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Robust Heteroclinic Cycles

M. Krupa
Institut für Angewandte und Numerische Mathematik, TU Wien, Wiedner Hauptstrasse 8-10/115/1, A-1040 Wien, Austria

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Summary. One phenomenon in the dynamics of differential equations which does not typically occur in systems without symmetry is heteroclinic cycles. In symmetric systems, cycles can be robust for symmetry-preserving perturbations and stable. Cycles have been observed in a number of simulations and experiments, for example in rotating convection between two plates and for turbulent flows in a boundary layer. Theoretically the existence of robust cycles has been proved in the unfoldings of some low codimension bifurcations and in the context of forced symmetry breaking from a larger to a smaller symmetry group. In this article we review the theoretical and the applied research on robust cycles.

Key words. heteroclinic cycles, robust, symmetry, stability, bifurcation, simulation, experiment

MSC numbers. 58F12, 58F14, 34-02, 34C37

PAC numbers. 02.30.Hq, 47.27.Ky, 47.27.Te, 47.27.Nz

1. Introduction

A *heteroclinic cycle* is a sequence of trajectories connecting a set of fixed points in a topological circle. The special case of a cycle consisting of one trajectory and one fixed point is usually called a *homoclinic* trajectory. Homoclinic trajectories are typically phenomena of codimension at least one while more complicated heteroclinic cycles typically are formed at singularities of higher codimension. Remarkably, however, differential equations with certain structure (for example symmetric systems or population models) may have heteroclinic cycles which are robust with respect to structure-preserving perturbations.

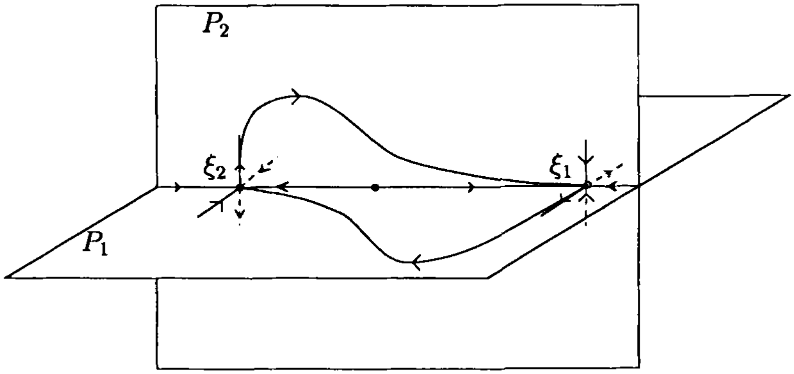

Fig. 1. An example of a robust heteroclinic cycle.

The existence of a *robust heteroclinic cycle* as considered in this article relies on the existence of a sequence of flow invariant spaces $P_1, \ldots, P_k$. The equilibria forming the vertices of the cycle can be numbered as $\xi_1, \ldots, \xi_k$ so that $\xi_j,\ \xi_{j+1} \in P_j$, $j = 1, \ldots, k-1$ and $\xi_k,\ \xi_1 \in P_1$. In other words every pair of consecutive equilibria lies in an invariant subspace. We further require that for the flow restricted to each of the invariant subspaces the consecutive equilibria are a saddle and a sink and are joined by a saddle-sink connection. An example is given in Figure 1. By allowing ξ_j to be more complicated invariant sets, we obtain a natural extension of this definition. Since saddle-sink connections cannot be broken by small perturbations it follows that every sufficiently small perturbation preserving the invariance of the spaces $P_1, \ldots, P_k$ also preserves the cycle.

For a symmetric system, invariant spaces arise naturally in the form of *fixed point spaces*, that is, spaces consisting of states invariant under some symmetries of the system. In particular, the cycle in Figure 1 can exist for a system having two reflection symmetries, one across the horizontal plane and one across the vertical plane. In this example the horizontal and the vertical planes are fixed point spaces of the corresponding reflections. For symmetric systems it is natural to define a heteroclinic cycle as a sequence of equilibria $\xi_1, \ldots, \xi_{k+1}$ joined by connecting orbits and such that ξ_{k+1} is obtained by applying a symmetry operation to ξ_1. A cycle is homoclinic if $k = 1$. The robust cycle depicted in Figure 1 is homoclinic if the underlying system is odd (equivalently $-\mathrm{Id}$ is a symmetry) and the two equilibria are equidistant from the origin.[1]

A notable feature of robust cycles is that they can be asymptotically stable (or possess some weaker form of stability). Intuitively, stability can be expected when the stable eigenvalues of the equilibria on the cycle are on the average stronger than the unstable ones. A stable cycle defines a mechanism of intermittency—a solution approaching it spends long periods near equilibria and makes fast transitions from one equilibrium to the next. In a perfectly symmetric system the return times increase monotonically and rapidly approach infinity, thus making the intermittent behavior uninteresting. However, under small, symmetry-breaking perturbations, the cycling behavior persists (even if

[1] For an odd system a cycle of the type shown in Figure 1 must be homoclinic.

there no longer is a cycle) and the transition times no longer converge to infinity. In many cases the transition times are either bounded or extremely long transition times are very infrequent. In a similar manner stochastic perturbations, or round-off errors in numerical computations, lead to boundedness of transition times. Hence, in applications, the existence of a stable heteroclinic cycle in the idealized model problem can be linked to the occurrence of intermittence.

Robust cycles have attracted the interest of a number of mathematicians and physicists. An important early work on cycles was the article of Busse and Clever [18] in 1979, who proposed a truncated ODE model based on only three modes in an effort to describe an aperiodic intermittent state in rotating Rayleigh-Bènard convection. In their ODE model they observed a heteroclinic cycle. Busse and Heikes [19] argued that the turbulent state observed in the experiment had gross features consistent with those of the heteroclinic cycle. Interestingly, in 1975 May and Leonard [68] had already observed the same cycle in a model for the nonlinear competition of three identical species. Here the robustness of the cycle is forced by the special structure of the Lotka-Volterra equations. The work of [68] stimulated the interest of mathematical biologists in robust cycles and led to significant research on the subject.

A wave of interest in robust cycles among mathematicians and physicists interested in equivariant bifurcation theory followed the article of Guckenheimer and Holmes [41] published in 1988. Bifurcations of systems with symmetry often lead to *spontaneous symmetry breaking*. A symmetry-breaking bifurcation occurs when a state possessing high symmetry loses stability, giving rise to states with less symmetry. The authors of [41] showed that robust heteroclinic cycles could naturally arise due to a low codimension symmetry-breaking bifurcation. Another milestone in the theory of robust cycles was the article of Lauterbach and Roberts [66], who showed that *forced symmetry breaking*, that is, slightly perturbing the equations so that some of the symmetries are broken, could naturally lead to the occurrence of robust cycles.

As indicated in the preceding paragraph there are three contexts in which robust cycles have been shown to exist: spontaneous symmetry breaking (SSB), forced symmetry breaking (FSB), and mathematical biology and game theory (MBGT). In this article we describe the research on cycles within these three contexts, putting more emphasis on SSB and FSB. We devote much attention to the issues concerning stability of cycles and their bifurcations. Finally, we describe the main experimental or theoretical situations in which there is evidence for the existence of robust cycles.

As noted robust heteroclinic cycles may connect invariant sets with stationary, periodic, or aperiodic dynamics. For most of the known examples the vertices are equilibria or periodic orbits. In a recent article Dellnitz et al. [27] showed a simple scenario leading to cycles for which the vertices can be chaotic (see [33] for a review and results on stability of such cycles). Cycles connecting chaotic sets may be present in many physical systems; evidence for the existence of such cycles in the context of turbulent flows in the wall region of a boundary layer was given by Sanghi and Aubry [79].

This article is divided into three parts. In Part I we consider two particularly simple robust cycles arising through the SSB and the FSB scenarios respectively. We try to give a comprehensive account of the issues concerning the existence and the stability of the two cycles. Part I is intended to introduce the main issues at a basic level. Part II is a detailed review of the theoretical research. Part III is a review of the applications. Readers who are mainly interested in applications can skip Part II and go directly to

Part III. The portions of Part II directly relevant to applications will be referred to in Part III and will be accessible to a broad audience.

In the remainder of this introduction we discuss the various aspects of the theory of cycles and the structure of this article in more detail.

Spontaneous Symmetry Breaking

As noted, Guckenheimer and Holmes [41] showed that SSB could lead to the occurrence of robust cycles. Their work inspired a fairly systematic search for robust cycles in low codimension problems. For steady state bifurcations of codimension one, cycles were found in special classes of symmetry groups [41], [34], [35], [42], [28]. A review of these examples can be found in the recent lecture notes of Field [33].

For generic (codimension 1) Hopf bifurcations, robust cycles are often found in bifurcations with the symmetries of planar lattices [85], [84], [23], [59], [60], [96]. Such bifurcation problems arise naturally in connection with Rayleigh–Bènard convection between horizontal plates.

Many examples of cycles are found in mode interactions involving a combination of steady state and Hopf modes. The symmetry groups considered were the trivial symmetry (the normal form symmetry of triple Hopf bifurcation suffices to force the existence of a robust cycle) [69], [63], the group $\mathbf{O}(2)$ [4], [72], [71], the group $\mathbb{D}_4$ [20], and the group $\mathbf{O}(3)$ [37], [3], [43].

Results on robust cycles related to SSB will be reviewed in Section 2 and in Section 5.

Forced Symmetry Breaking

Lauterbach and Roberts [66] observed that when symmetry is partially broken—that is, when equations commuting with the action of a symmetry group Γ, with dim Γ positive, are perturbed to a system of equations whose symmetries form a subgroup $\Delta \subset \Gamma$—cycles may appear along the old group orbits. (For this process to produce cycles the dimension of Δ must be less than the dimension of Γ. Indeed, the Lauterbach-Roberts examples broke symmetry from $\mathbf{O}(3)$ to a finite subgroup.) Formation of cycles through FSB was subsequently studied in [6], [65], [64], [49]. The results on cycles in FSB will be reviewed in Sections 3 and 6.

Stability

An important feature of robust heteroclinic cycles is that they may attract nearby dynamics. The strongest form of stability is *asymptotic stability*, occurring when an entire neighborhood of the cycle is attracted to it. Presently there are methods which establish necessary and sufficient conditions for asymptotic stability for most of the known cycles [61], [47], [35]. Unfortunately these methods cannot be summarized into a simply stated theorem.

Melbourne [70] noted that cycles might be stable in a sense weaker than asymptotic stability—indeed, this weaker form of stability may well be more the rule than the exception. Stability in this weaker sense occurs when an open subset of initial conditions in a neighborhood of the cycle is attracted to the cycle with the measure of that open set approaching full measure as the neighborhood shrinks to zero [70], [66], [62]. It turns out that yet weaker forms of stability are possible [14], [56], [28].

The issues concerning stability will be reviewed in Section 4.

Bifurcations from Cycles

A natural question is what happens when a cycle loses stability. Such bifurcations may lead to the appearance of long period periodic orbits, other heteroclinic cycles, and more complicated dynamics [45], [47], [2], [4], [21], [82], [24], [35].

Another type of bifurcation occurs when the cycle is broken due to forced symmetry breaking. Typically the cycle is replaced by an invariant set contained in a small tubular neighborhood of the cycle. The dynamics on the invariant set may be periodic [81], quasiperiodic [25], [26], [32], or chaotic [78], [42], [98].

The results on bifurcations will be reviewed in Section 8.

Systems of Mathematical Biology and Game Theory

Robust heteroclinic cycles also exist in models relevant to mathematical biology and game theory (MBGT). The May-Leonard model [68] noted previously is an example. A more comprehensive study of the existence and stability of robust cycles in MBGT is included in the book of Hofbauer and Sigmund [47]. The work of [47] inspired further investigations on the subject [45], [57], [46], [14], [46], [38]. The results on cycles in MBGT will be reviewed in Section 7.

Applications

Heteroclinic cycles were found in a number of applications. Rotating convection was mentioned previously. Other applications include: turbulent flow in a wall region of a boundary layer [10], [79], [11], [12], [13], convection in the presence of a magnetic field [23], [76], flow through an elastic hose pipe [88], the Kuramoto-Sivashinsky equation [5], [55], and Kolmogorov flows [83]. This list gives some idea of the number of physically motivated fluid dynamic models where heteroclinic phenomena have been observed. Part III of this article is devoted to applications. We will describe the cycles occurring in the following contexts: the dynamics of the Kuramoto-Sivashinsky equation (Section 9), rotating convection (Section 10), convection in the presence of a magnetic field (Section 11), turbulent flows in a boundary layer (Section 12), and flow through a hose pipe (Section 13).

PART I

2. Spontaneous Symmetry Breaking—An Example

The authors of [68], [18], and [41] consider the cubic differential equation

$$\begin{aligned} \dot{x}_1 &= x_1(\lambda + a_1x_1^2 + a_2x_2^2 + a_3x_3^2), \\ \dot{x}_2 &= x_2(\lambda + a_1x_2^2 + a_2x_3^2 + a_3x_1^2), \\ \dot{x}_3 &= x_3(\lambda + a_1x_3^2 + a_2x_1^2 + a_3x_2^2). \end{aligned} \tag{2.1}$$

The symmetries of (2.1) are generated by the reflections in coordinate planes κ_1, κ_2 and κ_3,

$$(x_1, x_2, x_3) \xrightarrow{\kappa_1} (-x_1, x_2, x_3),$$

$$(x_1, x_2, x_3) \xrightarrow{\kappa_2} (x_1, -x_2, x_3),$$

$$(x_1, x_2, x_3) \xrightarrow{\kappa_3} (x_1, x_2, -x_3),$$

and the cyclic permutation σ,

$$(x_1, x_2, x_3) \xrightarrow{\sigma} (x_3, x_1, x_2).$$

Remark 1. The symmetry group Γ of (2.1) is generated by two smaller groups, namely, $\mathbb{Z}_2^3 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by κ_1, κ_2, and κ_3, and $\mathbb{Z}_3$ generated by σ. The group Γ is not a direct product of $\mathbb{Z}_2^3$ and $\mathbb{Z}_3$; indeed the elements of the two groups do not commute, but other relations are in place, for example $\kappa_1\sigma = \sigma\kappa_3$. In algebraic terms Γ is a *semidirect product* of $\mathbb{Z}_2^3$ and $\mathbb{Z}_3$, denoted by $\Gamma = \mathbb{Z}_2^3 \cdot \mathbb{Z}_3$. (For a rigorous definition of semidirect products see [52, p. 99].) Equivalently $\Gamma = \mathbf{T} \oplus \mathbb{Z}_2$, where $\mathbf{T}$ is the tetrahedral group consisting of the orientation-preserving symmetries of a tetrahedron. Yet another description of Γ is as $\mathbb{Z}_2 \wr \mathbb{Z}_3$, where $\wr$ denotes *wreath product*. Consider (2.1) as a system of three coupled cells, the coordinate x_j corresponding to the jth cell. On every cell there is an identical $\mathbb{Z}_2$ action given by $x_j \to -x_j$. This action is called *internal*—indeed it consists of symmetries affecting only the given cell. An *external* symmetry is a permutation of different cells. In the case of (2.1) external symmetries are given by the cycle σ and its iterates. For more details and rigor on wreath products see Section 5 and [29].

In this section we analyze the existence and stability of cycles for (2.1) with $\lambda > 0$. Rather than repeat the analysis of [41] we combine the approach of [78] (existence) and [61] (stability) to obtain somewhat sharper results than [41]. Namely we derive a necessary and sufficient condition for the existence of the cycle and show that the condition of [41] is—in a sense to be specified below—necessary and sufficient for asymptotic stability. The results we derive hold for a large class of generic steady-state bifurcations with symmetry Γ (as shown in [41]). In the remainder of this section we prove the existence of the cycle, analyze its stability, and indicate how the obtained results can be generalized to the context of a bifurcation with symmetry Γ.

2.1. Existence

Recall that a heteroclinic cycle in the symmetric context was defined as a sequence of equilibria $\xi_1, \dots, \xi_{k+1}$ joined by connecting orbits and such that $\xi_{k+1} = \gamma\xi_1$ for some $\gamma \in \Gamma$, $k > 0$. The robust cycle we are going to find is homoclinic, that is, $k = 1$. Hence we need to find a flow invariant plane P, and two equilibria $\xi_1, \xi_2 \in P$ with the following properties:

(i) ξ_1 is a saddle and ξ_2 is a sink for the flow of (2.1) restricted to P.
(ii) there is a saddle-sink connection in P from ξ_1 to ξ_2.
(iii) there exists an element $\gamma \in \Gamma$ such that $\gamma\xi_1 = \xi_2$.

The plane P is chosen to be the x_1x_2 plane,

$$P = \{x \in \mathbb{R}^3\colon\ x_3 = 0\}.$$

The equations in (2.1) show that P is flow invariant. Note also that P is the fixed point space of the reflection κ_3. We look for ξ_1 in invariant line $L = \{x \in \mathbb{R}^3\colon\ x_2 = x_3 = 0\}$ (the x_1-axis). Assuming $a_1 < 0$, so that the bifurcation is supercritical, we get $\xi_1 = (\sqrt{-\lambda/a_1}, 0, 0)$. We also define $\xi_2 = \sigma\xi_1 = (0, \sqrt{-\lambda/a_1}, 0)$ and requirement (iii) is satisfied with $\gamma = \sigma$.

One checks easily that

$$\begin{aligned} Df(\xi_1) &= \lambda \cdot \operatorname{diag}\left(-2, \frac{a_1 - a_3}{a_1}, \frac{a_1 - a_2}{a_1}\right), \\ Df(\xi_2) &= \lambda \cdot \operatorname{diag}\left(\frac{a_1 - a_2}{a_1}, -2, -\frac{a_1 - a_3}{a_1}\right). \end{aligned} \tag{2.2}$$

Condition (i) is satisfied by assuming

$$a_3 > a_1 \qquad \text{and} \qquad a_2 < a_1. \tag{2.3}$$

Note that (2.3) is a necessary condition for the existence of a robust cycle.

Checking condition (ii) is in general a nontractable problem; existence of connections must be checked using numerics. If $\dim(P) = 2$, analytic results can be obtained using the Poincaré-Bendixson theorem [41], [71]. Sandstede and Scheel [78] obtain optimal conditions on (a_1, a_2, a_3), for which a cycle exists in (2.1). The argument of [78] is presented below.

Consider the restriction of (2.1) to P:

$$\begin{aligned} \dot{x}_1 &= x_1(\lambda + a_1x_1^2 + a_2x_2^2), \\ \dot{x}_2 &= x_2(\lambda + a_1x_2^2 + a_3x_1^2). \end{aligned} \tag{2.4}$$

We find conditions on (a_1, a_2, a_3) under which the flow of (2.4) has the following properties:

(a) (2.4) admits no equilibria other than the origin, $\pm\xi_1$ and $\pm\xi_2$.
(b) The unstable manifold $W^u(\xi_2)$ remains within $\mathcal{O}(\sqrt{\lambda})$ from the origin.

Properties (a) and (b), combined with the Poincaré-Bendixson theorem, imply the existence of the connection $\xi_1 \to \xi_2$. Moreover, the connecting trajectory is within $\mathcal{O}(\sqrt{\lambda})$ of the origin. Thus the cycle approaches the origin as $\lambda \to 0$. A straightforward computation shows that if ξ_1 is a saddle and ξ_2 a sink for (2.4) (or vice versa), then (a) holds. Hence (a) follows from (2.3).

We now show that (b) follows from $a_1 < 0$ and (2.3). Let $x(t) = (x_1(t), x_2(t))$ be the branch of $W^u(\xi_2)$ contained in the positive quadrant. Note that $0 > a_1 > a_2$ implies the existence of a constant $C > 0$ such that $0 \leq x_1(t) \leq C\sqrt{\lambda}$ for all $t \in \mathbb{R}$. Hence $\frac{\dot{x}_2(t)}{x_2(t)} \leq \lambda(1 + C^2a_3) + a_1$. Since $a_1 < 0$, $x_2(t)$ remains $\mathcal{O}(\sqrt{\lambda})$ and the assertion follows.

We have proved the following proposition.

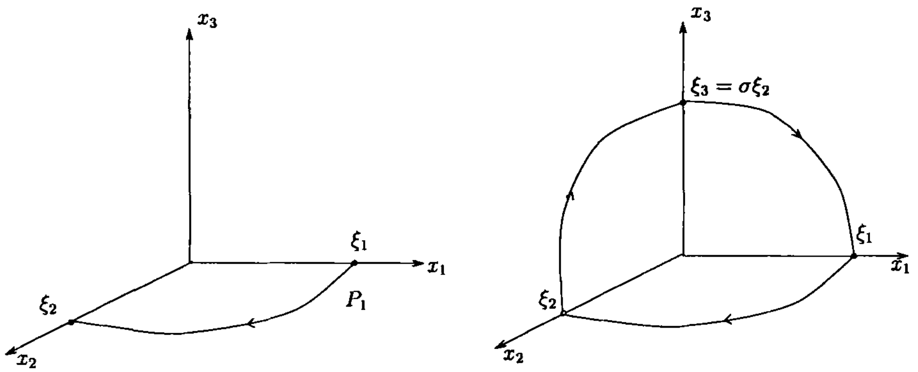

Fig. 2. The cycle existing for the system (2.1).

Proposition 1. *The cycle exists if and only if* $a_1 < 0$ *and condition (2.3) holds.*

Remark 2. In this example the existence of a homoclinic cycle in the symmetric sense implies the existence of a heteroclinic cycle in the usual sense. Indeed, the existence of the connection $\xi_1 \to \xi_2 = \sigma\xi_1$ implies the existence of two other connections, namely $\sigma\xi_1 \to \sigma^2\xi_1$ and $\sigma^2\xi_1 \to \sigma^3\xi_1 = \xi_1$. The union of these three connections forms a heteroclinic cycle in the usual sense; see Figure 2b. This implication holds true for any finite symmetry group, but may not hold for infinite groups.

Remark 3. An ODE in $\mathbb{R}^3$ whose symmetries are generated by the reflections κ_1, κ_2, and κ_3 may have a heteroclinic cycle as shown in Figure 2b [69]. In this case there is no homoclinic cycle as in Figure 2a, since σ is not a symmetry.

2.2. Stability

In this section we prove the following stability theorem.

Theorem 1. *The cycle found in the preceding section is stable if* $2a_1 > a_2 + a_3$. *For a generic choice of* (a_1, a_2, a_3) *the cycle is unstable when* $2a_1 < a_2 + a_3$.

Theorem 1 was partially proved in [41] (sufficiency) and follows from the results of [69]. Our proof follows the ideas of [61]. The advantage of this approach is its applicability to a wide class of robust cycles.

To prove Theorem 1 we derive the lowest order approximation of the return map around the cycle. We begin by defining the appropriate sections of the flow.

Let $\delta > 0$ be a small real number. Let x_0, y_0, $z_0 \in \mathbb{R}^3$ belong to the connecting orbits $\sigma^{-1}\xi_1 \to \xi_1$, $\xi_1 \to \xi_2$, and $\xi_1 \to \xi_2$, respectively, with $|x_0 - \xi_1| = |y_0 - \xi_1| =$

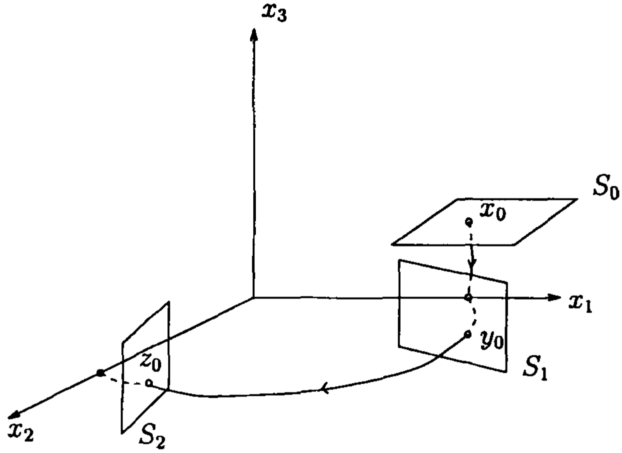

Fig. 3. The sections S_0, S_1, and S_2.

$|z_0 - \xi_2| = \delta$ (see Figure 3). We consider the following sections of the flow near the cycle (see Figure 3):

$$\begin{aligned} S_0 &= x_0 + \{(u, w, 0)\colon\ u^2 + w^2 < \eta\}, \\ S_1 &= y_0 + \{(u, 0, v)\colon\ u^2 + v^2 < \eta\}, \\ S_2 &= z_0 + \{(0, u, w)\colon\ u^2 + w^2 < \eta\}, \end{aligned}$$

where $\eta > 0$ is a small real number. Let U be a neighborhood of the origin (x_0) in S_0. If U is sufficiently small then all trajectories starting in U either converge to ξ_1 (this happens for all points with $w = 0$) or hit one of the sections S_2 and $\kappa_2 S_2$. Let h denote the first hit map from U to S_2. Note that h is defined on the set $U^{>0} = \{x \in U\colon\ w > 0\}$. Define $g\colon\ U^{>0} \to S_0^{>0}$ by $g = \sigma^{-1} h$. By a slight abuse of notation we identify the elements of S_0 with points in $\mathbb{R}^2$, that is, we write $g(u, w)$, rather than $g(x_0 + (u, w, 0))$. Let g_1 and g_2 denote the u and w components of g, respectively.

We now extend the definition of g to all of U, by requiring that $g(u, w) = (g_1(u, -w), -g_2(u, -w))$. In particular $g(u, 0) = (0, 0)$. Note that the resulting map is continuous. This follows from the properties of flow near hyperbolic equilibria and from the invariance of the plane $\sigma^{-1}P$ (the x_1x_3-plane). Indeed, for any $\epsilon > 0$ there exists ϵ_1 such that for any $x = (u, w) \in U$ the estimate $0 < |w| < \epsilon_1$ implies that the flow trajectory of x intersects $S_2 \cup \kappa_2 S_2$ at a point no further than ϵ away from $\{z_0, \kappa_2 z_0\}$. The stability analysis is based on the following result (see [61] for a more general version).

Proposition 2.

(i) *The cycle is asymptotically stable if* 0 *is asymptotically stable as a fixed point of* g.
(ii) *The cycle is unstable if* 0 *is a fixed point of* g.

Proof. Note that for all initial conditions in $U \setminus \sigma^{-1}P$ the map g^3 is the return map generated by the flow along the heteroclinic cycle (the connections $\xi_1 \to \xi_2 \to \sigma\xi_2 \to \xi_1$). The result follows. □

It is known that under a finite number of algebraic conditions (nonresonance conditions) the flow can be linearized [90]. The linearizing tansformation does not affect the symmetry properties [7]. Hence we assume that the flow is linear in small neighborhoods of the equilibria ξ_1 and ξ_2 and that the sections S_0 to S_2 are contained in these neighborhoods. It follows that $x_0 = \xi_1 + \delta e_3$, $y_0 = \xi_1 + \delta e_2$, and $z_0 = \xi_2 + \delta e_1$, where $\{e_1, e_2, e_3\}$ is the euclidean basis of $\mathbb{R}^3$. The oddness of g in w implies that it suffices to work out g for the elements of $U^{>0}$. Hence we assume that $w > 0$.

From (2.2) we read off the eigenvalues of $Df(\xi_1)$ and introduce the following notation:

$$r = -\lambda, \qquad e = \lambda\frac{a_1 - a_3}{a_1}, \qquad c = \lambda\frac{a_2 - a_1}{a_1}.$$

The letters r, e, and c signify the *radial*, the *expanding*, and the *contracting* directions. The radial direction is the x_1-axis and corresponds to the fixed point space of the group generated by κ_2 and κ_3. The radial direction contains the equilibrium ξ_1. It can also be described as $\sigma^{-1}P \cap P$. It will become clear below that the radial eigenvalue plays no role in determining stability.

In the following result we give a lowest order expression of g which will suffice to analyze its stability properties.

Proposition 3. *The map g has the form*

$$g(u, w) = (a(u, w), b(u, w)w^{c/e}),$$

for some continuous functions a and b, $b(u, w) > 0$ for $w > 0$. Moreover there exist constants K, $\alpha > 0$ such that

$$|a(u, w)| \le Kw^{\alpha}; \qquad |b(u, w)| \le K, \qquad (u, w) \in U.$$

For a generic choice of (a_1, a_2, a_3) there exists a constant $L > 0$ such that $L \le b(u, w)$ for $(u, w) \in U$.

Proof. Let ψ and ϕ denote the first hit maps from $U^{>0}$ to S_1 and from a neighborhood of y_0 in S_1 to S_2. The map ψ is (after linearization) given by

$$(u, w, \delta) \xrightarrow{\psi} (\Delta_1 w^{r/e}, \delta, \Delta_2 w^{c/e}),$$

where Δ_1 and Δ_2 are positive constants. The map ϕ is a diffeomorphism, since the flow between S_1 and S_2 has no singularity. Note that the invariance of P under the flow implies that ϕ maps $P \cap S_1$ to $P \cap S_2$. Hence ϕ has the following form:

$$(u, \delta, v) \xrightarrow{\phi} (\delta, a_0(u, v), b_0(u, v)v),$$

where a_0 and b_0 are smooth functions. The condition $a_0(0, 0) \neq 0$ can be easily expressed as a condition on the variational equation (Melnikov integral) for the connection $\xi_1 \to \xi_2$, and it is satisfied for an open and dense set of (a_1, a_2, a_3). The assertions of the proposition follow from the fact that $g = \sigma^{-1}\phi \circ \psi$. □

To establish the stability properties of the origin for g, we need the following elementary lemma.

Lemma 1. *Let B, $\rho > 0$, $\rho \neq 1$ be constants. Let k: $\mathbb{R}^+ \to \mathbb{R}^+$ be defined by $k(w) = Bw^\rho$. There exists $R > 0$ such that the following statements hold:*

(i) If $\rho > 1$ then $w < R$ implies that the sequence $\{k^n(w)\}_{n=1,2,\ldots}$ is decreasing and converges to 0 as n goes to infinity.
(ii) If $\rho < 1$ then for every $0 < w < R$ there exists an n such that the sequence $\{k^j(w)\}_{j=1,2,\ldots n}$ is increasing and $k^n(w) \geq R$.

Proof.

(i) Let $R \leq (\frac{1}{2B})^{\frac{1}{\rho-1}}$. Then $w < R$ implies $k(w) < \frac{1}{2}w$. The assertion follows.
(ii) Let $R \leq (\frac{B}{2})^{\frac{1}{1-\rho}}$. Then $w < R$ implies $k(w) > 2w$. The assertion follows. □

Proof of Theorem 1. Suppose $2a_1 > a_2 + a_3$. This is equivalent to $c/e < 1$. Let L be as defined in Proposition 3. Let $k(w) = Lw^{c/e}$. Proposition 3 implies that $(g^n)_2(u, w) \geq k(w)$ for a generic choice of (a_1, a_2, a_3). It follows from Lemma 1 that the origin is unstable for g. By Proposition 2 the cycle is unstable.

Suppose $2a_1 < a_2 + a_3$. Let K and α be as introduced in Proposition 3. Let $k(w) = Kw^\alpha$ and let $0 < R < \frac{\eta}{2}$ be such that the assertion of Lemma 1 holds and $KR^\alpha < \frac{\eta}{2}$. It follows that the set $U_R = \{(u, w) \in U\colon\ w < R\}$ is forward invariant for g and that $(g^n)_2(u, w) \leq k^n(w)$. It follows that the origin is asymptotically stable for g. Consequently the cycle is asymptotically stable. □

Remark 4. Note that the radial eigenvalues play no role in determining stability.

2.3. The Generic Case

Consider a dynamical system with symmetry $\Gamma = \mathbb{Z}_2^3 \cdot \mathbb{Z}_3$ and depending on a parameter λ. Suppose that for some value of λ there is a steady-state bifurcation of a state with full symmetry. Denote the critical eigenspace by E. We make the following assumption:

(A1) The eigenspace E is isomorphic to $\mathbb{R}^3$ with the action of Γ as described previously.

A generic bifurcation for which **(A1)** holds admits a homoclinic cyclu analogous to the one described in the preceding paragraph. This follows from the following facts:

1. Using the center manifold theorem one can reduce the bifurcation problem to a problem posed on the space E with symmetry Γ [40].
2. For a generic bifurcation problem it suffices to establish the existence of the cycle for the third order truncation (to see this, rescale the variables and time and treat the higher order terms as a small perturbation).

3. Forced Symmetry Breaking—An Example

In this section we present an example of a robust cycle arising through forced symmetry breaking. Recall that such an example was first found by Lauterbach and Roberts [66]. Here we present the much simpler example of Hou and Golubitsky [50]. Our presentation is less complete than in the previous section—we concentrate on the ideas behind the example of [50] and omit the computations.

3.1. Existence

The authors of [50] consider a differential equation

$$\dot{z} = f(z, \epsilon), \qquad z = (z_1, z_2) \in \mathbb{C}^2, \qquad \epsilon \in \mathbb{R}, \tag{3.1}$$

where f is smooth as a function of z and ϵ. Equivalently we think of (3.1) as posed on $\mathbb{R}^4$. Let $\mathbb{T}^2$ denote a two-torus and $\mathbb{D}_4$ the group of symmetries of the square. For $\epsilon = 0$ the symmetries of (3.1) are generated by the following operations:

$$\begin{aligned} (z_1, z_2) &\overset{(\phi,\psi)}{\longrightarrow} (e^{i\phi} z_1, e^{i\psi} z_2), \\ (z_1, z_2) &\overset{\kappa}{\to} (z_2, x_1), \\ (z_1, z_2) &\overset{\kappa_1}{\to} (\bar{z}_1, z_2). \end{aligned} \tag{3.2}$$

Let $\kappa_2 = \kappa\kappa_1$. The action of κ_2 is, clearly,

$$(z_1, z_2) \overset{\kappa_2}{\to} (z_1, \bar{z}_2).$$

For $\epsilon \neq 0$ most of the symmetries of f are broken. The remaining symmetries are generated by the reflections κ_1 and κ_2.

Remark 5. Let Γ be the group of symmetries of the unperturbed system ($\epsilon = 0$) and Δ the group of symmetries of the perturbed system. The group Γ consists of all the symmetries of a square lattice in the plane. Equivalently $\Gamma = \mathbb{T}^2 \cdot \mathbb{D}_4$, where $\cdot$ denotes semidirect product. The group Δ is the four element group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, also denoted by $\mathbb{D}_2$.

We now consider the unperturbed system ($\epsilon = 0$). Suppose that (3.1) has an equilibrium with symmetry $\mathbb{D}_4$, that is, of the form $\xi_0 = (x, x)$, $x \in \mathbb{R}$. The group orbit $M = \Gamma\xi_0$ is given by

$$M = \{(e^{i\phi}x, e^{i\psi}z_2)x\colon\ (\phi, \psi) \in \mathbb{T}^2\},$$

and is clearly diffeomorphic to $\mathbb{T}^2$. Moreover, M is $\mathbb{D}_4$-invariant. The symmetry of (3.1) implies that $f \mid M \equiv 0$. Hence $Df(x) \mid T_x M = 0$ for all $x \in M$. (This means that ξ_0 cannot be a hyperbolic equilibrium in the usual sense.) The symmetry of (3.1) also implies that the eigenvalues of $Df(x)$ are the same for every $x \in M$ and are equal to the eigenvalues of $Df(\xi_0)$. It follows that if all the eigenvalues of $Df(\xi_0)$, except for the ones corresponding to directions in the tangent space to M, are all in the left half of the complex plane then M is a locally attracting, normally hyperbolic invariant manifold. In this case we say that ξ_0 is a *hyperbolic equilibrium*. Consequently, one is interested in finding stable hyperbolic equilibria with symmetry $\mathbb{D}_4$, that is, of the form (x, x), $x \in \mathbb{R}$. The work of Field [30] implies that there exists an open set of smooth, Γ-symmetric vector fields of the form (3.1) with $\epsilon = 0$ which possess a stable hyperbolic equilibrium of the required form. The authors of [50] make, using local bifurcation theory, a particular choice of such a set. By making this choice they have enough information about the dynamics of the perturbed vector field ($\epsilon \neq 0$) to be able to prove the existence of stable cycles.

As noted above, M is a locally attracting, normally hyperbolic invariant manifold. The consequence of this is well known; for $\epsilon \neq 0$ the equation (3.1) has an invariant manifold M_ϵ. Moreover, M_ϵ has the following properties:

(i) M_ϵ retains the "unbroken" symmetries of M, that is, every symmetry of M which remains a symmetry for f when $\epsilon \neq 0$ is a symmetry of M_ϵ.
(ii) M_ϵ is diffeomorphic to M. This diffeomorphism preserves the symmetry properties of M_ϵ.

We conclude that the flow on M_ϵ can be described in terms of near-identity Δ-symmetric flow on M, or, by identifying M with $\mathbb{T}^2$, with a near-identity, Δ-symmetric flow on $\mathbb{T}^2$. It follows from (3.2) that the action of Δ on $\mathbb{T}^2$ is generated by

$$(\phi, \psi) \stackrel{\kappa_1}{\to} (-\phi, \psi),$$
$$(\phi, \psi) \stackrel{\kappa_2}{\to} (\phi, -\psi).$$

This action has the following fixed-point sets (no longer vector spaces, since the action is on the manifold):

$$\begin{aligned} C_1 &= \mathrm{Fix}(\kappa_1) = \{(0, \psi)\colon \psi \in S^1\} \cup \{(\pi, \psi)\colon \psi \in S^1\}, \\ C_2 &= \mathrm{Fix}(\kappa_2) = \{(\phi, 0)\colon \phi \in S^1\} \cup \{(\phi, \pi)\colon \phi \in S^1\}, \\ E &= \mathrm{Fix}(\Delta) = \{\xi_1 = (0,0), \xi_2 = (\pi, 0), \xi_3 = (\pi, \pi), \xi_4 = (0, \pi)\}. \end{aligned}$$

The sets C_1, C_2, and E must be invariant for the flow. (It is easy to check directly that these sets are invariant for any vector field on $\mathbb{T}^2$ with the above specified symmetries.) The structure of the invariant sets on $\mathbb{T}^2$ is shown in Figure 4. It is clear that the points $\xi_1, \ldots, \xi_4$ are equilibria.

Proposition 4. [50] *For an open set of families of Δ-symmetric vector fields $f(z, \epsilon)$, $\epsilon \approx 0$, the flow on C_1 and C_2 is as shown in Figure 4. Consequently there exists a heteroclinic cycle connecting $\xi_1 \to \xi_2 \to \xi_3 \to \xi_4$.*

Sketch of the proof. The existence of the cycle is a consequence of the following properties:

(i) The equilibria $\xi_1, \ldots, \xi_4$ have stable and unstable directions as shown in Figure 4,
(ii) $\xi_1, \ldots, \xi_4$ are the only equilibria for the flow on $\mathbb{T}^2$.

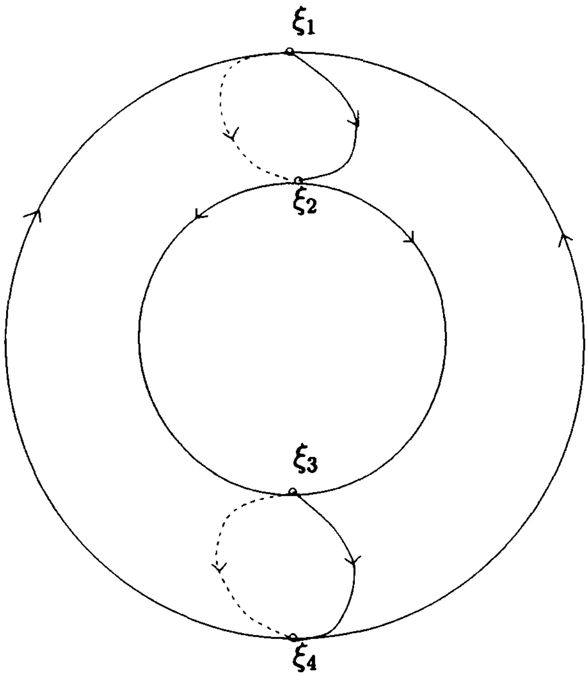

Fig. 4. The structure of the invariant sets on the invariant torus.

Hou and Golubitsky [50] prove that these conditions hold for an open set of families $f(z, \epsilon)$. □

Remark 6. The role of the planes P_j is played by the connected components of C_1 and C_2. The cycle is heteroclinic—the equilibria $\xi_1, \ldots, \xi_4$ are not symmetry related.

3.2. Stability

Recall that M_ϵ uniformly attracts nearby dynamics. Consequently the cycle is stable if and only if it is stable for the flow restricted to M_ϵ. Since M_ϵ is two-dimensional one needs to deal only with contracting and expanding eigenvalues, which makes the stability computation significantly simpler than in the previously described example of [41].

We now describe the relevant stability condition. For the flow restricted to M_ϵ let $-c_j$ be the stable eigenvalue of ξ_j and e_j the unstable eigenvalue of ξ_j, $j = 1, \ldots, 4$. Then the cycle is stable if

$$c_1 c_2 c_3 c_4 > e_1 e_2 e_3 e_4. \tag{3.3}$$

Generically the cycle is unstable if $c_1 c_2 c_3 c_4 < e_1 e_2 e_3 e_4$ (see Section 4 and [73] for details).

Proposition 5. [50] *For an open set of families of flows defined by (3.1) the cycle shown in Figure 4 exists and is asymptotically stable.*

Proposition 5 is proved by showing that (3.3) holds for an open subset of the set of vector fields found in Proposition 4.

PART II

In Part II we review the more theoretical research on robust cycles. We begin by summarizing the results on the stability of cycles (Section 4). The sections on spontaneous symmetry breaking, forced symmetry breaking, and cycles in mathematical biology and game theory follow (Sections 5, 6, and 7). In Section 8, which is the last section of Part II, we review the results on bifurcations from robust cycles.

4. Stability of Robust Cycles

Many of the early articles on robust cycles contain some stability results, but these results are often relevant only in a narrow context and are not always optimal. A systematic approach to stability computations in the context of systems with symmetry was developed by Krupa and Melbourne [61], who also obtained a sharp stability condition for a class of cycles. Hofbauer [45] and Hofbauer and Sigmund [47] developed methods for stability computations in the context of mathematical biology and game theory. The method of [47] turned out to be applicable for a class of cycles in systems with symmetry and was rediscovered by Field and Swift [35].

The goal of this section is to provide the reader with an overview of the stability issues. To this end we review the methods of computing stability and present the state of the art stability conditions in an abstract context. In the forthcoming sections, when reviewing stability results for particular examples, we refer back to the discussion of the present section. We concentrate on the case when the vertices of the cycle are equilibria, although the results described here sometimes generalize to the case of more complicated invariant sets. Field [33, Chapter 7] develops a stability condition for cycles connecting chaotic sets.

Consider a trajectory following a robust heteroclinic cycle. Clearly it will spend large amounts of time near the equilibria and the passages from one equilibrium to another will be relatively short. Hence the relative size of the eigenvalues of the linearizations at the equilibria will be the factor determining stability. The simplest example where this type of stability analysis has been applied is a pair of homoclinic cycles in the plane forming a 'figure 8'; see Figure 5. Let $-c$ and e denote respectively the contracting and the expanding eigenvalues at the equilibrium. The 'figure 8' configuration is asymptotically stable if $c > e$, and is unstable if $c < e$; see dos Reis [74].

In general we consider ODE's

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{4.1}$$

commuting with the action of a symmetry group Γ. We assume that (4.1) has a robust

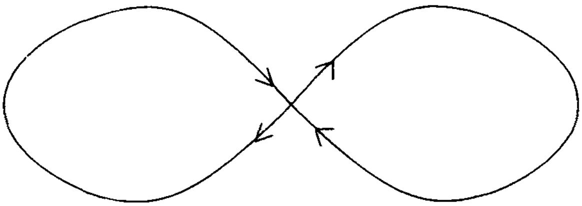

Fig. 5. The 'figure 8' configuration.

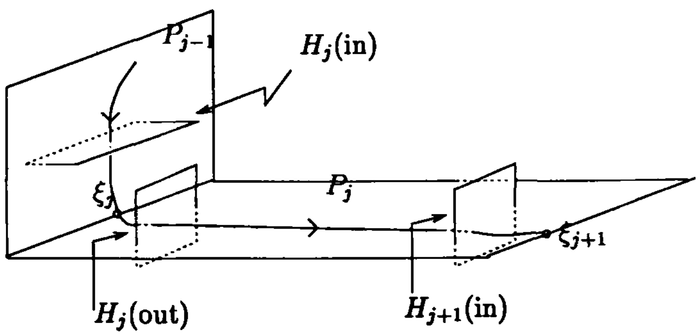

Fig. 6. The sections of the flow near ξ_j and ξ_{j+1}.

cycle and discuss its stability properties. We consider only the case when the vertices of the cycle are equilibria. Some generalizations of the stability theory to the case of vertices of other types will be discussed in the forthcoming sections on a case-by-case basis.

The approach to stability analysis reviewed in this section is to compute the return map for trajectories following the cycle. An alternative method using *average Liapunov functions* was developed by Hofbauer [45]. This method has only been applied in the context of MBGT and will be reviewed in Section 7.

4.1. Computation of the Return Map

Most stability computations are based on the derivation of a return map around the cycle. This map is composed of *local maps*, given by the flow near the equilibria, and *connecting diffeomorphisms* given by the flow between the vicinity of one equilibrium and the vicinity of the next one. For the derivation of the jth local map one often uses the linear flow determined by $Df(\xi_j)$ relying on the fact that a typical vector field can be linearized near equilibria [90] and that the linearizing transformation does not affect the symmetry properties [7]. We now sketch the derivation of the return map following the presentation of [61]. We use the system of sections of the flow defined in Figure 6.

Let

- λ_j^c be the eigenvalue of $df(\xi_j)$ restricted to $P_{j-1}\backslash P_j$ with maximal real part.[2] Let $-c_j = \mathrm{Re}(\lambda_j^c)$ ($c_j > 0$ by assumption).
- Let λ_j^e be the eigenvalue of $df(\xi_j)$ restricted to $P_j\backslash P_{j-1}$ with maximal real part. Let $e_j = \mathrm{Re}(\lambda_j^e)$ ($e_j > 0$ by assumption).
- Let λ_j^t be the eigenvalue of $df(\xi_j)$ restricted to $(P_j + P_{j-1})^\perp$ (an invariant space for $df(\xi_j)$) with maximal real part. Let $t_j = \mathrm{Re}\,(\lambda_j^t)$ and assume that

$$t_j < 0, \qquad j = 1, \dots, k. \tag{4.2}$$

If $P_j + P_{j-1} = \mathbb{R}^n$, then set $t_j = -\infty$.

Remark 7. If t_j were positive for some j then there would be trajectories leaving every sufficiently small neighborhood of the cycle. Hence (4.2) is necessary for asymptotic stability. If $t_j > 0$ for some j, weaker forms of stability are possible as discussed in Section 4.4.

Remark 8. It may happen that $P_{j-1} \subset P_j$ [76] (in fact this case was overlooked in [61]). The contracting eigenvalue λ_j^c is then not defined. For the sake of simplicity we do not consider this case here. It can be handled using similar methods.

Let ϕ_j denote the first hit map from $H_j(\mathrm{in})$ to $H_j(\mathrm{out})$ and ψ_j the connecting diffeomorphism from $H_j(\mathrm{out})$ to $H_{j+1}(\mathrm{in})$. In general the expression for ϕ_j may be rather complicated. Let us assume the following hypothesis:

(H1) $\dim(W^u(\xi_j)) = 1, \qquad j = 1, \dots, k.$

The hypothesis (H1) implies that $\lambda_j^e = e_j$. The derivation of ϕ_j is quite tedious, although elementary. In particular a choice of coordinates near ξ_j is necessary. We will write down an approximate formula for $g_j = \psi_j \cdot \phi_j$ using the coordinates w_j and z_j which are defined as follows: w_j is a (one-dimensional) coordinate along $W^u(\xi_j)$, and z_j is a multidimensional coordinate corresponding to the directions in $(P_{j-1} + P_j)^\perp$. These coordinates must be understood as local to the section $H_j(\mathrm{in})$. We write $z_j = (z_{j0}, z_{j1})$, where z_{j0} corresponds to the directions in the eigenspace of λ_j^t and z_{j1} corresponds to the union of the other eigenspaces. It turns out that only the components of g_j transverse to P_j matter in the stability computation. In particular the so-called radial directions corresponding to $P_{j-1} \cap P_j$ in the domain of g_j and to $P_j \cap P_{j+1}$ in the range of g_j do not matter (this was conjectured in [2] and proved in [61]). The mechanism of this phenomenon is clear in the stability computation in Section 2. For more details see [61]. We make the generic assumption that for both λ_j^c and λ_j^t no generalized eigenvectors exist. (Note that the eigenspaces of $df(\xi_j)$ may be forced by symmetry to be multidimensional.)

[2] More precisely, we consider eigenvalues of the linear map induced by $df(\xi_j)$ on $P_{j-1}/P_{j-1} \cap P_j$.

The components of g_j orthogonal to $P_j \cap P_{j+1}$ are given as follows:

$$(A_j|w_j|^{\lambda_j^c/e_j} + B_j|w_j|^{-\lambda_j^t/e_j} z_{j0} + \cdots) \qquad \text{the } w_{j+1}\text{th component,}$$
$$(C_j|w_j|^{\lambda_j^c/e_j} + D_j|w_j|^{-\lambda_j^t/e_j} z_{j0} + \cdots) \qquad \text{the } z_{j+1}\text{th component.} \qquad (4.3)$$

Remark 9.

(i) Both components of g_j vanish for $(w_j, z_j) = (0, 0)$. This follows from the flow-invariance of the planes P_j.

(ii) If λ_j^c and λ_j^t are real then the exponentiation in the terms $|w_j|^{\lambda_j^c/e_j}$ and $|w_j|^{\lambda_j^t/e_j}$ should be understood in the usual way. If, for example, λ_j^c is genuinely complex then $|w_j|^{\lambda_j^c/e_j}$ denotes a rotation of some vector v_0 by an angle $\theta = \text{const} \cdot \ln|w|$ followed by multiplication by $|w_j|^{c_j/e_j}$.

(iii) The entities A_j, B_j, C_j, and D_j are matrices depending on the point in the section $H_j(\text{in})$. It follows from (H1) that A_j and B_j are row vectors. The matrices C_j and D_j may have arbitrary dimensions.

(iv) The terms ... are related to coordinates corresponding to the z_{j1} directions. In many cases it can be shown that these terms are of higher order. The action of Γ may pose restrictions on the matrices A_j, B_j, C_j, D_j and sometimes even force some of them to vanish.

If λ_j^c and λ_j^t are real, then (4.3) assumes the following simplified form:

$$(A_j|w_j|^{c_j/e_j} + B_j|w_j|^{-t_j/e_j} z_0 + \cdots) \qquad \text{the } w_{j+1}\text{th component,}$$
$$(C_j|w_j|^{c_j/e_j} + D_j|w_j|^{-t_j/e_j} z_0 + \cdots) \qquad \text{the } z_{j+1}\text{th component.} \qquad (4.4)$$

The return map g has the form

$$g = g_k \cdot g_{k-1} \cdot \cdots \cdot g_1.$$

To guarantee asymptotic stability we need, apart from condition (4.2), a condition guaranteeing that the intersection of the cycle with $H_1(\text{in})$ attracts nearby initial conditions under iteration of g. The simplest case occurs when the symmetry forces one of the leading terms in every component and for every j to drop out. In this case the *transition matrix* method, which will be reviewed below, is applicable. Equivalently the method of *average Liapunov functions* can be used [46]. If no or less stringent symmetry restrictions on A_j, B_j, C_j, and D_j are present, one can try to single out a leading term in (4.3) for a subset of the initial conditions and show that this set is invariant for the return map. We refer to this approach as the *invariant cones method*. General results in this direction have been obtained by [61].

We now review the transition matrix method and the invariant cones method.

4.2. The Transition Matrix Method

For simplicity we consider the cases for which the transverse components of the maps g_j are as in (4.4) and that the cycle is in $\mathbb{R}^4$, which means that z_j is (at most) one-dimensional.

We are interested in the special cases when the maps g_j have the form

$$\begin{pmatrix} a w_j^{\alpha} z_j^{\beta} \\ b w_j^{\gamma} z_j^{\delta} \end{pmatrix}. \tag{4.5}$$

This happens when, for example, $A_j = D_j = 0$. This is the case most commonly encountered in the literature.

We now define the jth transition matrix as follows:

$$M_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}. \tag{4.6}$$

Let $M = M_k M_{k-1} \dots M_1$. It can now be easily computed that g has the form

$$\begin{pmatrix} a w^{\alpha} z^{\beta} \\ b w^{\gamma} z^{\delta} \end{pmatrix}, \tag{4.7}$$

where α, β, γ, and δ are the entries of M. When the entries of M are positive then its eigenvalues are real, one positive and one negative. We denote them by λ_+ and λ_-. The following result was obtained independently by [47] and by [35].

Theorem 2. *The cycle for which g has the form (4.7) is stable when, in addition to (4.2), $\lambda_+ > 1$. If $a(0,0)$, $b(0,0) \neq 0$, then the cycle is unstable when $\lambda_+ < 1$.*

Sketch of the proof. If $\lambda_+ > 1$ then there exists a positive integer k such that the sum of both rows of M^k is greater than 1. It is then easy to see, using polar coordinates for (w, z), that the cycle is stable.

If $\lambda_+ < 1$ one can show that the sums of rows of M^k converge to 0 as $k \to \infty$. Using this fact and polar coordinates one concludes that the cycle is unstable. □

Remark 10. The transition matrix method can be generalized to the case when, for all j, $z_j = (z_{j1}, \dots, z_{jm})$ and each component of g_j has the form $* \cdot w_j^{\alpha} z_l^{\beta}$, where $*$ is some function of (w, z), α and β are exponents depending on j and the component of g_j and $l \in \{1, \dots, k\}$.

The transition matrix method is applicable for cycles in mathematical biology and game theory [45], [47], and in some problems with symmetry [35], [63].

4.3. The Invariant Cone Method

Krupa and Melbourne [61] show that the following condition (in addition to (4.2)) guarantees asymptotic stability:

$$\prod_{j=1}^{k} \min(c_j, e_j - t_j) > \prod_{j=1}^{k} e_j. \tag{4.8}$$

A natural question is whether the condition

$$\prod_{j=1}^{k} \min(c_j, e_j - t_j) < \prod_{j=1}^{k} e_j \tag{4.9}$$

guarantees instability for a typical f. In general the answer is no. In particular in the cases when the maps g_j have the form (4.5) the necessary and sufficient condition for stability obtained using the transition matrix method (see Theorem 2) is weaker. Krupa and Melbourne [61] observe that the answer to the question whether (4.9) is sufficient depends on the action of symmetry. Let Σ_j denote the subgroup of Γ whose elements fix all points in P_j. Clearly this group leaves $H_j(\text{out})$ and $H_j(\text{in})$ invariant, and consequently ψ_j commutes with the action of Σ_j. This action may restrict the form of the matrices A_j, B_j, C_j, D_j. The following condition, formulated by [61], guarantees that no such restrictions take place:

(H2) The eigenspaces of λ_j^c, λ_j^t, λ_{j+1}^e, and λ_{j+1}^t lie in the same isotypic component of the action of Σ_j.

Remark 11. Recall that an action of a compact group can be decomposed into isotypic components, each being a direct sum of a number of copies of an irreducible representation. The hypothesis (H2) means that each of the four eigenspaces is a direct sum of a number of copies of the same irreducible representation.

Additionally [61] make the following hypothesis:

(H1′) $Df(\xi_j)$ has only one eigenvalue with a positive real part and every two eigenvectors contained in the corresponding eigenspaces can be mapped to each other by a symmetry element fixing ξ_j.

The authors of [61] prove that (4.9) typically implies instability of the cycle when (H1′) and (H2) hold.

Hypothesis (H1′) reduces to (H1) when $\dim \Gamma = 0$ and is a natural extension of (H1) to the cases with continuous symmetries. In most of the known examples (H1′) is satisfied; an example of a heteroclinic cycle where it fails can be obtained by modifying the cycle found by Swift and Barany [96]. It is quite natural to conjecture that cycles are typically unstable when (4.9) holds even if (H1′) is not satisfied. Proving such a result is an open problem.

Hypothesis (H2) implies that there are no symmetry restrictions on A_j, B_j, C_j, and D_j (see (4.3)). It holds for many of the cycles found in the Hopf-steady state and Hopf-Hopf mode interactions [71] as well as for the cycles found in Hopf bifurcation problems with the symmetry of planar lattices, that is, $D_k \cdot \mathbb{T}^2$, $k = 2, 4, 6$ [85], [84], and [23]. These examples will be discussed in more detail in Section 5. Hypothesis (H2) fails in many known cases, in particular whenever the transition matrix method is applicable (see Section 4.2).

The arguments of [61] are based on the existence of a g-invariant cone in $H_1(\text{in})$ consisting of the points for which

$$c_1|w_1| \le |z_{10}| \le c_2|w_1|, \tag{4.10}$$

where c_1 and c_2 are positive constants. The existence of the invariant cone relies on the hypothesis (H2). Inside the cone it is not hard to choose the leading term: It is $|w_j|^{\lambda_j^c/e_j}$ if $c_j < e_j - t_j$ and $|w_j|^{-\lambda_j^t/e_j} z_{j0}$ if the opposite inequality holds.

In the case when no transverse eigenvalues are present, condition (4.8) reduces to $\prod_{j=1}^k c_j > \prod_{j=1}^k e_j$. This condition was known to Dulac for cycles in the plane and appeared in various contexts in the work of [44], [75], [74], and [69].

4.4. Nonasymptotic Stability and Heteroclinic Networks

An interesting situation happens when $W^u(\xi_j) \not\subset P_j$ (equivalently, $t_j > 0$) for some choice of j. In this case asymptotic stability is impossible but the cycle can be stable in the following sense: For all initial conditions near the cycle excluding a cusp-shaped set of small measure, the corresponding trajectories approach the cycle [70], [66], and [62].[3] This stability property is called *almost asymptotic stability* (the original terminology introduced in [70] is *essential asymptotic stability*). Cycles satisfying (H2) and (H1′) are almost asymptotically stable if

$$\prod_{j=1}^{k} \min(c_j, e_j - t_j) > \prod_{j=1}^{k} e_j,$$
$$t_j < e_j, \qquad j = 1, \ldots, k. \tag{4.11}$$

The condition (4.11) is necessary and sufficient—if one of the inequalities is opposite, a strong form of instability takes place.

The case when g_j's have the form (4.5) was studied by Brannath [14] and Kirk and Silber [56]. In this case forms of stability weaker than almost asymptotic stability are possible. The authors of [14] and [56] consider the situation occurring when, at least for one j, the equilibrium ξ_j belongs to two or more different robust cycles. Such cycles are a part of a *heteroclinic network* (which may consist of yet more cycles). Brannath [14] and Kirk and Silber [56] show that the cycles forming the network can be simultaneously stable (in a weak sense). When the network is asymptotically stable then trajectories tend to a stable subcycle and no consistent 'switching' between different cycles within the network can occur.

Ashwin and Chossat [9] consider homoclinic cycles with $\dim(W^u(\xi_j)) > 1$ and all expanding eigenvalues real. They show that, if the cycle is asymptotically stable, there exists a subcycle tangent to the strong unstable direction which attracts most of the trajectories, but is not asymptotically stable. For complex contracting eigenvalues the authors of [9] obtain numerical evidence that the basin of attraction of the subcycle is riddled [1], [8].

Remark 12. Alexander et al. [1] and Ashwin et al. [8] show that chaotic attractors contained in a single fixed-point space exhibit complicated kinds of stability including almost asymptotic stability.

[3] As pointed out by Brannath [14], in a rigorous definition of almost asymptotic stability the shape of the small neighborhood of the cycle from which the interior of the cusp is removed must be specified. It suffices to assume that for all j such that $t_j > 0$ the intersection of this neighborhood with $H_j(\text{in})$ is a disc.

5. Creation of Cycles through Spontaneous Symmetry Breaking

Robust heteroclinic cycles have been found in a number of bifurcation problems of the spontaneous symmetry breaking type. Below we review the analysis of many of these problems, classifying them by codimension. We begin with some background on local bifurcations with symmetry. For details see [40].

Consider a parameter dependent ODE,

$$\dot{x} = f(x, \lambda), \qquad x \in \mathbb{R}^n, \qquad \lambda \in \mathbb{R}^k, \tag{5.1}$$

for some n and k. Assume that f commutes with the linear action of a compact Lie group Γ and that $p_0 \in \mathrm{Fix}(\Gamma)$ is an equilibrium of (5.1). One is interested in studying the steady-state and Hopf bifurcations of p_0. Since $Df(p_0)$ also commutes with Γ it follows that the eigenspaces of $Df(p_0)$ must be Γ-invariant. Generically the eigenspaces of $Df(p_0)$ corresponding to real eigenvalues are *absolutely irreducible* and the eigenspaces corresponding to complex eigenvalues are Γ-*simple* (see [40] for the definitions). Spontaneous symmetry breaking occurs if the action of Γ on the critical eigenspace is nontrivial.

Using the equivariant center manifold theorem one can reduce the bifurcation problem to an equivariant problem posed on the eigenspaces of the critical eigenvalues, that is, on the eigenspace of p_0 for a generic steady state bifurcation or on the eigenspace corresponding to the pair of purely imaginary eigenvalues for a generic Hopf problem. Hence generic steady-state bifurcation problems are posed on absolutely irreducible spaces and generic Hopf bifurcation problems on Γ-simple spaces. In mode interactions one considers direct sums of the relevant spaces.

5.1. Codimension 1 Problems

5.1.1. Steady State Bifurcations. The existence of a homoclinic cycle in an unfolding of a generic bifurcation was first shown by Guckenheimer and Holmes [41]. The bifurcation problem studied by [41] is relevant to the dynamics of rotating convection between two plates (see [19] and Section 10). It was analyzed in Section 2.

For a number of symmetry groups equivariant vector fields on absolutely irreducible spaces are determined and gradient at lowest order, which means that robust heteroclinic cycles do not occur for generic steady-state bifurcations. Known examples of robust cycles occurring through generic steady-state bifurcations, other than one example given in [41], are in the work of Field and Richardson [34], Field and Swift [35], and Guckenheimer and Worfolk [42]. In this section we discuss the results on robust cycles found in these articles. A more extensive review as well as a number of new examples are given in the recent book by Field [33].

Field and Richardson [34] and Field and Swift [35] consider bifurcation problems with symmetries of $(\mathbb{Z}_2)^k \cdot \mathbb{Z}_k$ acting on $\mathbb{R}^k$, $k \geq 4$. The groups $(\mathbb{Z}_2)^k \cdot \mathbb{Z}_k$ and their actions on $\mathbb{R}^k$ are the natural generalization of $(\mathbb{Z}_2)^3 \cdot \mathbb{Z}_3$ and its action on $\mathbb{R}^3$, see Section 2.

A helpful tool in equivariant bifurcation theory is the *invariant sphere* theorem of Field [31] (see also [36] and [95]). This theorem asserts the existence of an attracting invariant sphere following the loss of stability of the trivial equilibrium. The invariant sphere exists under an assumption on the cubic terms guaranteeing the attractivity of a

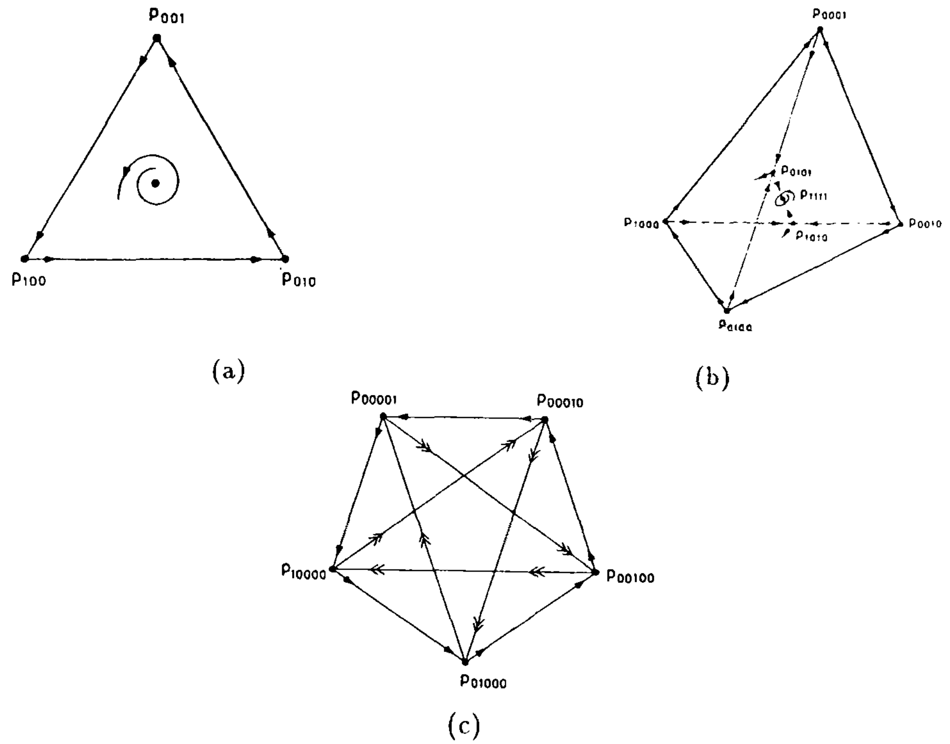

Fig. 7. The edge cycles (copied from a figure in [34]): (a) a heteroclinic cycle on Λ_2; (b) a heteroclinic cycle joining four equilibria on Λ_3; (c) possible heteroclinic cycles on Λ_4.

small neighborhood of 0. For the $(\mathbb{Z}_2)^k \cdot \mathbb{Z}_k$ symmetric problems we assume, following [35] and [34], the existence of such an invariant sphere and denote it by Λ_{k-1}.

Note that the action of $(\mathbb{Z}_2)^k$ implies the invariance of the positive quadrant $\{(u_1, \dots, u_k)\colon u_j \geq 0, j = 1, \dots k\}$. Consider the invariant simplex $\mathbf{S}_{k-1}$ given by the intersection of Λ_{k-1} and the positive quadrant. The edges and faces of $\mathbf{S}_{k-1}$ are also invariant, due to the action of $(\mathbb{Z}_2)^k$. It is not difficult to study the dynamics on the edges and to find cycles consisting of the edges of $\mathbf{S}_{k-1}$. Some of these edge cycles are shown in Figure 7. Field and Swift [35] compute the stability of the edge cycles on $\mathbf{S}_3$ using the transition matrix method. They also found a different type of cycle on $\mathbf{S}_3$, namely cycles joining equilibria contained in the interior of the edges.[4] Field and Richardson [34] find complicated 'face cycles' using the Poincaré-Hopf theorem.

Remark 13. By modifying the cycles on $\mathbf{S}_{k-1}$ one could obtain heteroclinic networks analogous to the ones studied by Brannath [14] and Kirk and Silber [56].

[4] There is a strong analogy between the cycles in $(\mathbb{Z}_2)^k \cdot \mathbb{Z}_k$ symmetric systems and the cycles found in the systems of mathematical biology and game theory—see Section 7.3 for a comparison.

Guckenheimer and Worfolk study the bifurcation problem with symmetry group Γ consisting of the orientation-preserving elements of $(\mathbb{Z}_2)^4 \cdot \mathbb{Z}_4$. The cycles they find are analogous to the cycles of [35]. The main difference in the dynamics is that the two-dimensional faces of $\mathbf{S}_3$ are no longer flow invariant. As a result the trajectories can spiral around the edges of the cycle, which leads to the occurrence of complicated dynamics. In their numerical investigations the authors of [42] find a period doubling cascade and a Shilnikov-type homoclinic orbit. Some of this complicated dynamics may be due to a bifurcation from the cycle studied by Worfolk [98]. The work of [98] will be reviewed in Section 8.

A fascinating feature of the bifurcation studied by [42] is that it leads directly to chaotic dynamics. The authors of [42] named this phenomenon *instant chaos*. Another instance of instant chaos is described by Field and Peng [33, Appendix A].

It turns out that for all the vector fields discussed in this section the symmetry group has wreath product structure $\Delta \wr \mathbb{Z}_k$, where Δ is some Lie group and k is a positive integer. Let Σ be any subgroup of S_k (the group of permutations of k symbols) and let Δ act on $\mathbb{R}^n$. The action of $\Delta \wr \Sigma$ on $\mathbb{R}^{nk}$ is given as follows:

$$(\delta, \sigma)(x_1, \ldots, x_k) = (\delta x_{\sigma^{-1}(1)}, \delta x_{\sigma^{-1}(2)}, \ldots, \delta x_{\sigma^{-1}(k)}).$$

For more details on wreath products and the related bifurcation problems see [29]. Preliminary results of Dionne, Field, and Krupa [28] show that robust cycles are ubiquitous in steady-state bifurcations with symmetry $\Delta \wr \mathbb{Z}_k$.

5.1.2. Hopf Bifurcations. In generic Hopf bifurcations robust heteroclinic cycles have been found for the symmetries of planar lattices, namely $\mathbb{D}_k \cdot \mathbb{T}^2$, $k = 2, 4, 6$ [85], [84], and [23], respectively, and for problems with symmetry $\mathbb{Z}_k \cdot \mathbb{T}^2$, $k = 2, 4, 6$ [60], [59], and [96], respectively. These bifurcations are the Hopf bifurcation analogues of the Kuppers-Lortz instability studied by Busse and Heikes [19]. In this section we review the work of Silber et al. [85], Silber and Knobloch [84], Knobloch and Silber [59], [60], and Swift and Barany [96]. The article of Clune and Knobloch [23] will be discussed in connection with cycles in magnetoconvection in Section 11.

The normal form of a Γ-symmetric system near a Hopf bifurcation has additional $\mathbb{S}^1$ symmetry corresponding to phase shift [40]. Hence, near a Hopf bifurcation point, one is effectively studying a bifurcation problem with symmetry $\Gamma \times \mathbb{S}^1$. In Hopf bifurcations one encounters solutions with purely spatial symmetry, that is, with symmetry groups contained in $\Gamma \times \{\mathbf{1}\}$, and solutions having some spatio-temporal symmetry of the form (γ, σ), $\sigma \neq \mathbf{1}$. The former are called standing waves (patterns) and the latter traveling or rotating waves (patterns). In problems with lattice symmetry one usually refers to patterns having some translational symmetry (a nonzero $\mathbb{T}^2$ component) as rolls. Other types of patterns are usually called, according to their symmetry, hexagons, squares, or rectangles.

Silber et al. [85] classify the periodic solutions of the relevant $\mathbb{D}_2 \cdot \mathbb{T}^2 \times \mathbb{S}^1$ equivariant normal form system. They prove the existence of a heteroclinic cycle joining three types of these periodic orbits, namely traveling rolls, standing squares, and alternating rolls. Using the stability condition of [61] they prove that under certain conditions on the system's parameters and the normal form coefficients the cycle is asymptotically stable. Silber and Knobloch [84] carry out similar analysis for Hopf bifurcations with $\mathbb{D}_4 \cdot \mathbb{T}^2$ symmetry. They find a heteroclinic cycle joining three types of periodic solutions, namely alternating rectangles, standing rectangles, and standing squares. The cycle cannot be stable. They conjecture the existence of a stable heteroclinic cycle involving a quasiperiodic solution.

Knobloch and Silber [60] study Hopf bifurcations $\mathbb{Z}_4 \cdot \mathbb{T}^2$ finding a homoclinic cycle to a traveling roll solution. The cycle can be stable. Knobloch and Silber [59] find similar cycles in problems with symmetry $\mathbb{Z}_2 \cdot \mathbb{T}^2$ and $\mathbb{Z}_6 \cdot \mathbb{T}^2$. In the $\mathbb{Z}_6 \cdot \mathbb{T}^2$ symmetric problem another cycle, namely a homoclinic cycle to a standing square solution, was found by Swift and Barany [96]. This cycle is quite interesting for the following reason. Let ξ and $\gamma\xi, \gamma \in \mathbb{Z}_6 \cdot \mathbb{T}^2$ be two consecutive equilibria in the cycle and let Fix Δ be the fixed-point space containing the connection $\xi \to \gamma\xi$. The contracting eigenvalue of $\gamma\xi$ turns out to be complex. Consequently the cycle is of Shilnikov type and chaotic dynamics arise in its vicinity. The cycle obeys the stability condition of [61], that is, $c/e > 1$ is the necessary condition for stability. For $c/e \approx 1$ numerical experiments indicate the existence of a strange attractor.

5.2. Problems of Codimension Higher than 1

Codimension-two bifurcation problems provide a number of examples of robust heteroclinic cycles; see [4], [72], [71], [3], [20], [69], and [63]. In this section we review these examples concentrating on the work of Armbruster et al. [4], Melbourne et al. [71], Armbruster and Chossat [3], and Melbourne [69].

5.2.1. Steady-State Mode Interactions with O(2) Symmetry. Armbruster, Guckenheimer, and Holmes [4] and Proctor and Jones [72] were the first to find an example of a robust cycle in a codimension-two problem. They proved the existence of a cycle in steady-state mode interactions with $\mathbf{O}(2)$ symmetry. A related problem of steady state mode interactions with $\mathbb{D}_4$ symmetry was considered by Campbell and Holmes [20]. In this section we outline the proof of the existence of a cycle for the $\mathbf{O}(2)$ symmetric problem.

Consider a parametrized system with $\mathbf{O}(2)$ symmetry having an $\mathbf{O}(2)$-invariant equilibrium. Since the linearization around a symmetric equilibrium commutes with the $\mathbf{O}(2)$ action it follows that eigenspaces are $\mathbf{O}(2)$-invariant. Typical real eigenspaces on which at least some reflections and some rotations from $\mathbf{O}(2)$ act nontrivially can be identified with $\mathbb{C}$ and admit the following $\mathbf{O}(2)$ actions:

$$z \xrightarrow{\theta} e^{ik\theta}z \quad \text{for} \quad \theta \in \mathbf{SO}(2), \quad k \in \mathbb{Z}^+,$$

$$z \xrightarrow{\kappa} \bar{z}, \quad \kappa \text{ is a reflection.}$$

A *steady-state mode interaction* occurs when (at least) two eigenvalues corresponding to symmetry unrelated eigenvectors simultaneously pass through 0. Here we consider the situation when two real eigenspaces merge forming a generalized eigenspace. Armbruster et al. [4] consider the steady-state mode interaction of type $(1, 2)$ $(1 : 2$ resonance) occurring when the critical generalized eigenspace has the form $\mathbb{C} \times \mathbb{C}$ with the following action:

$$(z_1, z_2) \xrightarrow{\theta} (e^{i\theta} z_1, e^{2i\theta} z_2) \qquad \text{for} \quad \theta \in \mathbf{SO}(2),$$
$$(z_1, z_2) \xrightarrow{\kappa} (\bar{z}_1, \bar{z}_2), \qquad \kappa \text{ is a reflection.} \tag{5.2}$$

Note that any matrix commuting with this action must be diagonal. This implies that the Jordan block of the 0 eigenvalue at criticality is semisimple and hence must be the 0 matrix.

Remark 14. For PDE's the Fourier decomposition provides a decomposition into invariant spaces of the $\mathbf{O}(2)$ action—the k-th Fourier mode corresponds to the $z \xrightarrow{\theta} e^{ik\theta} z$ action. In the context of Fourier decomposition the bifurcation studied by [4] corresponds to the Fourier modes 1 and 2 simultaneously becoming unstable.

The center manifold theorem for symmetric systems [40] implies that the bifurcation problem may be reduced to an ODE on $\mathbb{C}^2$ commuting with the action (5.2) [4]. The relevant vector field, after suitable rescalings, has the normal form,

$$\dot{z}_1 = \bar{z}_1 z_2 + z_1(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2) + \text{hot},$$
$$\dot{z}_2 = \pm z_1^2 + z_2(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2) + \text{hot}. \tag{5.3}$$

Consider the groups: $\mathbb{Z}_2 = \{1, \kappa\}$ and $\tilde{\mathbb{Z}}_2 = \{1, \kappa\pi\}$. The corresponding fixed-point spaces are

$$\text{Fix}(\mathbb{Z}_2) = \{(z_1, z_2) = (x, y), x, y \text{ real}\},$$

and

$$\text{Fix}(\tilde{\mathbb{Z}}_2) = \{(z_1, z_2) = (ix, y), x, y \text{ real}\}.$$

Consider (5.3) with the higher order terms neglected. Armbruster et al. [4] formulate conditions on the coefficients e_j, $j = 1, \ldots, 4$ and μ_j, $j = 1, 2$, implying the existence of a saddle point $(0, y_*) \in \text{Fix}(\mathbb{Z}_2) \cap \text{Fix}(\tilde{\mathbb{Z}}_2)$. Additional conditions on e_j and μ_j imply the existence of an attracting neighborhood of 0 containing the $\mathbf{O}(2)$ group orbit of $(0, y_*)$ and no other equilibria. The existence of a homoclinic cycle joining $\xi_1 = (0, y_*)$ to $\xi_2 = (0, -y_*)$ and, by symmetry, ξ_2 to ξ_1 follows from the Poincaré-Bendixson theorem provided that additional conditions on μ_j and e_j are satisfied. Such a cycle is shown in Figure 8. The cycle can still exist even if these additional conditions fail as long as ξ_1 is a saddle point. Recently Sandstede and Scheel [78] determined exactly the region of the existence of the homoclinic cycle. To obtain this result they rescaled (5.3) and applied singular perturbation analysis.

5.2.2. Hopf-Steady State and Hopf-Hopf Mode Interactions with O(2) Symmetry. Melbourne, Chossat, and Golubitsky [71] consider Hopf-steady state and Hopf-Hopf mode interactions with $\mathbf{O}(2)$ symmetry. Recall that a generic Hopf bifurcation in a system with $\mathbf{O}(2)$ symmetry gives rise to two periodic solutions, a standing wave and a rotating

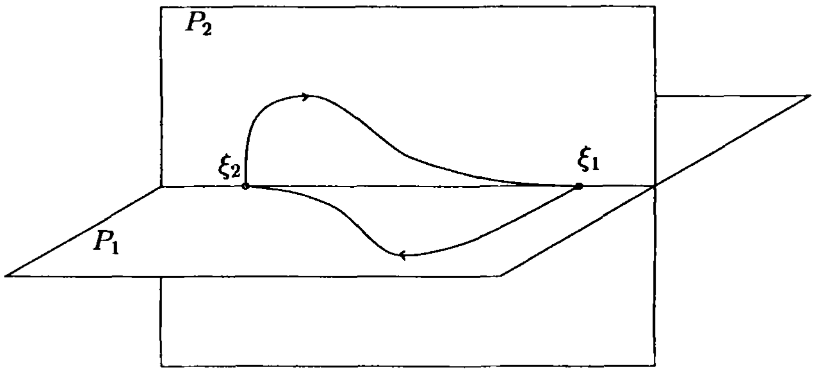

Fig. 8. The homoclinic cycle in the steady-state mode interaction with **O**(2) symmetry.

wave. More precisely there are two group orbits of periodic solutions, one corresponding to rotating waves and one to standing waves. The symmetry of an individual rotating wave as a set is **SO**(2) and a standing wave has the symmetry of a reflection. For more details see [40].

(a) Steady State-Hopf. As mentioned in Section 5.2.1 eigenvalues of the linearization at an invariant equilibrium for problems with **O**(2) symmetry may be double. For a steady state-Hopf interaction the critical eigenspace can have the form $\mathbb{C} \oplus \mathbb{C}^2$, with the **O**(2) action given by

$$(z_1, z_2, z_3) \xrightarrow{\theta} (e^{li\theta} z_1, e^{mi\theta} z_2, e^{-mi\theta} z_3) \qquad \text{for} \quad \theta \in \mathbf{SO}(2),$$

$$(z_1, z_2, z_3) \xrightarrow{\kappa} (\bar{z}_1, z_3, z_2), \qquad \kappa \text{ is a reflection.}$$

Heteroclinic cycles occur when the critical eigenspace has this form. The primary bifurcating solutions are an equilibrium, a standing wave, and a rotating wave. Melbourne et al. [71] prove the existence of a heteroclinic cycle joining the steady state and the standing wave when $l = m$.

(b) Hopf-Hopf. In a Hopf-Hopf mode interaction cycles may occur when the critical eigenspace is of the form $\mathbb{C}^2 \oplus \mathbb{C}^2$ with the action of **O**(2) given by

$$(z_1, z_2, z_3, z_4) \xrightarrow{\theta} (e^{li\theta} z_1, e^{-il\theta} z_2, e^{mi\theta} z_3, e^{-mi\theta} z_4) \qquad \text{for} \qquad \theta \in \mathbf{SO}(2),$$

$$(z_1, z_2, z_3, z_4) \xrightarrow{\kappa} (z_2, z_1, z_4, z_3), \qquad \kappa \text{ is a reflection.}$$

The primary solutions are two pairs of standing waves and rotating waves, a pair corresponding to each mode. For $l = m$ Melbourne et al. [71] find three types of heteroclinic cycles, namely a cycle connecting two rotating waves, a cycle connecting two standing waves, and a cycle connecting all four periodic solutions.

In both the steady state-Hopf and the Hopf-Hopf mode interactions the spaces P_j are group orbits of two-dimensional vector spaces under the product of the actions of **O**(2)

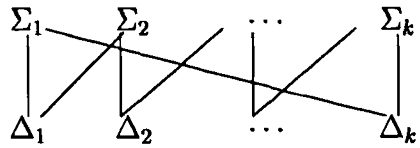

Fig. 9. Lattice connections suggesting the existence of a heteroclinic cycle.

and the normal form symmetry ($\mathbb{S}^1$ in the steady state-Hopf case and $\mathbb{T}^2$ in the Hopf-Hopf case). Using this property Melbourne et al. [71] are able to apply the Poincaré-Bendixson theorem and derive sufficient conditions for the existence of robust heteroclinic cycles. These conditions are not necessary. When they fail, phase plane simulation can be used to provide evidence for the existence of cycles.[5]

The cycles described in this section were found for the normal form equations which have additional $\mathbb{S}^1$ symmetry (steady state-Hopf) or $\mathbb{T}^2$ symmetry (Hopf-Hopf). Since the normal form symmetry is only approximate, in the full system the cycles are expected to be replaced by nearby intermittent dynamics. The situation is similar to that in [69]; see also Section 5.2.4.

Using $\mathbb{S}^1$ (respectively $\mathbb{T}^2$) symmetry of the normal form, one can realize periodic orbits as relative equilibria, or, in other words, flow-invariant group orbits. This allows for an easy generalization of the stability methods of [61]. It turns out that all cycles can be asymptotically stable. Moreover, Krupa and Melbourne [62] show that the cycle occurring in the steady state-Hopf interaction and two of the cycles occurring in the Hopf-Hopf interaction, namely the cycle joining the rotating waves and the cycle joining the standing waves, could be almost asymptotically stable, without being asymptotically stable. In Hopf-Hopf interaction almost asymptotic bistability of the two mentioned cycles is possible.

Remark 15. Melbourne et al. [71] find a useful algebraic indicator for the existence of robust cycles. Recall that the isotropy group of a point $x \in \mathbb{R}^n$ is defined as

$$\Sigma_x = \{\sigma \in \Gamma \colon \sigma x = x\}.$$

The equivalence classes of isotropy subgroups with inclusion form a partially ordered set, often referred to as the *isotropy lattice*. Melbourne et al. [71] notice that certain configurations in the isotropy lattice suggested the existence of heteroclinic cycles. Let $\xi_1, \dots, \xi_k$ be the equilibria in the cycle and $\Sigma_1, \dots, \Sigma_k$ their isotropy subgroups. Suppose $P_1, \dots, P_k$ are fixed point spaces of the groups $\Delta_1, \dots, \Delta_k$. Clearly $\mathrm{Fix}\,\Sigma_i \subset \mathrm{Fix}\,\Delta_{i-1} \cap \mathrm{Fix}\,\Delta_i$. It often happens that Δ_{i-1} and Δ_i are maximal isotropy subgroups properly contained in Σ_i. The isotropy lattice then contains a configuration of the type shown in Figure 9. Conversely, finding such configurations in the isotropy lattice suggests the existence of heteroclinic cycles.

[5] Sharp conditions could probably be obtained using the methods of Sandstede and Scheel [78].

5.2.3. Steady State Mode Interactions with O(3) Symmetry. Armbruster and Chossat [3] consider steady-state mode interactions with $\mathbf{O}(3)$ symmetry. Their article follows up on the work of Friedrich and Haken [37] who consider steady-state interactions for a convection problem in a spherical shell and numerically locate a number of heteroclinic cycles. These and other cycles were found analytically in the work of [3].

Recall that the irreducible representations of $\mathbf{O}(3)$ are absolutely irreducible and are given by the spaces V_l generated by spherical harmonics of order l (for background on bifurcations with $\mathbf{O}(3)$ symmetry see [40]). Armbruster and Chossat [3] consider an equilibrium with two eigenvalues 0 and the eigenspace of 0 given by $V_1 \oplus V_2$ and study the unfolding of this singularity. They find three types of heteroclinic cycles. In order to rigorously establish the existence of the cycles they vary a tertiary parameter (a coefficient of a higher order term of the normal form). The three types of cycles found in [3] are:

(a) A heteroclinic cycle joining two axisymmetric solutions with maximal isotropy types. This cycle joins equilibria having the same isotropy type but not lying on the same group orbit. The cycle is not stable in the region where it is theoretically found, but Armbruster and Chossat [3] locate it numerically for other values of the coefficients, where it is asymptotically stable. A sufficient condition of [71] is used to confirm asymptotic stability.
(b) A heteroclinic cycle joining a maximal solution and a submaximal solution, both axisymmetric. The cycle is stable. Here the condition of [71] is too strong and stability follows from the condition of [61]. Since this condition was not known at the time, Armbruster and Chossat [3] conclude stability based on numerical results and the conjecture of Armbruster [2]. The submaximal equilibrium can lose stability in a Hopf bifurcation leading to the appearance of a heteroclinic cycle between an equilibrium and a periodic orbit.
(c) A homoclinic cycle involving a submaximal nonaxisymmetric solution. This cycle is analogous to the one found by [4]; see Figure 8. The cycle can be stable. Armbruster and Chossat [3] conjecture that this cycle can explain the mechanism of the aperiodic reversal of the Earth's magnetic dipole field in the geological times.

The work of [3] has recently been extended by Guyard [43], who studied mode interactions with $\mathbf{O}(3)$ symmetry for different irreducible representations. He looked for heteroclinic cycles of the type shown in Figure 8 and obtained a complete classification for the eigenspaces of 0 equal to $V_l \oplus V_{l+1}$.

5.2.4. Triple and Quadruple Hopf Bifurcations. Melbourne [69] and Krupa, Melbourne, and Scheel [63] consider triple and quadruple Hopf bifurcations. Here the systems under consideration are not assumed to have any symmetry, but still they behave very similarly to symmetric systems. The reason is that the Birkhoff normal forms of the considered systems have symmetry, $\mathbb{T}^3$ in the case of a triple Hopf bifurcation and $\mathbb{T}^4$ in the case of a quadruple Hopf bifurcation (the symmetry of a three torus and a four torus, respectively). Near the bifurcation points the vector field can be written as a small perturbation of the normal form. For the normal-form vector field one can prove the existence of a heteroclinic cycle. Upon the addition of the perturbation breaking the normal-form symmetry the cycle is replaced by a nearby invariant set supporting intermittent dynamics similar to this on the cycle. The main difference is in the transition times which, for the perturbed system, are erratic and tend to be bounded (extremely long return times occur infrequently).

We now sketch the analysis for the triple Hopf bifurcation. Center manifold and Birkhoff normal form techniques lead to a vector field on $\mathbb{C}^3$. Under finitely many nondegeneracy conditions (corresponding to the absence of strong resonances) this vector field, truncated at cubic order, decouples into phase-amplitude equations. The amplitude equations give a three-dimensional vector field with $\mathbb{Z}_2^3 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetry, each copy of $\mathbb{Z}_2$ being the remnant of the phase-shift symmetry from one of the Hopf modes. Coordinates (x_1, x_2, x_3) may be chosen so that the group of symmetries is generated by the reflections

$$(x_1, x_2, x_3) \rightarrow (\pm x_1, \pm x_2, \pm x_3).$$

Then the equivariance of the vector field guarantees that each coordinate axis and plane is flow-invariant and it is easy to check that for an open set of equivariant vector fields, there is a heteroclinic cycle between three equilibria lying on the coordinate axes. This cycle lifts to a cycle between three periodic solutions for the six-dimensional truncated vector field. Finally, if the cycle is asymptotically stable, then it is possible to deduce heteroclinic-like behavior for the full bifurcation problem.

Analogous results may be obtained in the case of a quadruple Hopf bifurcation. For this case a number of heteroclinic cycles can be found, in particular planar, that is, joining equilibria in three coordinate planes, and nonplanar, that is, joining equilibria in four coordinate planes. The reduced $\mathbb{Z}_2^4$ symmetric case was studied by Kirk and Silber [56], who addressed mainly the question of the existence of heteroclinic networks and their stability. The situation is analogous as in the case of heteroclinic cycles on the simplex for the replicator equations of mathematical biology and game dynamics (Section 7); see Brannath [14] for the most up to date account.

Similar considerations hold for a k-tuple Hopf bifurcation, where k is a positive integer.

6. Creation of Cycles through Forced Symmetry Breaking

Consider a differential equation commuting with a symmetry group Γ and suppose that small terms are added to the equation which break the symmetry of Γ. In other words the group of symmetries of the perturbed equation is a proper subgroup of Γ. This phenomenon is called *forced symmetry breaking*. As illustrated in Section 3, heteroclinic cycles may be created as a result of forced symmetry breaking. More specifically, one considers an invariant group orbit M (a relative equilibrium), which, under a symmetry-breaking perturbation of the vector field, remains invariant for the flow but is not a single group orbit for the action of the smaller group. The manifold M is equivariantly diffeomorphic to Γ/K, where K is the isotropy subgroup of the elements of M under the action of Γ. As indicated in Section 3 such perturbations may lead to the occurrence of heteroclinic cycles on M. Lauterbach and Roberts [66] consider forced symmetry breaking from the symmetry of $\mathbf{SO}(3)$. They study the following three cases: the dynamics on $\mathbf{SO}(3)/\mathbf{O}(2)$ under the symmetry breaking to $\mathbf{T}$ (the group of orientation-preserving symmetries of a tetrahedron), the dynamics on $\mathbf{SO}(3)/\mathbf{T}$ under the symmetry breaking to $\mathbf{O}(2)$, and the dynamics on $\mathbf{SO}(3)/\mathbf{T}$ under the symmetry breaking to $\mathbb{D}_n$. In the first two cases they find homoclinic cycles which could be asymptotically stable. In the third case, depending on the parity of n, they find a pair of homoclinic cycles or a homoclinic network consisting of two homoclinic cycles. Of the pair of homoclinic cycles one has to be unstable and the other could be asymptotically stable. In the heteroclinic network each of the two cycles, but not both simultaneously, can be almost asymptotically stable (see also [70]).

Armbruster and Ihrig [6] recover some of the results of [66] using topological methods.

The work of [66] was extended by Lauterbach et al. [65] and Lauterbach and Maier-Paape [64]. Let L denote the group of the remaining symmetries acting on $M = \Gamma/K$. Lauterbach et al. [65] define graphs $G_{(L,\Gamma/K)}$ whose vertices are equilibria and edges connected components of fixed point spaces of the action of L on G/K. They also consider projected graphs $G^p_{(L,G/K)}$ obtained from $G_{(L,G/K)}$ by reducing the action of L. Finding heteroclinic cycles is equivalent to finding closed paths in such graphs. Lauterbach et al. [65] classify the graphs $G^p_{(L,G/K)}$ for $\Gamma = \mathbf{O}(3)$. A consequence of this classification is that homoclinic cycles are admitted for $(L, K) \in \{(\mathbf{T}, \mathbf{T}), (\mathbf{T}, \mathbb{O}), (\mathbb{O}, \mathbf{T}), (T, \mathbf{O}(2)), (\mathbf{O}(2), \mathbf{T})\}$, where $\mathbb{O}$ denotes the group of orientation-preserving symmetries of the octahedron. Lauterbach and Maier-Paape [64] consider reaction diffusion equations on a sphere in $\mathbb{R}^3$. They show that for some perturbations breaking the symmetry to $\mathbf{T}$ there exists a homoclinic cycle.

Golubitsky and Hou [50] (see also [49]) present a simple and elegant example of a heteroclinic cycle arising through forced symmetry breaking. For the review of their work see Section 3.

7. Cycles in Mathematical Biology and Game Theory

Robust heteroclinic cycles can be found in the models coming from biology and game theory (MBGT). The first example of such a cycle was given by May and Leonard [68]. In this section we describe the mechanism of the existence of cycles using the example of the Lotka-Volterra equations and the replicator equations. The exposition is based on the presentation in Hofbauer and Sigmund [47] and in Hofbauer [46].

7.1. The Existence of Cycles

The Lotka-Volterra equations,

$$\dot{x}_i = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right), \qquad i = 1, \ldots, n, \tag{7.1}$$

model various phenomena in biology, e.g., predator-prey systems. The variables x_j correspond to different species of a biological system and are assumed to be nonnegative. After a suitable change of coordinates one can transform (7.1) to the replicator equations,

$$\dot{x}_i = x_i \left[(Ax)_i - x \cdot Ax \right], \qquad i = 1, \ldots, n, \tag{7.2}$$

which are posed on the simplex $\mathbf{S}_n = \{x \in \mathbb{R}^n \;\; \sum_{i=1}^n x_i = 1\}$. The equations (7.2) are also used as a model in game theory.

Note that the coordinate hyperplanes

$$H_{j_1 \dots j_k} = \{(x_1, \dots, x_n) \ : \ x_{j_1} = \dots = x_{j_k} = 0\}$$

are invariant for the flow of (7.1) and (7.2). There are many examples of heteroclinic cycles and networks joining equilibria in the hyperplanes $H_{j_1 \dots j_k}$ [45], [47], [57], [46], [14], including the already mentioned cycle of May and Leonard [68]. The occurrence of heteroclinic cycles and networks depends of course on the signs and relative sizes of r_i and $a_{ij}, i, j = 1, \dots, n$. The following is an example of a heteroclinic cycle occurring for (7.2). Note that adding a number to a column of A does not alter (7.2) restricted to $\mathbf{S}_n$. Hence we may assume with no loss of generality that $a_{ii} = 0, \quad i = 1, \dots, n$. Let $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0)$, etc. Clearly e_j is an equilibrium of (7.2). Note also that the eigenvalues of the linearization of (7.2) at e_j in the directions tangent to $\mathbf{S}_n$ are given by the nondiagonal entries of A in the j-th row. Under some additional assumptions on A implying the nonexistence of equilibria on the one-dimensional edges joining e_j to e_{j+1} there exists a heteroclinic cycle joining $e_1 \to e_2 \to \dots \to e_n \to e_1$. For $n = 3$ this is the cycle of May and Leonard [68].

An interesting property of robust cycles in general was discovered in the context of MBGT by Gaunersdorfer [38], who showed that the time averages of trajectories converging to heteroclinic cycles do not converge. The work of [38] was extended by Takens [99], who classified the topological equivalence classes for the dynamics local to a simple heteroclinic cycle.

7.2. Stability

As reported in Section 4 stability of cycles can be computed using Poincaré sections and return maps. For (7.1), (7.2), and their generalizations the transition matrix method, whose special case was described in Section 4, is applicable. This method was used by [47] to obtain asymptotic stability conditions and by [14] to analyze nonasymptotic stability for heteroclinic networks on $\mathbf{S}_n$. Hofbauer [44] presented an alternative method based on average Liapunov functions. Hofbauer [46] obtained various stability criteria, in particular the following explicit criterion for the cycle described in the preceding paragraph:

Theorem 3. *Let A' be the matrix obtained from A by reordering the columns so that only the diagonal has positive entries. Suppose* $\det A' \neq 0$. *The cycle is unstable if all leading principal minors of A' are positive. The cycle is stable if at least one of the leading principal minors is negative.*

Sketch of the proof. If all leading principal minors of A' are positive then A' is an M matrix. An equivalent characterization of this property is that there exists a vector $p > 0$ (that is, $p = (p_1, \dots, p_n)$ and all p_j's are positive) such that $A'p > 0$. Conversely, if at least one leading principal minor of A' is negative (A' is not an M matrix) then there exists $p < 0$ such that $A'p > 0$. Since $A = A'S$ for some permutation matrix S, analogous properties hold for A and the vector $q = S^{-1}p$.

Suppose that there exists $p > 0$ such that $Ap > 0$. Consider the *average Liapunov function* $P(x) = \prod x_i^{p_i}$. Let $x(t)$ be a trajectory of the system passing nearby the cycle.

Then $\frac{\dot{P}}{P} = \sum_{i=1}^{n} p_i \frac{\dot{x}_i}{x_i}$. It follows that when $x(t)$ is near ξ_j then $\frac{\dot{P}}{P} \approx (Ap)_j > 0$. Since $x(t)$ spends most of the time near the vertices ξ_j it must be repelled from the cycle.

Suppose that there exists $p < 0$ such that $Ap > 0$. Consider $P(x) = \prod x_i^{-p_i}$. Then $\frac{\dot{P}}{P} = -\sum_{i=1}^{n} p_i \frac{\dot{x}_i}{x_i}$. It follows that when $x(t)$ is near ξ_j then $\frac{\dot{P}}{P} \approx -(Ap)_j < 0$. Since $x(t)$ spends most of the time near the vertices ξ_j it must be attracted to the cycle. □

An interesting issue related to stability is *permanence* of the biological system. It is important to note that the equilibria in the hyperplanes $H_{j_1 \ldots j_k}$ correspond to some species being extinct. Near such equilibria deterministic models are usually replaced by stochastic ones. This puts the question of stability of cycles in a different light. One may consider the problem of permanence of a biological system, that is, whether all the species are going to survive. Certainly a necessary condition for permanence is that the heteroclinic cycles are *unstable*.

7.3. Comparison between Cycles Found in Systems of MBGT and in Systems with Symmetry

Consider a dynamical system on $\mathbb{R}^n$ commuting with the action of $(\mathbb{Z}_2)^n$ given by reflections in the coordinate axes (e.g., the amplitude equations for the n-tuple Hopf bifurcation [69], [63]) or with the action of $(\mathbb{Z}_2)^n \cdot \mathbb{Z}_n$, where the action of $\mathbb{Z}_n$ is generated by the cycle $(x_1, \ldots, x_n) \to (x_n, x_1, \ldots, x_{n-1})$ (e.g., the steady-state bifurcation problem considered in [34] and [35]). Note that for these actions the coordinate hyperplanes must be invariant, just as in the case of the equations (7.1) and (7.2). Conversely the equation (7.1) can be converted to an equation with $\mathbb{Z}_2^n$ symmetry by letting $x_j = u_j^2$. Consequently there are many similarities in the geometry of the flow for the systems with these types of symmetry and for the flow of (7.1) or (7.2). In particular cycles joining the vertices of the simplex are analogous to the edge cycles on the invariant sphere of the steady-state bifurcation problem [34], [35] and to the cycles in n-tuple Hopf bifurcations [69], [63]. The transition matrix method of determining stability is also analogous.

8. Bifurcations from Robust Cycles

Bifurcations from robust heteroclinic cycles are a natural problem to consider and have been given some attention even in the earliest research. Two types of bifurcations have attracted the attention of researchers. The first type of problem is to analyze the bifurcation arising as a result of a change in the dynamics near the cycle, for example the loss of stability of the cycle. The second type of problem is related to forced symmetry breaking. The question is: What happens to the dynamics, if due to small, symmetry breaking perturbations the invariance of the planes P_j is destroyed? In this section we review the research on the two types of bifurcations.

8.1. A Change in the Dynamics near the Cycle

The most significant example of this type of bifurcation is the loss of asymptotic stability of the cycle. It follows from the discussion in Section 4 that there are two ways in which a cycle can lose asymptotic stability. The first possibility occurs when one of the transverse eigenvalues (t_j's) becomes positive. This is called a *transverse bifurcation*. The second possibility occurs when the intersection of the cycle with the appropriate section of the flow becomes a repellor for iteration of the return map. Depending on the type of the cycle this is equivalent to a change of the direction of inequality in either (4.8), or in the condition given in Theorem 2, or in some other algebraic inequality. For this reason this type of bifurcation is called a *resonant bifurcation*.

The first result on resonant bifurcations was obtained by Hofbauer [45] and Hofbauer and Sigmund [47] for heteroclinic cycles in systems of mathematical biology and game theory (see Section 7). They show that the loss of stability of a certain cycle led to the appearance a periodic orbit with a long period.

Armbruster, Guckenheimer, and Holmes [4] analyze resonant bifurcations of the heteroclinic cycle occurring in the mode interaction problem they considered (see Section 5.2.1). They show, using elliptic integrals, that, as the homoclinic cycle they find looses stability, it gives rise to a modulated traveling wave. Their analysis relies on the presence of a two-dimensional symmetry group (the underlying $\mathbf{O}(2)$ symmetry and the normal-form symmetry $\mathbf{SO}(2)$). Armbruster [2] considers a generalization of the example of [4] and shows that, as the equilibria in the cycle undergoes a Hopf bifurcation, other cycles are created, joining periodic solutions of standing wave type. Scheel and Chossat [82] consider resonant bifurcations from a homoclinic cycle of the same configuration as the cycle studied in [4] (see Figure 8) occurring in the presence of $\mathbb{D}_4$ symmetry. They pose the problem in an abstract setting not related to any specific problem and show that, for the class of cycles for which $c < e$ typically implies instability, a resonant bifurcation leads to the appearance of a long-period periodic orbit. The distance of the periodic orbit from $\mathrm{Fix}(\Delta)$ is a flat function of $1 - c/e$.

Campbell and Holmes [21] consider the example of [2] and study transverse bifurcations. They conjecture the existence of at least three branches of quasiperiodic solutions near the original cycle. Their conjecture is based on a mixture of theoretical considerations and numerical results.

Chossat et al. [24] consider transverse bifurcations of homoclinic cycles in $\mathbb{R}^4$. For cycles satisfying (H2) (cf. Section 4) a flat bifurcation to a long-period periodic orbit is found. For cycles violating (H2) the bifurcation leads to the occurrence of other heteroclinic cycles. A classification of resonant and transverse bifurcations of heteroclinic cycles in $\mathbb{R}^4$ will be given in the forthcoming work of [63].

Guckenheimer and Worfolk [42] consider a steady-state bifurcation problem with symmetry of an index two subgroup of $(\mathbb{Z}_2)^4 \times \mathbb{Z}_4$ (equivalently $\mathbb{Z}_2 \wr \mathbb{Z}_4$, cf. Section 5.1.1). Worfolk [98] extends the work of [42], showing that the cycle found by [42] undergoes a bifurcation of *inclination-flip type* [48]. Worfolk [98] shows that, under some genericity assumptions, the bifurcation leads to the existence of a horseshoe and provides numerical evidence supporting the validity of these genericity assumptions.

8.2. Destruction of a Cycle through Forced Symmetry Breaking

Swift [95] and Soward [86] show that forced symmetry breaking for the cycle of Busse and Heikes [19] leads to the occurrence of a long-period periodic orbit. Generalizing the work of [95] and [86], Scheel [81] considers homoclinic cycles with symmetry $\mathbb{Z}_2^3 \cdot \mathbb{Z}_3$

(Figure 2) and $\mathbb{D}_4$ (Figure 8). He projects the equations of motion onto the orbit space and applies the method of Lin [67]. In the case of $\mathbb{Z}_2^3 \cdot \mathbb{Z}_3$ (this case is the abstract realization of the bifurcation studied by [95] and [86]) he considers symmetry breaking to $\mathbf{T}$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$, finding periodic orbits with spatio-temporal symmetry $\mathbb{Z}_3$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$, respectively. In the case of $\mathbb{D}_4$ he considers symmetry breaking to $\mathbb{D}_2$, $\mathbb{Z}_2$, and $\mathbb{Z}_4$, obtaining periodic orbits with symmetry $\mathbb{Z}_2$, $\mathbb{Z}_2$, and $\mathbb{Z}_4$, respectively.

Sandstede and Scheel [78] extend the work of Scheel [81]. They study the same types of forced symmetry breaking as considered in [81] and obtain precise results on the number of existing periodic orbits and their stability as well as information on the existence of horseshoes and strange attractors. They exploit the presence of *inclination-flip* and *orbit flip* degeneracies for the unbroken homoclinic cycles and apply the expanded version of Lin's method worked out in the thesis of Sandstede [80].

Chossat [25] studies the example of [4] (see Section 5 and Figure 8) and considers forced symmetry breaking $\mathbf{O}(2) \to \mathbf{SO}(2)$. He applies the orbit space reduction and the method of Lin. He proves the existence of an invariant two torus. One of the frequencies of the motion is given by slow drift along the $\mathbf{SO}(2)$ orbit. Chossat and Field [26] consider the same example and study forced symmetry breaking $\mathbf{O}(2) \to \mathbf{SO}(2)$ and $\mathbf{O}(2) \to \mathbb{Z}_2(\kappa)$ (the group $\mathbb{Z}_2(\kappa)$ is generated by a reflection in $\mathbf{O}(2)$). They apply polar blowing up, a transformation leading to an enlarged phase space with a less singular action. Polar blowing up may simplify the geometry of the flow. It also renders the orbit space reduction more effective, since the orbit space is a manifold. Chossat and Field [26] recover the result of [25] using elementary phase plane analysis. In the case of $\mathbf{O}(2) \to \mathbb{Z}_2(\kappa)$ they state a result asserting the existence of a long-period periodic orbit. The geometry of the periodic orbit depends on the ratio of the contracting and the expanding eigenvalues (for notation see Section 4). The proof of the result requires an application of Shilnikov coordinates and will be given in the forthcoming article of Field [32]. Near the critical ratio of the eigenvalues chaotic dynamics is observed.

PART III

Part III is devoted to experimental and numerical applications. We discuss the following topics: dynamics of the Kuramoto-Sivashinsky equation (Section 9), rotating convection (Section 10), convection in the presence of a magnetic field (Section 11), turbulent flows in a boundary layer (Section 12), and flow through a hose pipe (Section 13). In these contexts there is good evidence for the existence of robust cycles.

9. The Kuramoto-Sivashinsky Equation

In this section we summarize the results of Armbruster, Guckenheimer, and Holmes [5] and Kevrekidis, Nicolaenko, and Scovel [55] concerning the existence of a robust homoclinic cycle for the Kuramoto-Sivashinsky (KS) equation. The KS equation

$$u_t + \alpha u_{xxxx} + u_{xx} + \frac{1}{2}(u_x)^2 = 0; \qquad x \in \mathbb{R}, \tag{9.1}$$

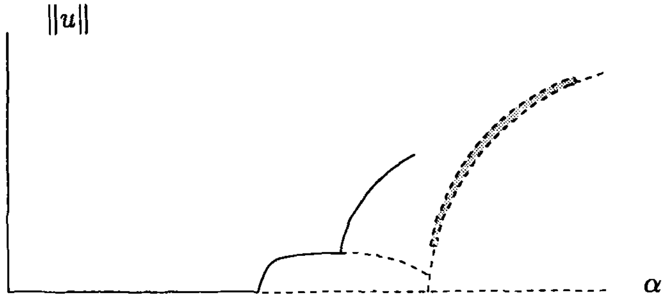

Fig. 10. Schematic bifurcation diagram for the Kuramoto-Sivashinsky equation.

has been used to model flame fronts in combustion, directional solidification, and weak two-dimensional turbulence (for references see [5]). This equation has symmetries given by translations and reflections in the state variable, that is,

$$u(x,t) \stackrel{\theta}{\to} u(x+\theta,t), \qquad \theta \in \mathbb{R},$$

$$u(x,t) \stackrel{\kappa}{\to} u(-x,t).$$

Hence (9.1) commutes with the action of $\mathbb{R}\cdot\mathbb{Z}_2$. We impose periodic boundary conditions,

$$u(0,t) = u(L,t), \qquad u_x(0,t) = u_x(L,t), \ldots,$$

where $L > 0$ is a constant. Consider the rescaling

$$x \to Lx, \qquad t \to L^2 x, \qquad \alpha \to L^2\alpha. \tag{9.2}$$

The rescaling (9.2) does not alter the equation (9.1), but leads to the rescaling of the boundary conditions, which become 2π periodic. The period L is now involved in the definition of the parameter α. The symmetry group of (9.1) with periodic boundary conditions is $\mathbf{O}(2)$. This follows, since the translation symmetries now have to be taken modulo 2π and form the group $\mathbf{SO}(2) = \mathbb{R}/[0, 2\pi]$.

Note that the function $u \equiv 0$ is an equilibrium solution of (9.1). When L increases, this solution undergoes a sequence of bifurcations to equilibria with symmetry $\mathbb{D}_k$, $k = 1, 2, \ldots$, where $\mathbb{D}_1 \stackrel{\text{def}}{=} \mathbb{Z}_2$. The bifurcations occur as the Fourier modes $\{sin(kx), cos(kx)\}$ become unstable. More abstractly this means that the action of $\mathbf{SO}(2)$ on the eigenspace of the zero eigenvalue is $\theta z = e^{ik\theta}z$. We refer to the bifurcating equilibria as the *k-modes*. The first two bifurcations of this sequence are shown in Figure 10. Near the onset of the mode with $\mathbb{D}_2$ symmetry a heteroclinic-like structure was found numerically by Hyman and Nicolaenko [53] and Hyman et al. [54]. In Figure 10 we show a schematic bifurcation diagram based on the results of Kevrekidis et al. [55]. The locus of the existence of the heteroclinic cycles is indicated by the shaded strip. Armbruster et al. [5] and Kevrekidis et al. [55] obtain evidence of the existence of a robust homoclinic cycle near the onset of the two-mode using two different approaches. Kevrekidis et al. [55] discretize (9.1) using an eight-mode and a sixteen-mode Galerkin projection. In

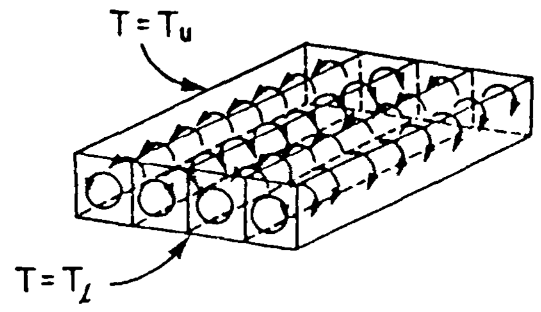

Fig. 11. The convection rolls (copied from a figure in [40]).

their computations they find the homoclinic cycle and describe its structure, i.e., that the connections were contained in invariant spaces and were robust. Armbruster et al. [5] study the dynamics of a fourth order truncation of (9.1) reduced to the center unstable manifold of the two-mode near its onset. For this system they prove the existence of a robust heteroclinic cycle as well as reproduce the diagram shown in Figure 10. Their analysis of the reduced equation is very similar to the analysis of the reduced problem of [4]; see Section 5.2.1 for a review.

The work of [5] and [55] provides good evidence of the existence of cycles but the result is not rigorously proved in either of the articles. It is unlikely that an analytical proof can be found since global information is required.

10. Rotating Rayleigh-Bénard Convection

Busse and Clever [18] (see also [15], [19], [95]) argue that the dynamics of the rotating Rayleigh-Bénard convection observed at some values of the system's parameters has gross features consistent with those of a robust homoclinic cycle. One is interested in the following experimental situation. A viscous fluid is contained in a rectangular box, whose side walls are insulated and whose upper and lower faces are held at constant temperature. The box is rotating around the vertical axis with angular velocity Ω. Suppose the horizontal faces of the box are very large compared to its height. Then one can assume that the horizontal walls extend to infinity and thus consider convection between two infinite plates.

A natural parameter used in the context of the convection experiment is the Rayleigh number Ra. It measures the temperature difference across the layer. When Ra is small the pure conduction state PC is observed. As Ra grows PC loses stability to convection rolls (see Figure 11). For Ω very small the rolls are stable, but for slightly higher values of Ω they are unstable and a weakly turbulent state is observed. Figure 12, which appeared in [19], shows a number of snapshots of the experiment. (For a more recent account of the experiment see [100].) The pictures were made using the shadowgraphic technique. The dark areas represent rising fluid and the light areas represent falling fluid. Following the evolution of a patch of rolls one sees that it remains stationary for some time and then switches fairly quickly to rolls rotated by an angle of roughly 60 degrees.

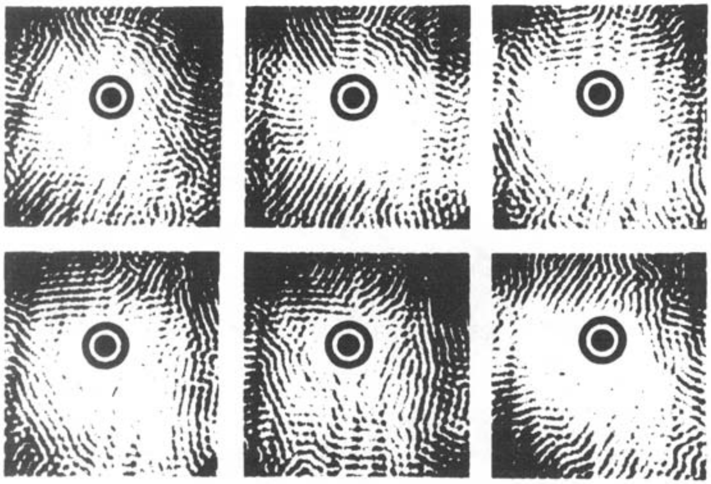

Fig. 12. Homoclinic cycles in convection. From [19]. Reprinted with permission from F. M. Busse and K. E. Heikes. Convection in a rotating layer: A simple case of turbulence. *Science* **208**, 173–175. Copyright 1980 American Association for the Advancement of Science.

Busse and Clever [18] consider the bifurcation corresponding to the loss of stability of the pure conduction state and derive a three-dimensional ordinary differential equation with cubic nonlinearity using asymptotic expansion. In this model they numerically verify the existence of the cycle. Guckenheimer and Holmes [41] prove the existence of a homoclinic cycle for the system of [18] using the Poincaré-Bendixson theorem. For an extensive analysis of the system studied by [18] and [41], see Section 2 of this article. Goldstein et al. [39], under certain idealizing assumptions, reduce the bifurcation problem for the Boussinesq equation to a bifurcation problem for the system of [18] using the center manifold theorem and the methods of equivariant bifurcation theory. We now review the results of [39].

Goldstein et al. [39] assume that the box laterally extended to infinity, and they make the idealizing assumption that the centrifugal force is balanced by pressure. Under these assumptions the experiment has the symmetry of the special Euclidian group $\mathbf{SE}(2) = \mathbf{SO}(2) \cdot \mathbb{R}^2$, $\mathbf{SO}(2)$ corresponding to rotations and $\mathbb{R}^2$ to translations in the horizontal directions. An additional symmetry of midplane reflection is due to the Boussinesq approximation and the assumption that the boundary conditions at the bottom and the top plates are identical.

One is now interested in studying a steady-state bifurcation problem with symmetry $\mathbf{SE}(2)$. Note however that $\mathbf{SE}(2)$ is not a compact group (the effect of this is discussed in [40]). It turns out that the eigenspace of the zero eigenvalue associated to the instability is infinite dimensional. This problem may be partially solved by reducing the phase space to the space of functions periodic on a planar lattice. Goldstein et al. [39] consider the space

$$H_6 = \{\text{functions periodic on a hexagonal lattice}\}.$$

Reduced to H_6 the problem has symmetry of $\mathbb{Z}_6 \cdot \mathbb{T}^2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_6 \cdot \mathbb{T}^2$ is the subgroup of $\mathbf{SE}(2)$ leaving H_6 invariant and $\mathbb{Z}_2$ corresponds to the midplane reflection. Goldstein et al. [39] derive the relevant bifurcation problem on the center manifold. Thus the ODE obtained, posed on $\mathbb{C}^3$ ($\mathbb{R}^6$), has a three-dimensional fixed-point space invariant under the action of $\mathbb{Z}_3 \cdot (\mathbb{Z}_2)^3$. The ODE restricted to this space is identical to the system studied by [18] and [41] and thus possesses a robust homoclinic cycle. For idealized boundary conditions the authors of [39] symbolically compute the normal coefficients (the coefficients a_1, a_2, and a_3 in (2.1); see Section 2) and show they corresponded to the regime of the existence of a stable cycle. Clune [22] computes these coefficients for experimental boundary conditions, reaching the same conclusion.

It remains to explain what happens in the experiment. It is clear from Figure 12 that the dynamics observed are only very roughly approximated by a homoclinic cycle. The two main discrepancies are the following:

1. The intermittent roll patterns are fragmented (confined to small patches of the plane).
2. The return times for the patches of intermittent rolls do not grow to infinity but remain bounded and display erratic behavior.

Based on these two points there has been much criticism of the Busse-Clever model [51], [97]. Below we present some arguments in support of the model.

The first discrepancy between the model and the experiment expresses the limitation of the assumption that solutions are periodic on a lattice. The same problem is observed for steady-state patterns—using the method of looking for solutions defined on a periodic lattice, a number of steady state patterns (rolls, hexagons) are found, yet this approach fails to predict the fragmentation of the patterns that occurs in experiments.

The second discrepancy is most likely caused by imperfections of the model. It may be due to nondeterministic effects, but an equally likely reason is symmetry imperfections. Busse [16], [17] argues that under stochastic perturbations of the model the behavior seen int he experiment can be predicted. Swift [95] and Soward [86] explore the second point of view, that adding symmetry imperfections to the model equations may result in breaking the heteroclinic cycle and lead to similar, yet physically realistic dynamics. These authors exploit the fact that for a non-Boussinesq fluid the midplane reflection is broken. They consider the system of [18] and introduce terms breaking the midplane reflection, which is equivalent to breaking the invariance of the plane containing the cycle. They show the existence of periodic solutions close to the original cycle and having the same symmetry properties.

Dynamics yet closer to the experiment would likely be found if one considered perturbations of the rotational symmetry (which is not perfect in the experiment). The resulting dynamics would, most likely, no longer be periodic, but chaotic, yet would closely resemble the homoclinic cycle [69]. The main difference would be in transition times, which, rather than growing to infinity, would behave erratically. Stone and Holmes [93] (see also [69]) show in a similar context that deterministic perturbations lead to a probability distribution of return times. Very long return times become very unlikely, i.e., the probability of the occurrence of a return time larger than $T_0 > 0$ is of the order $\mathcal{O}(e^{-\lambda_u T_0})$, where λ_u is the unstable eigenvalue of the equilibrium (roll solution). The results of [93] are in qualitative agreement with the rotating convection experiment; indeed, the transition times between laminar phases display erratic behavior.

11. Convection in the Presence of a Magnetic Field

In this section we consider convection between two infinite plates in the presence of a magnetic field acting in the vertical direction. The experiment has symmetry $\mathbf{E}(2) = \mathbf{O}(2) \cdot \mathbb{R}^2$ (rotations and reflections in the plane). Clune and Knobloch [23] show that the pure conduction state can lose stability through a Hopf bifurcation. The analysis of the full $\mathbf{E}(2)$-symmetric Hopf bifurcation problem is not possible due to the noncompactness of $\mathbf{E}(2)$. Clune and Knobloch [23] study Hopf bifurcations on the square lattice ($\mathbb{D}_4 \cdot \mathbb{T}^2$ symmetry) and on the hexagonal lattice ($\mathbb{D}_6 \cdot \mathbb{T}^2$ symmetry); see Section 5.1.2 for a review of related work. They compute the normal form coefficients from the model equations using a symbolic computation program, thus making sure that the parameter regions considered are relevant for the physical system. For the hexagonal lattice problem they find a cycle joining a number of periodic solutions. Based on the necessary condition of [61] they conclude that the cycle was unstable, but are able to see it as a transient in numerical computations. The dynamic behavior of trajectories starting near the heteroclinic cycle involves switching between the cycle and its symmetric iterate.

Rucklidge and Matthews [76], [77] consider a Galerkin truncation of the model equations involving 44 modes. Using a scaling and elimination procedure they further reduce the number of equations to ten. In their numerical simulations they find a heteroclinic cycle involving two types of rolls (equilibrium solutions). An interesting feature of this cycle is that the isotropy group of one of the roll solutions is contained in the isotropy group of the other one. As a result one observes a transient equilibrium state with certain symmetry, then a transition to an equilibrium with less symmetry and then again a transition to an equilibrium with more symmetry. More specifically one first sees convection rolls, then convection rolls with shear and then convection rolls again. The cycle of Rucklidge and Matthews clearly does not come from a Hopf bifurcation—it joins steady-state solutions. Thus it is not related to the cycles of [23]. Moreover it appears to be stable for a large parameter range. It is known that no heteroclinic cycles are present in steady state bifurcation problems with $\mathbb{D}_4 \cdot \mathbb{T}^2$ symmetry; this means that it is not possible to realize the cycles of [76], [77] in the context of spontaneous symmetry breaking. An interesting problem would be to find the origin of those cycles, in the same way as most known cycles can be traced back to a spontaneous or forced symmetry breaking bifurcation.

Rucklidge and Matthews [76], [77] predict the existence of a number of other heteroclinic cycles based on the form of the reduced ODE system, but do not find them numerically in the 44-mode model.

12. Robust Heteroclinic Cycles in Turbulent Fluid Flows

Consider fluid flow along a flat plate or in a circular pipe. Such domains have symmetry which is preserved by the equations of motion (the Navier-Stokes equations). Clearly the symmetry must have an influence on the dynamics and may lead to the occurrence of robust heteroclinic cycles. There is a considerable amount of research providing evidence of the existence of cycles in turbulent flows. In this section we give a summary of this research concentrating on the pioneering work of Aubry et al. [10].

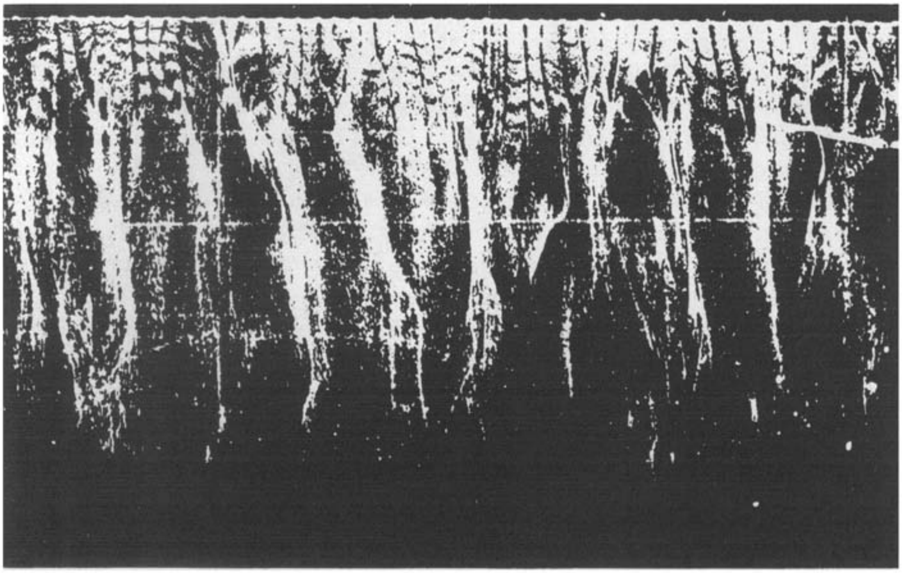

Fig. 13. The streak-like eddies observed by Kline et al. [58] (S. J. Kline, W. C. Reynolds, W. C. Schraub, and P. W. Rundstadler. The structure of turbulent boundary layers. *Journal of Fluid Mechanics* **30**, 741–773. © 1990 Cambridge University Press. Reprinted with permission of Cambridge University Press).

Aubry et al. [10] consider the dynamics of fluid flowing along a flat plate (wall). Their work is motivated by the experiments of Kline et al. [58], who observed intermittent structures in the dynamics of the flow. More specifically, they saw large eddies in the form of streaks stretched along the streamwise direction (see Figure 13). The streaks underwent violent bursting events characterized by the ejection of slow moving fluid from the wall and the arrival of fast moving fluid to the wall region. After a bursting event the streaks formed again and the configuration was a translate of the previous one. Aubry et al. [10] study a Galerkin approximation of the flow, using Fourier modes for the directions parallel to the wall and a Karhunen-Loève (KL) eigenfunction [11] in the direction normal to the wall. The KL eigenfunction is derived using experimental data. In order to justify a truncation involving relatively few modes, the flow only in the wall region is considered and a time-dependent pressure term representing the influence of the outer flow is introduced. This term is estimated as of lower order and is neglected in the initial analysis. The symmetry of the reduced equations is $\mathbf{O}(2)$ corresponding to translations and reflections in the spanwise direction (the direction parallel to the wall and normal to the direction of the stream). In the reduced equations Aubry et al. [10] find an asymptotically stable homoclinic cycle connecting an equilibrium to its translate by π (in the model domain of width 4π). The equilibria contained in the cycle are reconstructed to the corresponding velocity fields which in a satisfactory manner resemble the velocity field of the streaklike structure. The feature of the streaks rearranging in a translated formation after the event corresponds to the equilibria on the cycle being translates of each other by π.

The pressure term modeling the effect of the flow in the outer level can be seen as a small, time-dependent forcing. To reflect the influence of the pressure field in the outer level, a stochastic perturbation is also included in the model. The cycle is found in the absence of these two perturbations; upon their addition it is replaced by intermittent dynamics. Stone and Holmes [91], [92], [93] consider systems possessing a homoclinic cycle and study the effects of a stochastic forcing or a deterministic periodic forcing. Recall that in the absence of a perturbation the passage times (the duration of the passages near the equilibrium) for a trajectory following the cycle tends to infinity. The perturbation causes the solution to leave the vicinity of the equilibrium sooner—extremely long return times are no longer expected. Stone and Holmes [91], [92], [93] show that the passage times have a characteristic distribution $P(T)$ with exponential tails of the order $\mathcal{O}(e^{-\lambda_u T})$ where λ_u is the unstable eigenvalue. The mean passage time is of the order $\mathcal{O}(\frac{1}{\lambda_u}|ln(\epsilon)|)$, where ϵ is the size of the perturbation. The results of [91], [92], [93] hold for deterministic as well as stochastic perturbations and are in good qualitative agreement with the experimental observations.

The model of [10] is quite crude and it was a surprise that it so well reproduced the experimental data. In particular the use of just the constant Fourier mode to model the spanwise direction led to a number of problems in the modeling [79], [11]. The surprising accuracy of the low dimensional model was explained by Berkooz et al. [12], [13].

Sanghi and Aubry [79] consider Galerkin approximations involving more modes, in particular nonconstant Fourier modes in the streamwise direction. In the numerical studies of the equations thus obtained they find evidence of heteroclinic cycles joining time-dependent (periodic, quasiperiodic, chaotic) invariant sets. The gross features of the dynamics are the same as in the work of [10].

Sometimes, in order to reduce drag, riblets—that is, small streaks parallel to the mean flow—are introduced. Mathematically this means $\mathbf{O}(2) \to \mathbb{D}_n$, $n \in \mathbb{Z}$, forced symmetry breaking in the spanwise direction. Such forced symmetry breaking with $n = 4$ was considered by Campbell and Holmes [20], who proved the existence of a heteroclinic cycle analogous to the one found by Armbruster, Guckenheimer, and Holmes [4] (since $\mathbb{D}_4$ is finite, the cycles of [20] are isolated in the phase space unlike the cycles of [4], which lie on a continuous group orbit).

13. Flow through an Elastic Tube with Symmetric Support

Steindl [88] (see also [87]) considers fluid flow through an elastic tube supported by n symmetrically positioned springs; see Figure 14. The symmetry of the system is D_n. The $\mathbf{O}(2)$ symmetry is obtained in the limit when $n \to \infty$ or the springs become infinitely long. Steindl [88] studies the stability of the $\mathbf{O}(2)$ symmetric equilibrium solutions corresponding to the tube hanging down. He varies four parameters:

- the fluid velocity ρ,
- the stiffness of the support c,
- the height of the support ξ,
- the mass ratio $\beta = \frac{m_F}{m_F+m_T}$, where m_F is the mass of the fluid and m_T is the mass of the tube.

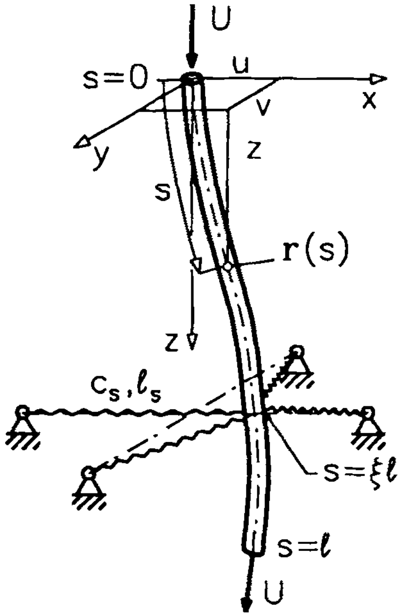

Fig. 14. Fluid conveying elastic tube with D_4 symmetric support. (This figure was made by A. Steindl.)

When the fluid velocity ρ grows from 0, the symmetric equilibrium loses stability. For smaller c this occurs through a Hopf bifurcation and for larger c through a steady-state bifurcation. Steindl [88] shows numerically that for a large region in the (β, ξ) plane a Hopf-steady state mode interaction took place for a moderate value of c. Moreover β and ξ could be adjusted in such a way that the unfolding of the mode interaction contained an almost asymptotically stable heteroclinic cycle. The cycle was found for small values of β, which corresponds to the choice of a very thick tube. In another work Steindl [89] numerically locates Hopf-Hopf mode interactions and shows the existence of a cycle joining two standing waves in the unfolding. For general results on cycles in steady-state Hopf and Hopf-Hopf mode interactions see Section 5.2.2 and [71].

It is quite interesting to imagine the motion of the tube corresponding to a trajectory following the cycles. Consider the cycle joining an equilibrium to a standing wave. Both the equilibrium and the standing wave have spatial symmetry $\mathbb{Z}_2$ corresponding in the physical space to a reflection across an invariant plane. The equilibrium is a buckled state contained in an invariant plane and the standing wave is an oscillatory motion also in an invariant plane. A trajectory following the cycle may first approach the buckled state, then oscillate out of the invariant plane approaching an oscillation in the perpendicular invariant plane. Next it approaches the buckled state in the same invariant plane. Subsequently the whole process repeats itself.

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References

[1] J. D. Alexander, I. Kan, J. A. Yorke, and Zhiping You. Riddled Basins. *Int. J. Bifur. Chaos* **2** (1992), 795–813.

[2] D. Armbruster. More on structurally stable H-orbits, in *Proceedings of the International Conference on Bifurcation Theory and Its Numerical Analysis*, Li Kaitai et al., eds., Xian Jiaotong University Press, Xian, China (1989).

[3] D. Armbruster and P. Chossat. Heteroclinic orbits in a spherically invariant system. *Physica D* **50** (1991), 155–176.

[4] D. Armbruster, J. Guckenheimer, and P. Holmes. Heteroclinic cycles and modulated waves in systems with $\mathbf{O}(2)$ symmetry. *Physica D* **29** (1988), 257–282.

[5] D. Armbruster, J. Guckenheimer, and P. Holmes. Kuramoto-Sivashinsky dynamics on the center unstable manifold. *SIAM J. Appl. Math.* **49** (1989), 676–691.

[6] D. Armbruster and E. Ihrig. Topological constraints for explicit symmetry breaking. *Lectures in Applied Mathematics* **29** (1993), 37–47.

[7] D. G. Aronson, S. A. van Gils, and M. Krupa. Homoclinic twist bifurcations with $\mathbb{Z}_2$ symmetry. *J. Nonlin. Sci.* **4** (1994), 195–219.

[8] P. Ashwin, J. Buescu, and I. N. Stewart. From attractor to a chaotic saddle: a tale of transverse instability. *Nonlinearity* **9** (1996), 703–737.

[9] P. Ashwin and P. Chossat. Attractors for robust heteroclinic sets with a continuum of connections. Preprint **INLN** Université de Nice 96.6 (1996).

[10] N. Aubry, P. Holmes, J. Lumley, and E. Stone. The dynamics of coherent structures in the wall region of a turbulent boundary layer. *J. Fluid Mech.* **192** (1988), 115–173.

[11] G. Berkooz, P. Holmes, and J. L. Lumley. The proper orthogonal decomposition in the analysis of turbulent flows. *Ann. Rev. Fluid Mech.* **25** (1993), 539–575.

[12] G. Berkooz, P. Holmes, and J. L. Lumley. Intermittent dynamics in simple models of the turbulent wall layer. *J. Fluid Mech.* **230** (1991), 75–95.

[13] G. Berkooz, P. Holmes, and J. L. Lumley. On the relation between low-dimensional models and the dynamics of coherent structures in the turbulent wall layer. *Theoret. Comput. Fluid Dyn.* **4** (1993), 255–269.

[14] W. Brannath. Heteroclinic networks on the simplex. *Nonlinearity* **7** (1994), 1367–1384.

[15] F. M. Busse. Transition to turbulence in Rayleigh-Bènard convection. In *Hydrodynamic Instabilities and the Transition to Turbulence* (H. L. Swinney and J. P. Gollub eds.) Springer-Verlag, New York (1981), 97–137.

[16] F. M. Busse. Transition to turbulence via the statistical limit cycle route. In *Chaos and Order in Nature* (H. Haken ed.), Springer-Verlag, New York (1981).

[17] F. M. Busse. Transition to turbulence via the statistical limit cycle route. In *Turbulence and Chaotic Phenomena in Fluids* (T. Tatsumi ed.), North-Holland, Amsterdam (1984).

[18] F. M. Busse and R. M. Clever. Nonstationary convection in a rotating system. *Recent Development in Theoretical and Experimental Fluid Mechanics* (U. Müller, K. G. Roessner, and B. Schmidt, eds.) Springer-Verlag, New York (1979), 376–385.

[19] F. M. Busse and K. E. Heikes. Convection in a rotating layer: A simple case of turbulence. *Science* **208** (1980), 173–175.

[20] S. Campbell and P. Holmes. Heteroclinic cycles and modulated traveling waves in a system with $\mathbb{D}_4$ symmetry. *Physica D* **59** (1992), 52–78.

[21] S. Campbell and P. Holmes. Bifurcation from **O**(2) symmetric heteroclinic cycles with three interacting modes. *Nonlinearity* **4** (1991), 697–726.

[22] T. Clune. Thesis, Pattern selection in convective systems. University of California-Berkeley, Berkeley, 1993.

[23] T. Clune and E. Knobloch. Pattern selection in three-dimensional magnetoconvection. *Physica D* **74**, 151–176 (1994).

[24] P. Chossat, M. Krupa, I. Melbourne, and A. Scheel. Transverse bifurcations of homoclinic cycles. To appear in *Physica D* (1996).

[25] P. Chossat. Forced reflectional symmetry breaking of an **O**(2) symmetric homoclinic cycle. *Nonlinearity* **6** 723–731 (1993).

[26] P. Chossat and M. J. Field. Geometric analysis of the effect of symmetry breaking perturbations on an **O**(2) invariant homoclinic cycle. In *Normal Forms and Homoclinic Chaos* (W. F. Langford and W. Nagata eds.) Fields Institute Communications, American Mathematical Society, Providence, RI (1995), Vol. 4, pp. 21–42.

[27] M. Dellnitz, M. J. Field, M. Golubitsky, A. Hohmann, and J. Ma. Cycling chaos. *Int. J. Bifur. Chaos* **5**(4) (1995) 1243–1247.

[28] B. Dionne, M. J. Field, and M. Krupa. Heteroclinic cycles in problems with wreath product symmetry. In preparation.

[29] B. Dionne, M. Golubitsky, and I. Stewart. Coupled cells with internal symmetries. Part I: wreath products. *Nonlinearity* **9** (1996), 559–574.

[30] M. J. Field. Equivariant dynamical systems. *Trans. Am. Math. Soc.* **259** (1980), 185–205.

[31] M. J. Field. Equivariant bifurcation theory and symmetry breaking. *J. Dyn. Diff. Eqs.* **1** (1989), 369–421.

[32] M. J. Field. Bifurcation and symmetry breaking from a homoclinic cycle with zero eigenvalues. In preparation.

[33] M. J. Field. *Lectures on Bifurcations, Dynamics and Symmetry*, Res. Notes in Math. 356, Addison-Wesley–Longman, Harlow, Essex (1996). [Appendix A coauthored with Peng].

[34] M. J. Field and R.W. Richardson. Symmetry breaking and branching patterns in equivariant branching theory II. *Arch. Rat. Mech. Anal.* **120** (1992), 147–190.

[35] M. J. Field and J. W. Swift. Stationary bifurcation to limit cycles and heteroclinic cycles. *Nonlinearity* **4** (1991), 1001–1043.

[36] M. J. Field and J. W. Swift. The Hopf bifurcation and the Hopf fibration. *Nonlinearity* **7** (1994), 385–402.

[37] R. Friedrich and H. Haken. Static, wavelike, and chaotic thermal convection in spherical geometries. *Phys. Rev. A* **34** (1986), 2100–2120.

[38] A. Gaunersdorfer. Time averages for heteroclinic attractors. *SIAM J. Appl. Math.* **52** (1992), 1476–1489.

[39] H. F. Goldstein, E. Knobloch, and M. Silber. Planform selection in rotating convection: Hexagonal symmetry. *Phys. Rev. A* **46** (1992), 4755–4761.

[40] M. Golubitsky, I. N. Stewart, and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol. II, Appl. Math. Sci. Ser. **69**, Springer-Verlag, New York (1988).

[41] J. Guckenheimer and P. Holmes. Structurally stable heteroclinic cycles. *Math. Proc. Cambridge Phil. Soc.* **103** (1988), 189–192.

[42] J. Guckenheimer and P. Worfolk. Instant chaos. *Nonlinearity* **5** (1992), 1211–1222.

[43] P. Guyard. Interaction de modes dans les problèmes de bifurcation avec symétrie sphérique. Thesis, Université de Nice, 1994.

[44] J. Hofbauer. A general cooperation theorem for hypercycles. *Monatsh. Math.* **91** (1991), 233–240.
[45] J. Hofbauer. Heteroclinic cycles on the simplex. *Proc. Int. Conf. Nonlinear Oscillations*, Janos Bolyai Math. Soc., Budapest (1987).
[46] J. Hofbauer. Heteroclinic cycles in ecological differential equations. *Proc. Equadiff 8., the Czech-Slovak Conference on Differential Equations and Their Applications.* (P. Brunovsky and M. Medved eds.) Tatra Mountains Mathematical Publications, Bratislava (1994), 105–116.
[47] J. Hofbauer and K. Sigmund. *The Theory of Evolution and Dynamical Systems.* Camb. Univ. Press, Cambridge (1988).
[48] A. Homburg, H. Kokubu, and M. Krupa. The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic orbit. *Ergodic Theory and Dynamical Systems* **14** (1994), 667–693.
[49] C. Hou. Thesis, Symmetry breaking and heteroclinic cycles. University of Houston, Houston, 1995.
[50] C. Hou and M. Golubitsky. An example of symmetry breaking to heteroclinic cycles. *J. Diff. Eqs.* To appear.
[51] Y. Hu, R. E. Ecke, and G. Ahlers. Time and length scales in rotating Rayleigh-Bénard convection. *Phys. Rev. Lett.* **74** (1995), 5040–5043.
[52] T. Hungerford. *Algebra*, **GTM** 73, Springer-Verlag, New York (1980).
[53] J. M. Hyman and B. Nicolaenko. The Kuramoto-Sivashinsky equation: A bridge between PDE's and dynamical systems. *Physica D* **18** (1986), 113–126.
[54] J. M. Hyman, B. Nicolaenko, and S. Zaleski. Order and complexity in the Kuramoto-Sivashinsky model of weakly turbulent interfaces. Report **LA-UR-86-1947**, LANL, Los Alamos NM (1986).
[55] I. G. Kevrikedis, B. Nicolaenko, and J. C. Scovel. Back in the saddle again: A computer assisted study of the Kuramoto-Sivashinsky equation. *SIAM J. Appl. Math.* **21** (1990), 760–790.
[56] V. Kirk and M. Silber. A competition between heteroclinic cycles. *Nonlinearity* **7** (1994), 1605–1621.
[57] G. Kirlinger. Permanence in Lotka-Volterra equations: Linked predator-prey systems. *Math. Biosci.* **82** (1986), 165–191.
[58] S. J. Kline, W. C. Reynolds, W. C. Schraub, and P. W. Rundstadler. The structure of turbulent boundary layers. *J. Fluid Mech.* **30** (1990), 741–773.
[59] E. Knobloch and M. Silber. Hopf bifurcation with $\mathbb{Z}_4 \times \mathbb{T}^2$ symmetry. *Bifurcation and Symmetry* (E. Allgower et al. eds.) ISNM 104, Birkhauser, Basel (1992), 241–252.
[60] E. Knobloch and M. Silber. Oscillatory convection in a rotating layer. *Physica D* **94** (1993), 213–232.
[61] M. Krupa and I. Melbourne. Asymptotic stability of heteroclinic cycles in systems with symmetry. *Erg. Th. Dyn. Sys.* **15** (1995), 121–147.
[62] M. Krupa and I. Melbourne. Nonasymptotically stable attractors in $\mathbf{O}(2)$ mode interactions. In *Normal Forms and Homoclinic Chaos.* (W. F. Langford and W. Nagata eds.) Fields Institute Communications, American Mathematical Society, Providence, RI (1995), Vol. 4, pp. 219–232.
[63] M. Krupa, I. Melbourne, and A. Scheel. Stability and bifurcations of heteroclinic cycles in systems with symmetry. In preparation.
[64] R. Lauterbach and S. Maier-Paape. Heteroclinic cycles for reaction diffusion systems by forced symmetry breaking. *Trans. Am. Math. Soc.*, submitted.
[65] R. Lauterbach, S. Maier-Paape, and Ernst Reißner. A systematic study of heteroclinic cycles in dynamical systems with broken symmetry. *Proc. Roy. Soc. Edinburgh* **126A** (1996), 885–909.
[66] R. Lauterbach and M. Roberts. Heteroclinic cycles in dynamical systems with broken spherical symmetry. *J. Diff. Eq.* **100** (1992), 22–48.
[67] X. B. Lin. Using Melnikov's method to solve Shilnikov's problems. *Proc. Roy. Soc. Edinburgh* **116A** (1990), 295–325.

[68] R. M. May and W. Leonard. Nonlinear aspects of competition between three species. *SIAM J. Appl. Math.* **29** (1975), 243–252.
[69] I. Melbourne. Intermittency as a codimension three phenomenon. *Dyn. Diff. Eqs.* **1** (1989), 347–367.
[70] I. Melbourne. An example of a nonasymptotically stable attractor. *Nonlinearity* **4** (1991), 835–844.
[71] I. Melbourne, P. Chossat, and M. Golubitsky. Heteroclinic cycles involving periodic solutions in mode interactions with **O**(2) symmetry. *Proc. Roy. Soc. Edinburgh* **113A** (1989), 315–345.
[72] M. R. E. Proctor and C. A. Jones. The interaction of two spatially resonant patterns in thermal convection. Part 1. Exact 1:2 resonance. *J. Fluid Mech.* **188** (1988), 301–335.
[73] G. L. dos Reis. Structural stability of equivariant vector fields. *An. Acad. Brasil. Ciênc.* **50** (1978), 273–176.
[74] G. L. dos Reis. Structural stability of equivariant vector fields on two manifolds. *Trans. Am. Math. Soc.* **283** (1984), 633–43.
[75] J. W. Reyn. A stability criterion for separatrix polygons in the plane. *Nieuw Arch. Wisk.* **27** (1979), 238–254.
[76] A. M. Rucklidge and P. C. Matthews. Shearing instability in magnetoconvection. *Double-Diffusive Convection* (A. Brandt and H. J. S. Fernando eds.) American Geophysical Union, Washington, DC (1995), 171–184.
[77] A. M. Rucklidge and P. C. Matthews. Analysis of the shearing instability in nonlinear analysis and magnetoconvection. *Nonlinearity* **9** (1996).
[78] B. Sandstede and A. Scheel. Forced symmetry breaking of homoclinic cycles. *Nonlinearity* **8** (1995), 333–365.
[79] S. Sanghi and N. Aubry. Mode interaction models for near-wall turbulence. *J. Fluid Mech.* **247** (1993), 455–488.
[80] B. Sandstede. Verzweigungstheorie homokliner Verdopplungen. Report No. 7, Institut für Angewandte Analysis und Stochastik, Berlin (1993).
[81] A. Scheel. Bifurcation d'orbites périodiques à partir de cycles homoclines en présence de symétrie. *Memoire de DEA*. Université de Nice, Nice (1991).
[82] A. Scheel and P. Chossat. Bifurcation d'orbites pèriodiques á partir d'un cycle homocline symètrique. *C. R. Acad. Sci. Paris* **314** (1992), 49–54.
[83] Z. S. She and B. Nicolaenko. Temporal intermittency and turbulence production in the Kolmogorov flow. *Topological Fluid Mechanics* (H. K. Moffat ed.), Cambridge University Press, Cambridge **256** (1990).
[84] M. Silber and E. Knobloch. Hopf bifurcation on a square lattice. *Nonlinearity* **4** (1991), 1063–1106.
[85] M. Silber, H. Riecke, and L. Kramer. Symmetry-breaking Hopf bifurcation in anisotropic systems. *Physica D* **61** (1992), 260–272.
[86] A. M. Soward. Bifurcation and stability of finite amplitude convection in a rotating layer. *Physica D* **14** (1985), 227–241.
[87] A. Steindl. Hopf/steady-state mode interaction for a fluid conveying elastic tube with $\mathbb{D}_4$ symmetric support. *International Series of Numerical Mathematics* **104** (1992), 305–315.
[88] A. Steindl. In preparation.
[89] A. Steindl. Heteroclinic cycles in the dynamics of a fluid conveying tube. Proceedings of ICIAM/GAMM 1995. Special issues of *Zeitschrift für Angewandte Mathematik und Mechanik*, issue **4** (Edwin Kreuzer and Oskar Mahrenholtz eds.) Academie Verlag, Berlin (1996), 529–532.
[90] S. Sternberg. On the structure of local homeomorphisms of Euclidean n-space, II. *Am. J. Math.* **80** (1958), 623–631.
[91] E. Stone and P. Holmes. Noise induced intermittency in a model of turbulent boundary layer. *Physica D* **37** (1989), 20–32.
[92] E. Stone and P. Holmes. Random perturbations of heteroclinic attractors, *SIAM J. Appl. Math* **50** (1990), 726–743.
[93] E. Stone and P. Holmes. Unstable fixed points, heteroclinic cycles, and exponential tails in turbulence production. *Phys. Lett. A* **155** (1991), 29–41.

[94] J. W. Swift. Hopf bifurcation with the symmetry of the square. *Nonlinearity* **1** (1988) 333–377.

[95] J. W. Swift. Convection in a rotating fluid layer. *Contemporary Mathematics* **28**, AMS (1984), 435–448.

[96] J. W. Swift and E. Barany. Chaos in the Hopf bifurcation with tetrahedral symmetry: Convection in a rotating fluid with low Prandtl number. *Eur. J. Mech., B/Fluids* **10** (1991), 99–104.

[97] Y. Tu and M. C. Cross. Chaotic domain structure in rotating convection. *Phys. Rev. Lett.* **69** (1992), 2515–2518.

[98] P. Worfolk. An equivariant, inclination-flip, heteroclinic bifurcation. *Nonlinearity* **9** (1996), 631–647.

[99] F. Takens. Heteroclinic attractors: time averages and moduli of topological conjugacy. *Bol. Soc. Bras. Mat.* **25** (1994), 107–120.

[100] F. Zhong and R. Ecke. Pattern dynamics and heat transport in rotating Rayleigh-Bénard convection. *Chaos* **2** (1992), 163–171.