f_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the function defined in (1.1) computed with the entries a_{ij} of A. Let $\Sigma(A)$ denote the system $\dot{x} = f_A(x), y = h(x)$, and $x(t, x_0, A)$ be the solution of the diferential equation $\dot{x} = f_A(x)$, at time t, when the initial condition is x_0. Similarly, let $\bar{x}(x_0, A)$ denote the corresponding equilibrium, and let $E_{A,+} = \{\bar{x}(x_0, A) : x_0 \in \mathbb{R}^n_{>0}\}$.

Define also $\mathcal{E} = \bigcup_{A \in \mathcal{A}_{\geq 0}} E_{A,+}$ to be the set of all positive equilibrium points.

We will use the following notation, for any function $u : [0, +\infty) \to \mathbb{R}^n$:

$$\|u\|_T := \text{ess.sup. } \{|u(t)| : t \geq T\}$$

(if $T = 0$, we may simply drop the subscript "T").

Let us also introduce the following vector functions:

$$\vec{\rho}_n(x) = (\ln x_1, \ldots, \ln x_n)' \quad \text{and} \quad \mathrm{Exp}_n(v) = (e^{v_1}, \ldots, e^{v_n})'$$

defined on $\mathbb{R}^n_{>0}$ and on $\mathbb{R}^n$, respectively. (From now on, we will drop the subscript n, since its value is usually clear from the context.) Notice that, for $x \in \mathbb{R}^n_{>0}$,

$$\vec{\rho}(h(x)) = C\vec{\rho}(x) \;\; \text{and}\; h(x) = \mathrm{Exp}(C\vec{\rho}(x)).$$

Definition 2.1 *For any given matrix $A_0 \in \mathcal{A}_{\geq 0}$, define an A_0-type matrix to be any other element $A \in \mathcal{A}_{\geq 0}$ such that $a_{ij} = 0$ if and only if $a^0_{ij} = 0$.*

Let A_0 be a matrix in $\mathcal{A}_{\geq 0}$. A pair of compact sets K of $\mathcal{A}_{\geq 0}$ and P of $\mathcal{E}$ is said to satisfy the *property* $\mathcal{P}_0$ if the following hold:

(a) $A_0 \in K$,

(b) $\bar{x}(x_0, A_0) =: \bar{x}_0 \in P$ for some $x_0 \in \mathbb{R}^n_{>0}$, and

(c) for every $\bar{x} \in P$, $|h_i(\bar{x}_0) - h_i(\bar{x})| \leq \frac{1}{3} h_i(\bar{x}_0)$ for every $i = 1, \ldots, p$.

It is useful to define also the set

$$Q(P, K) = \{q \in \mathbb{R}^n_{>0} : \bar{x}(q, A) \in P \text{ for some } A \in K\}.$$

Definition 2.2 *A system $\dot{z} = g(z, h(x))$, evolving in a state space $\mathcal{X}$ (open set of $\mathbb{R}^n$ containing $\mathbb{R}^n_{>0}$) is a (full-state) observer for system $\Sigma(A_0)$ if, for each $x(0) \in \mathbb{R}^n_{>0}$, $z(0) \in \mathbb{R}^n_{\geq 0}$, the composite system has solutions defined for all $t > 0$, and $|z(t) - x(t)| \to 0$ as $t \to +\infty$.*

The system $\dot{z} = g(z, h(x))$ is a parameter A_0-robust observer for the system $\Sigma(A_0)$ if

(i) there exist functions $\beta = \beta_{A_0} \in \mathcal{KL}$ and $\varphi = \varphi_{A_0} \in \mathcal{K}_\infty$,

and for any pair of compact sets $K \subset \mathcal{A}_{\geq 0}$ and $P \subset \mathcal{E}$ that satisfy property $\mathcal{P}_0$

(ii) there exist functions $\tilde{\beta} = \tilde{\beta}_{K,P} \in \mathcal{KL}$ and $\zeta = \zeta_{K,P} \in \mathcal{K}$,

(iii) for each compact set $Q_0 \subset Q(P, K)$, there exists $T = T_{K, Q_0, \bar{x}_0}$

such that, for each $A \in K$ (A_0-type matrix) and $q \in Q_0$ with $\bar{x}(q, A) \in P$, the solution of the extended system

$$\begin{aligned} \dot{x} &= f_A(x), \quad x(0) = q \\ \dot{z} &= g(z, h(x)), \quad z(0) = z_0 \end{aligned}$$

satisfies

$$\begin{aligned} |z(t) - x(t, q, A)| \;\leq\; & \beta(|z(T) - \bar{x}_0|, t) + \tilde{\beta}(|q - \bar{x}(q, A)|, t) \\ & + \varphi(\|x(\cdot, q, A) - \bar{x}(q, A)\|_T) \\ & + \zeta(\|A_0 - A\|_F) + \zeta(|x_0 - q|), \end{aligned}$$

for all $t \geq T$ and all $z_0 \in \mathbb{R}^n_{\geq 0}$.

Remark 2.1 *The set of initial conditions, $Q(P,K)$, contains whole positive classes and thus can be unbounded. In particular, note that if $P \equiv \{\bar{x}_0\}$, and $K \equiv \{A_0\}$ the definition of a parameter A_0-robust observer reduces to that of a full-state observer.*

Given a system of the from (1.1) we know that $x(t,q,A)$ converges to $\bar{x}(q,A)$, so one may always choose T large enough so that the term $\varphi(\|x(\cdot,q,A)-\bar{x}(q,A)\|_T)$ becomes very small. The terms in β and $\tilde{\beta}$ also become very small, since these are $\mathcal{KL}$ functions. Eventually, the observer's estimates will be dominated only by the differences $\|A_0 - A\|_{\mathrm{F}}$ and $|x_0 - q|$.

From now on, assume that the output maps are such that one of the rows of the matrix C coincides with one of the columns (transposed) of B, for instance,

$$h_l(x) = x_1^{b_{1j}} x_2^{b_{1j}} \dots x_n^{b_{nj}} \tag{2.5}$$

for some $1 \le l \le p$ and some $1 \le j \le m$. This is simply saying that one of the measurements is one of the reaction rates of the network, which seems to be a reasonable choice. This condition together with $\mathcal{D} + \operatorname{im} C' = \mathbb{R}^n$ are sufficient to ensure detectability of the system (1.1) with outputs (1.2) (see [1, 2]). We will try to prove the following result.

Theorem 2.1 *Let the matrix B be fixed. Let $A_0 \in \mathcal{A}_{\geq 0}$ and let C be such that the system $\Sigma(A_0)$: $\dot{x} = f_{A_0}(x), y = h(x)$ is detectable and $h(x)$ satisfies (2.5). Then the system $\dot{z} = f_{A_0}(z) + C'(h(x) - h(z))$ is an A_0-robust observer for system $\Sigma(A_0)$.*

The difference between the real and ideal values of the equilibria points will be a major factor in deciding whether an observer is robust, and so whether a reasonable estimate for $x(t,q,A)$ is to be expected. In Section 3 we will show how the equilibria and trajectories of the original system behave as the parameter A varies.

Input-to-state stability estimates also play a very important role in establishing this Theorem, and are stated next.

2.1 An ISS-estimate

By an *input* $u(\cdot)$ we mean a measurable essentially bounded function $u : [0,+\infty) \to \mathbb{R}^p$, possibly restricted to take values in a set $\mathbb{U}$ of $\mathbb{R}^p$. In the papers [1, 2], we have studied the stability properties of the following system with inputs

$$\dot{z} \;=\; f_{A_0}(z) + C'(u - h(z)) := f^*_{A_0}(z,u) \tag{2.6}$$

from which the full-state observer (1.4) is obtained by letting $u(t) \equiv h(x(t,x_0,A_0))$. By profiting from the stability properties of (2.6) one would like to conclude that this observer is also A_0-robust. In other words, if the input is

$$u(t) \equiv h(x(t,x_0,A_0)) + \mathsf{error},$$

where $\texttt{error} = h(x(t,q,A)) - h(x(t,x_0,A_0))$ would be the difference between the actual and "ideal" outputs of the system, one may still obtain a reasonable estimate of $x(t,q,A)$. This is indeed true, as we will show for output maps h that satisfy (2.5).

For any fixed $\bar{x} \in E_{A_0,+}$, consider the function V (which plays the role of a Lyapunov function in [7, 2]), defined on $\mathbb{R}^n_{\geq 0}$:

$$V(z,\bar{x}) = \sum_{i=1}^{n} z_i(\ln z_i - \ln \bar{x}_i) + (\bar{x}_i - z_i) \tag{2.7}$$

V is continuous on $\mathbb{R}^n_{\geq 0}$ and differentiable on $\mathbb{R}^n_{>0}$, and satisfies the properties:

(i) For $z \in \mathbb{R}^n_{\geq 0}$, $V(z,\bar{x}) \geq 0$ and $V(z,\bar{x}) = 0 \Leftrightarrow z = \bar{x}$.

(ii) There exist two functions $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that $\nu_1(|z-\bar{x}|) \leq V(z,\bar{x}) \leq \nu_2(|z-\bar{x}|)$, for all $z \in \mathbb{R}^n_{\geq 0}$ (properness).

For any real number $0 < \theta < 1$, define the following set of inputs

$$\mathbb{U}_\theta = \{u \in \mathbb{R}^p : |u_k - h_k(\bar{x})| \leq \frac{\theta}{2} h_k(\bar{x}),\ k = 1, \dots, p\}.$$

It can be shown that V is an ISS-Lyapunov function, with input set $\mathbb{U}_\theta$, with respect to the point $\bar{x}$ and the input $h(\bar{x})$ (see [2] for the definition), for system (2.6), when the map h satisfies (2.5). By a usual argument, we can prove that the existence of an ISS-Lyapunov function implies ISS:

Proposition 2.1 *Assume that the map h is such that (2.5) holds and the system with outputs $\dot{x} = f_{A_0}(x), y = h(x)$, is detectable. Then the system with inputs $\dot{z} = f^*_{A_0}(z,u)$ is ISS with input set $\mathbb{U}_\theta$, with respect to the point $\bar{x}$ and the input $h(\bar{x})$, i.e., there exist functions $\beta \in \mathcal{KL}$ and $\varphi \in \mathcal{K}_\infty$ such that*

$$|z(t) - \bar{x}| \leq \beta(|z(0) - \bar{x}|, t) + \varphi(\|u - h(\bar{x})\|) \tag{2.8}$$

for all $z \in \mathbb{R}^n_{\geq 0}$ and all $u \in \mathbb{U}_\theta$.

3 Dependence of the system on the parameters

In this section we will consider the system $\dot{x} = f_A(x)$ and will start by studying the continuity of the equilibria $\bar{x} = \bar{x}(q,A)$ as a map from $\mathcal{A}_{\geq 0} \times \mathbb{R}^n_{>0}$ to the union of the equilibria $\mathcal{E}$, and so from now on assume that B has been fixed.

For each fixed A, the continuity with respect to initial conditions has already been established by Sontag in [7], through the following result:

Lemma 3.1 *Fix a matrix $A \in \mathcal{A}_{\geq 0}$. Then*

1. *For each q, w in $\mathbb{R}^n_{>0}$, there exists a unique $x = \varphi(q, w) \in \mathbb{R}^n_{>0}$ such that:*

$$x - q \in \mathcal{D} \quad \textit{and} \quad \vec{\rho}(x) - \vec{\rho}(w) \in \mathcal{D}^\perp.$$

   Furthermore, the map $q, w \mapsto \varphi(q, w)$ is of class C^1.

2. *For any $q \in \mathbb{R}^n_{>0}$ and any $w \in E_{A,+}$, the element $x = \varphi(q, w)$ corresponds to the (unique) positive equilibrium $\bar{x}(q, A)$.*

The differentiability of $\bar{x}(\cdot, A)$ with respect to initial conditions, follows from this Lemma by picking any equilibrium $\bar{z} \in E_{A,+}$ and observing that $\bar{x}(q, A) = \varphi(q, \bar{z})$ for all $q \in \mathbb{R}^n_{>0}$.

We will establish joint continuity, and in fact differentiability, with respect to q, A, and for that will first prove a general result which is valid for a wider class of matrices.

3.1 Irreducible matrices

Consider a matrix $G \in \mathbb{R}^{m\times m}$, $G = [g_{ij}]$, and define another matrix

$$M_G = \left(\frac{1}{1 + \sum g_{ii}^2} G + I\right)^{m-1}.$$

Also define

$$\phi(G) = \left(1 + \sum_{i=1}^m g_{ii}^2\right)^{-1},$$

so that $M_G = (\phi(G)G + I)^{m-1}$.

Introduce the following subset of $\mathbb{R}^{m\times m}$

$$\mathcal{G} = \{G \in \mathbb{R}^{m\times m} : M_G > 0 \text{ and } \vec{1}\,G = 0\},$$

where the inequality means that every entry of the matrix on the left hand side is strictly positive, and $\vec{1} \equiv [1\ 1 \ldots 1]$. Note that $\mathcal{G}$ contains all the irreducible matrices which have *nonnegative off-diagonal* entries and *arbitrary diagonal* entries. The set $\mathcal{G}$ may be seen as an open subset of the $m^2 - m$ dimensional linear subspace $\{G : \vec{1}\,G = 0\}$ of $\mathbb{R}^{m\times m}$.

To each matrix $A \in \mathcal{A}_{\geq 0}$, we associate a matrix in $\mathcal{G}$ as follows. Recall that we assumed that all the diagonal entries of A are zero (since their value does not enter in the computations of the vector field f_A), and we define

$$\tilde{A} = A + \begin{bmatrix} -\sum_{i=1}^m a_{i1} & 0 & \cdots & 0 \\ 0 & -\sum_{i=1}^m a_{i2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\sum_{i=1}^m a_{im} \end{bmatrix}.$$

Now clearly, $\vec{1}\tilde{A} = 0$ and, since it is always true that $|\tilde{a}_{ii}| \le 1 + \sum \tilde{a}_{ii}^2$, it follows that

$$\frac{1}{1 + \sum \tilde{a}_{ii}^2}\, \tilde{A} + I \ge 0.$$

Note that A is irreducible if and only if $\frac{1}{1+\sum \tilde{a}_{ii}^2}\, \tilde{A} + I$ is, so the latter is again irreducible. Thus $\tilde{A} \in \mathcal{G}$. For each $G \in \mathcal{G}$ observe that

$$\vec{1}\,(\phi(G)\, G + I)^{m-1} = \vec{1}$$

because $\vec{1}\, G = 0$ and $\vec{1}\,(\phi(G)\, G + I) = \vec{1}$. So, any nonnegative eigenvector, $v \in \mathbb{R}^n_{\ge 0}$ (but $v \ne (0, \ldots, 0)'$), of the matrix M_G must correspond to the eigenvalue $\mu = 1$ since

$$M_G v = \mu v \;\Rightarrow\; \vec{1}(M_G v) = \vec{1}(\mu v) \;\Leftrightarrow\; \vec{1}v = \mu \vec{1} v,$$

and $\vec{1}v$ is a positive scalar.

Since, by definition, M_G is irreducible and has all entries positive, by the Perron-Frobenius theorem we know that the spectral radius of M_G, $\sigma(M_G)$, is an eigenvalue of M_G, of algebraic (and hence geometric) multiplicity one. Moreover, an eigenvector associated to $\sigma(M_G)$ can be chosen to have all entries strictly positive. This eigenvector is usually called a Perron eigenvector of M_G. But, as we have just seen, any positive eigenvector of M_G corresponds to the eigenvalue $\mu = 1$, i.e.,

$$\sigma(M_G) \equiv 1, \quad \forall\, G \in \mathcal{G}.$$

Define $v_{\mathrm{P}} : \mathcal{G} \to \mathbb{R}^m_{>0}$ to be the map that assigns to each $G \in \mathcal{G}$, the unique Perron eigenvector of M_G, which has its first coordinate equal to 1.

By a *rational function everywhere defined on* $\mathcal{G}$ we mean a function $\psi : \mathcal{G} \to \mathbb{R}$ which is a quotient $\psi = p_{\mathrm{num}}/p_{\mathrm{den}}$ of two polynomial functions $p_{\mathrm{num}}, p_{\mathrm{den}} : \mathbb{R}^{m\times m} \to \mathbb{R}$ such that $p_{\mathrm{den}}(G) \ne 0$ for all $G \in \mathcal{G}$.

Proposition 3.1 *The map v_{P} is a rational function on $\mathcal{G}$. In particular, v_{P} is C^1 on $\mathcal{G}$.*

Proof. For each $G \in \mathcal{G}$, by abuse of notation, write v_{P} for $v_{\mathrm{P}}(G)$. We will also drop the subscript and let $M = M_G$, for simplicity. We have $Mv_{\mathrm{P}} = \sigma(M)v_{\mathrm{P}}$ or, equivalently, $(M - I)v_{\mathrm{P}} = 0$. The matrix $M - I$ has rank $m - 1$ because $\sigma(M) = 1$ is a simple root of the characteristic polynomial of M.

Put $M - I = [N_1\ N]$ where N_1 is the first column of $M - I$ and N is the remaining $m \times (m-1)$ matrix, put $v_{\mathrm{P}} = \begin{pmatrix} 1 \\ w_{\mathrm{P}} \end{pmatrix}$, and notice that

$$N w_{\mathrm{P}} = -N_1.$$

Claim. N has rank $m - 1$.

Suppose the claim is false. Then there exists an element u in the kernel of N, and one could write $N(w_{\mathrm{P}}+u) = -N_1$. But if this is true, then it also holds that

$$(M-I)\begin{pmatrix} 1 \\ w_{\mathrm{P}}+u \end{pmatrix} = 0$$

which implies $w_{\mathrm{P}}+u = w_{\mathrm{P}}$, because v_{P} is in fact the unique vector with first coordinate 1 in the kernel of $M-I$. So $u \equiv 0$, which proves the claim.

Applying the Moore-Penrose pseudoinverse of N yields

$$v_{\mathrm{P}} = \begin{pmatrix} 1 \\ w_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} 1 \\ -(N'N)^{-1}N'N_1 \end{pmatrix}.$$

(Note that $\det(N'N) \neq 0$ for every G.) This shows that v_{P} is a rational function on $\mathcal{G}$.

The Perron eigenvector of M_G, defined above for each $G \in \mathcal{G}$, is also an eigenvector of the matrix G, corresponding to the 0 eigenvalue.

Indeed, let the equation $\phi(G)\,G\,v = \lambda\,v$ define an eigenvalue of $\phi(G)\,G$ and its corresponding eigenvector. Then

$$(\phi(G)\,G + I)v = (\lambda+1)v \;\Rightarrow\; (\phi(G)\,G+I)^{m-1}v = (\lambda+1)^{m-1}v$$

So, the pair v, λ given by $v = v_{\mathrm{P}}$ and $(\lambda+1)^{m-1} = \sigma(M_G) = 1$ satisfies the eigenvalue equation. Moreover, since $\vec{1}(\phi(G)\,G+I)v_{\mathrm{P}} = \vec{1}v_{\mathrm{P}}$ is a positive scalar, it must hold that $(\lambda+1)\vec{1}v_{\mathrm{P}}$ is also a positive scalar, implying that $\lambda+1$ (hence λ) is a real number. The only real number that satisfies $(\lambda+1)^{m-1} = 1$ is $\lambda = 0$.

Furthermore, since $\phi(G) > 0$, we have

$$Gv_{\mathrm{P}} = 0, \quad \forall\; G \in \mathcal{G}.$$

And since $\sigma(M)$ has multiplicity one, the kernel of G has dimension 1 and is given by:

$$\ker(G) = \;\mathrm{span}\;\{v_{\mathrm{P}}(G)\}.$$

3.2 Continuity of equilibrium points

For each fixed $A \in \mathcal{A}_{\geq 0}$, note that system (1.1) can be written in yet another form (see [7]):

$$f_A(x) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}x_1^{b_{1j}}x_2^{b_{2j}}\ldots x_n^{b_{nj}}(b_i-b_j) = B\tilde{A}\theta_B(x) \tag{3.9}$$

where $\tilde{A} = A - \;\mathrm{diag}\;(\sum a_{i1}, \sum a_{i2}, \ldots, \sum a_{im})$. It is clear that $\tilde{A}$ is also an irreducible matrix and, in particular, $\tilde{A} \in \mathcal{G}$. The function $\theta_B : \mathbb{R}^n_{>0} \to \mathbb{R}^m_{\geq 0}$ is given by

$$\theta_B(x) = \mathrm{Exp}(B'\vec{\rho}(x)) = (e^{\langle b_1, \vec{\rho}(x)\rangle}, e^{\langle b_2, \vec{\rho}(x)\rangle}, \ldots, e^{\langle b_m, \vec{\rho}(x)\rangle})'.$$

This form provides an alternative characterization of the equilibria which follows from the fact that B' is an onto map from $\mathbb{R}^n$ to $\mathbb{R}^m$ (see Lemma V.1 of [7]):
Fact. The point $\bar{x} \in \mathbb{R}^n_{>0}$ is an equilibrium for (1.1) if and only if $\theta_B(\bar{x}) \in \ker\tilde{A}$.

Proposition 3.2 *The map $\mathbb{R}^n_{>0} \times \mathcal{A}_{\geq 0} \to \mathcal{E} \subset \mathbb{R}^n_{>0}$ given by $(x_0, A) \mapsto \bar{x}(x_0, A)$ is real-analytic, in particular, continuously differentiable.*

Proof. For each A consider the matrix $\tilde{A}$, constructed from A as indicated above. Then by the arguments above, $\ker \tilde{A} =$ span $\{v_{\mathrm{P}}(\tilde{A})\}$.

The fact then says that each equilibrium $\bar{x} \in E_{A,+}$ is characterized by

$$\theta_B(\bar{x}) = c\, v_{\mathrm{P}} \quad \Leftrightarrow \quad B' \vec{\rho}(\bar{x}) = \vec{\rho}(c\, v_{\mathrm{P}}),$$

where c is a positive constant.

Therefore, for each A, there exists some equilibrium point, $\bar{z}(A)$ given by

$$\vec{\rho}(\bar{z}(A)) = B(B'B)^{-1} \vec{\rho}(v_{\mathrm{P}}(\tilde{A})).$$

(Since B' has full column rank, this formula gives $B'\vec{\rho}(\bar{z}(A)) = B'B(B'B)^{-1}\vec{\rho}(v_{\mathrm{P}}(\tilde{A})) = \vec{\rho}(v_{\mathrm{P}}(\tilde{A}))$.) Now, by Proposition 3.1, the map v_{P} is rational on $\mathcal{G}$; the entries of $\tilde{A}$ are linear combinations of the entries of A; the functions $\mathrm{Exp}(\cdot)$ and $\vec{\rho}(\cdot)$ are analytic on $\mathbb{R}^n$ and $\mathbb{R}^n_{>0}$, respectively, so it follows that the map from $\mathcal{A}_{\geq 0}$ to $\mathcal{E}$ given by $\bar{z}(A) :=\mathrm{Exp}(B(B'B)^{-1}\vec{\rho}(v_{\mathrm{P}}(\tilde{A}))$ is also an analytic map.

Now let $q = x_0$ be any initial condition in $\mathbb{R}^n_{>0}$ and set $w = \bar{z}(A)$ in Lemma 3.1. It follows that $\varphi(q, w) \equiv \bar{x}(x_0, A)$, so we may we conclude that $\bar{x} : \mathbb{R}^n_{>0} \times \mathcal{A}_{\geq 0} \to \mathcal{E}$, given by $\varphi(\bar{z}(A), x_0)$ is a real analytic map.

Using the modulus of continuity one can show:

Corollary 3.1 *For each pair of compact sets $P \subset \mathcal{E}$ and $K \subset \mathcal{A}_{\geq 0}$, there exists a function ζ of class $\mathcal{K}$ such that*

$$|\bar{x}(x_0, A_0) - \bar{x}(q, A)| \leq \zeta(|x_0 - q|) + \zeta(\|A_0 - A\|_F)$$

for all $x_0, q \in Q(P, K)$ and all $A_0, A \in K$.

4 Uniform bounds

For a fixed matrix A, the convergence of system (1.1) was proved using the Lyapunov function (2.7). The function $V(\cdot, \cdot)$ is continuous when seen as a function $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{>0} \to \mathbb{R}$, and is continuously differentiable on $\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0}$.

In principle, the $\mathcal{K}_\infty$ functions ν_1, ν_2, that provide lower and upper bounds for V, depend on the point $\bar{x}$. However, for each compact set $P \subset \mathcal{E}$, one can find two class $\mathcal{K}_\infty$ functions $\nu_1 = \nu_{1,P}, \nu_2 = \nu_{2,P}$ such that property (ii) of V is satisfied for every $\bar{x} \in P$. Consider

$$\nu_1(r) = \inf\{V(x) : |x - \bar{x}| \geq r,\ x \in \mathbb{R}^n_{\geq 0},\ \bar{x} \in P\}$$

and

$$\nu_2(r) = r + \max\{V(x) : |x - \bar{x}| \leq r,\ x \in \mathbb{R}^n_{\geq 0},\ \bar{x} \in P\}.$$

Introduce the notation

$$\pi_j(x,\bar{x}) = \pi_j := \left[\frac{x_1}{\bar{x}_1}\right]^{b_{1j}} \left[\frac{x_2}{\bar{x}_2}\right]^{b_{2j}} \cdots \left[\frac{x_n}{\bar{x}_n}\right]^{b_{nj}},$$

$$q_j(x,\bar{x}) = q_j := \langle b_j, \rho(x) - \rho(\bar{x})\rangle,$$

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)' \in \mathcal{E}$ and π_j is defined for $x \in \mathbb{R}^n_{\geq 0}$ and q_j is defined for $x \in \mathbb{R}^n_{>0}$. Observe that $\pi_j = e^{q_j}$. Define the function $\mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{>0} \to \mathbb{R}_{\geq 0}$

$$\Psi(x,\bar{x}) := \sum_{i=1}^{m}\sum_{j=1}^{m} \left(e^{-\pi_i} - e^{-\pi_j}\right)^2.$$

Lemma 4.1 *(Lemma 2.8 in [2]) If $\bar{x} \in E_{A,+}$, then for all $x \in \mathbb{R}^n_{\geq 0}$:*

$$\Psi(x,\bar{x}) = 0 \;\Leftrightarrow\; x \in E_0 \cup E_{A,+}.$$

For a fixed $A \in \mathcal{A}_{\geq 0}$ and any element $\bar{x} \in E_{A,+}$, it is proved in Lemma 2.10 in [2] that

$$\begin{aligned} \nabla V(x,\bar{x})\; f_A(x) &= -\textstyle\sum_{i=1}^m \sum_{j=1}^m a_{ij}\, e^{\langle b_j, \rho(\bar{x})\rangle}\, e^{q_j}\; \left|q_i - q_j - (e^{q_i - q_j} - 1)\right| \\ &\leq -\tfrac{1}{2}\textstyle\sum_{i=1}^m \sum_{j=1}^m a_{ij}\, e^{\langle b_j, \rho(\bar{x})\rangle}\, \left(e^{-\pi_i} - e^{-\pi_j}\right)^2. \end{aligned}$$

Suppose $A_1 \in \mathcal{A}_{\geq 0}$ is a matrix with entries $a^1_{ij} = 1$ if $a_{ij} > 0$ and $a^1_{ij} = 0$ if $a_{ij} = 0$. Then we can write:

$$\nabla V(x,\bar{x})\; f_A(x) \leq -\frac{1}{2}\min_{a_{ij}>0}\{a_{ij}\}\; \min_j e^{\langle b_j, \rho(\bar{x})\rangle} \;\sum_{i=1}^{m}\sum_{j=1}^{m} a^1_{ij}\; \left(e^{-\pi_i} - e^{-\pi_j}\right)^2.$$

Since A_1 is irreducible, we may apply a result on quadratic forms given in Lemma VIII.1 in [7] to conclude that there exists a positive constant k such that

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a^1_{ij}\; \left(e^{-\pi_i} - e^{-\pi_j}\right)^2 \geq k\; \sum_{i=1}^{m}\sum_{j=1}^{m} \left(e^{-\pi_i} - e^{-\pi_j}\right)^2.$$

Thus we have:

Lemma 4.2 *(Lemma 2.10 in [2], adapted) There exists a positive constant k, a continuous function $c : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a function $\kappa : \mathcal{A}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by*

$$c(\xi) = \frac{1}{2}\min_j e^{\langle b_j, \rho(\xi)\rangle} \quad \textit{and} \quad \kappa(A) = k\;\min\{a_{ij} : a_{ij} \neq 0,\; i,j = 1,\ldots,m\}$$

such that, given any matrix $A \in \mathcal{A}_{\geq 0}$ and any element $\bar{x} \in E_{A,+}$:

$$\nabla V(x,\bar{x})\; f_A(x) \leq -\kappa(A)\, c(\bar{x})\, \Psi(x,\bar{x}), \tag{4.10}$$

for all $x \in \mathbb{R}^n_{>0}$.

Lemma 4.3 *Let $P \subset \mathcal{E}$ and $K \subset \mathcal{A}_{\geq 0}$ be compact sets. Then there exists a continuous positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\alpha = \alpha_{P,K}$, such that, given any pair $\bar{x} \in P$, $A \in K$ with $\bar{x} \in E_{A,+}$,*

$$\nabla V(x, \bar{x})\, f_A(x) \leq -\alpha(V(x, \bar{x}))$$

for all $x \in \mathbb{R}^n_{>0} \cap \mathcal{S}_{\bar{x}}$.

Proof. Let $c : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\kappa : \mathcal{A}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the functions given in Lemma 4.2. We will show that the following function, defined from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ is positive definite:

$$\alpha(r) = \inf\{\kappa(A)\, c(\bar{x})\, \Psi(x, \bar{x}) : A \in K, \bar{x} \in P, x \in \mathcal{C}_r \cap \mathcal{S}_{\bar{x}}\}$$

where

$$\mathcal{C}_r = \{x \in \mathbb{R}^n_{\geq 0} : V(x, \bar{x}) = r, \bar{x} \in P\}.$$

First, we show that $\mathcal{C}_r$ is a compact subset of $\mathbb{R}^n_{\geq 0}$. (a) Closed: let x^k be a sequence in $\mathcal{C}_r$ converging to a point $x \in \mathbb{R}^n_{\geq 0}$. We must show that $x \in \mathcal{C}_r$. For each x^k there exist $\bar{x}^k \in P$ with $V(x^k, \bar{x}^k) = r$. Since P is compact, the sequence $\{\bar{x}^k\}$ has a converging subsequence, $\bar{x}^{k_l} \to \bar{x}$. By continuity of V, $V(x^{k_l}, \bar{x}^{k_l}) \to V(x, \bar{x})$, hence $V(x, \bar{x}) = r$. So $x \in \mathcal{C}_r$ as wanted. (b) Bounded: let $x \in \mathcal{C}_r$. Then $\nu_{1,P}(|x - \bar{x}|) \leq r$ for all $\bar{x} \in P$, which implies $|x| \leq \nu_{1,P}^{-1}(r) + |\bar{x}|$. So, with $M = \max\{|\bar{x}| : \bar{x} \in P\}$, $\mathcal{C}_r$ is contained in the closed ball of radius $\nu_{1,P}^{-1}(r) + M$ centered at the origin.

Next show that the set

$$\tilde{\mathcal{C}}_r = \{(x, \bar{x}) : \bar{x} \in P, x \in \mathcal{C}_r \cap \mathcal{S}_{\bar{x}}\}$$

is also compact: since it is clearly a subset of the compact set $\mathcal{C}_r \times P$, it is enough to show that $\tilde{\mathcal{C}}_r$ is closed. Given a sequence of points $(x^k, \bar{x}^k) \in \tilde{\mathcal{C}}_r$ converging to a point $(x, \bar{x}) \in \mathcal{C}_r \times P$, we need to show that also $x \in \mathcal{S}_{\bar{x}}$. But, by definition of a class we have

$$\langle v_i, x^k - \bar{x}^k \rangle = 0, \quad \forall\, i = 1, \ldots, n - m + 1,$$

and by continuity it follows that $\langle v_i, x - \bar{x} \rangle = 0$ for all such i, that is, x belongs to the class $\mathcal{S}_{\bar{x}}$, as wanted.

It is clear $r = 0$ implies $\tilde{\mathcal{C}}_r = \{(\bar{x}, \bar{x}) : \bar{x} \in P\}$, so $\alpha(0) = 0$. The functions $\kappa(\cdot)$ and $c(\cdot)$ will have strictly positive minimum values on the compact sets K and P, respectively.

Now, we take any $r > 0$ and show that $\inf\{\Psi(x, \bar{x}) : (x, \bar{x}) \in \tilde{\mathcal{C}}_r\}$ is positive. To get a contradiction, assume that this infimum is zero for some $r > 0$. Then there exists an infinite sequence $(x^k, \bar{x}^k)$ such that $\Psi(x^k, \bar{x}^k) \to 0$. Since $\tilde{\mathcal{C}}_r$ is compact, there exists a comverging subsequence: $(x^{k_l}, \bar{x}^{k_l}) \to (x_0, \bar{x}_0) \in \tilde{\mathcal{C}}_r$. Then $\Psi(x_0, \bar{x}_0) = 0$ and by Lemma 4.1, $x_0 \in E_0 \cup E_{A_0,+}$ for some $A_0 \in \mathcal{A}_{\geq 0}$. But, under the no boundary equilibrium assumption, $E_0 \cap \mathcal{S}_{\bar{x}} = \emptyset$ for all $\bar{x}$. So, by uniqueness of the positive equilibrium in each class, $x_0 = \bar{x}_0$ which implies $r = V(x_0, \bar{x}_0) = 0$ and contradicts $r > 0$. Thus, $\alpha(r) > 0$ whenever $r > 0$.

Finally, by construction, α satisfies: $\kappa(A)\, c(\bar{x})\, \Psi(x,\bar{x}) \geq \alpha(V(x,\bar{x}))$ for all $\bar{x} \in P$, all $A \in K$ and all $x \in \mathcal{S}_{\bar{x}}$, and in particular, for all $x \in \mathbb{R}^n_{>0} \cap \mathcal{S}_{\bar{x}}$. Without loss of generality we may assume that α is continuous – otherwise, one can always construct another positive definite function $\tilde{\alpha}$, continuous, and satisfying $\alpha(r) \geq \tilde{\alpha}(r)$, for all r. This finishes the proof.

Corollary 4.1 *Given any compact sets $P \subset \mathcal{E}$ and $K \subset \mathcal{A}_{\geq 0}$, there exists a function $\tilde{\beta} = \tilde{\beta}_{P,K}$ of class $\mathcal{KL}$ such that for any pair $\bar{x} \in P$, $A \in K$ and all $q \in \mathbb{R}^n_{>0}$ with $\bar{x} = \bar{x}(q,A)$*

$$|x(t,q,A) - \bar{x}(q,A)| \leq \tilde{\beta}(|q - \bar{x}(q,A)|\,, t),$$

for all $t \geq 0$.

Proof. Pick any compact sets $P \subset \mathcal{E}$ and $K \subset \mathcal{A}_{\geq 0}$ and let α be the positive definite function given by Lemma 4.3. Consider the initial value problem

$$\dot{y} \leq -\alpha(y), \quad y(0) = y_0.$$

where $y_0 \in \mathbb{R}^n_{\geq 0}$. Then by a comparison result such as Lemma 4.4 of [6], there exists a function $\beta = \beta_\alpha$ of class $\mathcal{KL}$ such that $y(t) \leq \beta(y_0, t)$, for all $t \geq 0$. Using the functions $\nu_1 = \nu_{1,P}$, $\nu_2 = \nu_{2,P} \in \mathcal{K}_\infty$ define $\tilde{\beta}(r,t) = \nu_1^{-1}(\beta(\nu_2(r),t))$ which is again a $\mathcal{KL}$ function and depends only on P and K.

Now, pick any $\bar{x} \in P$ and $A \in K$. Let $q \in \mathbb{R}^n_{>0}$ be such that $\bar{x} = \bar{x}(q,A)$. Recall that $x(t,q,A)$ is the unique solution of the initial value problem $\dot{x} = f_A(x)$, $x(0) = q$, and we know that $x(t,q,A) \in \mathcal{S}_{\bar{x}}$ for all $t \geq 0$.

Define $y(t) := V(x(t,q,A), \bar{x}(q,A))$. From Lemma 4.3 the function $y(t)$ satisfies, for all $t \geq 0$, $\dot{y} \leq -\alpha(y)$, $y(0) = V(q, \bar{x}(q,A))$. Therefore, recalling property (ii) of V, we have

$$\nu_1(|x(t,q,A) - \bar{x}(q,A)|) \leq V(x(t,q,A), \bar{x}(q,A)) \leq \beta(\nu_2(|q - \bar{x}(q,A)|), t)$$

for all $t \geq 0$. This finishes the proof of the Corollary.

Lemma 4.4 *Let $s_0 > 0$ be any real number. Given any compact sets $P \subset \mathcal{E}$ and $K \subset \mathcal{A}_{\geq 0}$, and given any compact set of initial conditions $Q_0 \subset Q(P,K)$, there exists $T = T_{K,Q_0,s_0} > 0$ such that, for every $\bar{x} \in P$, $A \in K$ and $q \in Q_0$ with $\bar{x} = \bar{x}(q,A)$, the following hold*

$$\begin{aligned} |x(t,q,A) - \bar{x}(q,A)| &\leq 1 \\ |h_i(x(t,q,A) - h_i(\bar{x}(q,A)| &\leq s_0, \end{aligned}$$

for all $t \geq T$ and all $i = 1, \ldots, p$.

Proof. Let $P \subset \mathcal{E}$, $K \subset \mathcal{A}_{\geq 0}$ and $Q_0 \subset Q(P,K)$ be given compact sets. Let $\tilde{\beta}$ be the $\mathcal{KL}$ function given in Corollary 4.1. Put

$$M = \max_{q \in Q_0, A \in K} |q - \bar{x}(q,A)|\,,$$

and let $T_1 = T_1(M)$ be so that $\tilde{\beta}(M,t) \leq 1$ for all $t \geq T_1$. Consider the closed ball

$$\mathcal{B}_1 = \{x \in \mathbb{R}^n : |x| \leq 1 + \max_{\bar{x}\in P} |\bar{x}| \} \tag{4.11}$$

and let c_1 be a Lipschitz constant for the function h in the set $\mathcal{B}_1$. Next, pick $T = T_{K,Q_0,s_0} \geq T_1$, so that $\tilde{\beta}(M,t) \leq \frac{s_0}{c_1}$, for all $t \geq T$.

Now, pick any $\bar{x} \in P$, $A \in K$ and $q \in Q_0$ with $\bar{x} = \bar{x}(q,A)$. For all $t \geq T_1$, from Corollary 4.1 it follows that $|x(t,q,A) - \bar{x}(q,A)| \leq \tilde{\beta}(M,t) \leq 1$ and so $x(t,q,A) \in \mathcal{B}_1$ for all $t \geq T_1$.

We then have, using the fact that h is Lipschitz on $\mathcal{B}_1$,

$$|h_i(x(t,q,A)) - h_i(\bar{x}(q,A))| \leq c_1 |x(t,q,A) - \bar{x}(q,A)| \leq c_1\tilde{\beta}(M,t) \leq s_0$$

for all $t \geq T$, which finishes the proof.

5 Proof of Theorem 2.1

Given a chemical network characterized by matrices B as in Section 1 and $A_0 \in \mathcal{A}_{\geq 0}$, let C be such that the system $\Sigma(A_0)$: $\dot{x} = f_{A_0}(x)$, $y = h(x)$ is detectable. We will next show that $\dot{z} = f_{A_0}(z) + C'(h(x) - h(z))$ provides an A_0-robust observer for $\Sigma(A_0)$.

Consider the system $\dot{z} = f_{A_0}(z) + C'(u - h(z))$ and let $\beta = \beta_{A_0} \in \mathcal{KL}$ and $\varphi = \varphi_{A_0} \in \mathcal{K}_\infty$ be as in Proposition 2.1.

Pick any compact sets $P \subset \mathcal{E}$ and $K \subset \mathcal{A}_{\geq 0}$ that have property $\mathcal{P}_0$, so $A_0 \in K$ and $\bar{x}_0 = \bar{x}(x_0, A)$ for some $x_0 \in \mathbb{R}^n_{>0}$. Let $\tilde{\beta} = \tilde{\beta}_{P,K} \in \mathcal{KL}$ be as in Corollary 4.1, and $\zeta = \zeta_{P,K} \in \mathcal{K}$ as in Corollary 3.1. Pick any compact subset $Q_0 \subset Q(P,K)$ and let $T = T_{K,Q_0,\bar{x}_0}$ be the number given by Lemma 4.4, when $s_0 = \frac{1}{7}\min_{1\leq i\leq p} h(\bar{x}_0)$.

Pick any $A \in K$ (A_0-type) and any $q \in Q_0$ with $\bar{x}(q,A) \in P$. Consider the extended system $\dot{x} = f_A(x)$, $\dot{z} = f_{A_0}(z) + C'(h(x) - h(z))$ and let $(x(t,q,A), z(t))$ be its solution at time t, with initial condition $(q, z(0))$. Lemma 4.4 (with $s_0 = \frac{1}{7}\min_{1\leq i\leq p} h(\bar{x}_0)$), together with part (c) of property $\mathcal{P}_0$ and triangle inequality, imply

$$|h_i(x(t,q,A) - h_i(\bar{x}_0)| \leq \frac{10}{21} h_i(\bar{x}_0)$$

for all $t \geq T$, and all $i = 1, \ldots, p$. Let $\theta = 20/21$ and observe that $h(x(t,q,A)) \in \mathbb{U}_\theta$ for all $t \geq T$. Applying Proposition 2.1 with $u(t) \equiv h(x(t+T,q,A))$ and $z(T)$ as initial condition, yields

$$|z(t) - \bar{x}_0| \leq \beta(|z(T) - \bar{x}_0|, t) + \varphi(\|h(x(\cdot,q,A)) - h(\bar{x}_0)\|_T)$$

for all $t \geq T$.

Now, using the triangle inequality

$$|z(t) - x(t,q,A)| \leq |z(t) - \bar{x}_0| + |\bar{x}_0 - \bar{x}(q,A)| + |\bar{x}(q,A) - x(t,q,A)|$$

together with the ISS estimate, Corollary 3.1 and Corollary 4.1 obtain

$$\begin{aligned}|z(t) - x(t, q, A)| &\leq \beta(|z(T) - \bar{x}_0|, t) + \tilde{\beta}(|q - \bar{x}(q, A)|, t) + \varphi(\|h(x(\cdot, q, A)) - h(\bar{x}_0)\|_T) \\ &\quad + \zeta(\|A_0 - A\|_{\mathrm{F}}) + \zeta(|x_0 - q|)\end{aligned}$$

for all $t \geq T$.

Let c_1 be a Lipschitz constant of the function h on the closed ball $\mathcal{B}_1$ (defined in (4.11). Note that $x(t, q, A) \in \mathcal{B}_1$ for $t \geq T$ and also $P \subset \mathcal{B}_1$. Then, since φ is $\mathcal{K}_\infty$, we have

$$\begin{aligned}\varphi(\|h(x(\cdot, q, A)) - h(\bar{x}_0)\|_T) &\leq \varphi(c_1 \|x(\cdot, q, A) - \bar{x}_0\|_T) \\ &\leq \varphi(2c_1 \|x(\cdot, q, A) - \bar{x}(q, A)\|_T) + \varphi(2c_1 \|\bar{x}(q, A) - \bar{x}_0\|_T).\end{aligned}$$

Using Corollary 3.1 once more,

$$\varphi(2c_1 \|\bar{x}(q, A) - \bar{x}_0\|_T) \leq \varphi(4c_1 \, \zeta(|x_0 - q|)) + \varphi(4c_1 \, \zeta(\|A_0 - A\|_{\mathrm{F}}))$$

and renaming the functions $\varphi(2c_1 r) \to \varphi(r)$ and $\varphi(4c_1\zeta(r)) + \zeta(r) \to \zeta(r)$ (note that $\varphi(4c_1\zeta(\cdot))$ is still of class $\mathcal{K}$ and depends on K and P), the desired estimate for the difference $|z(t) - x(t, q, A)|$ follows.

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