The surface/surface intersection problems by means of matrix-based representation

LUU BA Thang
(Joint work with Laurent Busé)

INRIA Sophia Antipolis

SAGA, Vilnius, 27-30 September 2011

Outline

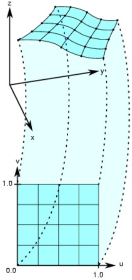
1. Matrix-based implicit representation
   - Implicit representation matrices of rational surfaces
   - Representation matrix of the intersection curve

2. Pencil of matrices and the surface/surface intersection problem
   - Reduction of a bivariate pencil of matrices
   - The surface/surface intersection problem

Part I.1 : Implicit Representation Matrices of Rational Surfaces

Motivations

- In geometric modeling, curves and surfaces are usually given under parameterized forms.
- An implicit representation is very useful in practice (e.g. decide if a point belongs to a curve or surface)



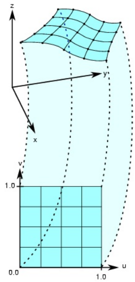
$\Rightarrow$ **Change of representation**:
elimination theory, implicitization problem

- The implicitization problem has become a very classical problem and several methods have been developed
  - methods based on Gröbner basis computations: output a polynomial
  - methods based on "resultant-like" computations: **eliminant matrices**

Motivations

- In geometric modeling, curves and surfaces are usually given under parameterized forms.
- An implicit representation is very useful in practice (e.g. decide if a point belongs to a curve or surface)

$\Rightarrow$ **Change of representation**:
elimination theory, implicitization problem



- The implicitization problem has become a very classical problem and several methods have been developed
  - methods based on Gröbner basis computations: output a polynomial
  - methods based on "resultant-like" computations: **eliminant matrices**

Matrix-based methods for implicitization

The **main difficulty** comes from the **base points**: points of the parameter space for which the parameterization is not well defined (typically division by zero).

- **Resultant-based methods**:
  - parameterization without base point over a toric variety (A. Khetan)
  - parameterization with prescribed base points (Busé, Eisenbud/Schreyer)
- **Square matrices filled with syzygies**:
  - Sederberg-Chen: the case of plane curves
  - Cox, Goldman and al.: moving planes and quadratics without base point
  - Cox, D'Andrea, Busé : moving planes and quadratics with base points
- **Non square matrices filled with approximation complexes**:
  - Busé, Chardin and Jouanolou: approximation complexes with less hypothesis, more general.

$$\text{Parameterized form} \xrightarrow{\text{Matrix representation}} \text{Implicit equation}$$

Matrix-based methods for implicitization

The **main difficulty** comes from the **base points**: points of the parameter space for which the parameterization is not well defined (typically division by zero).

- **Resultant-based methods**:
  - parameterization without base point over a toric variety (A. Khetan)
  - parameterization with prescribed base points (Busé, Eisenbud/Schreyer)
- **Square matrices filled with syzygies**:
  - Sederberg-Chen: the case of plane curves
  - Cox, Goldman and al.: moving planes and quadratics without base point
  - Cox, D'Andrea, Busé : moving planes and quadratics with base points
- **Non square matrices filled with approximation complexes**:
  - Busé, Chardin and Jouanolou: approximation complexes with less hypothesis, more general.

$$\text{Parameterized form} \xrightarrow{\text{Matrix representation}} \text{Implicit equation}$$

Matrix-based methods for implicitization

The **main difficulty** comes from the **base points**: points of the parameter space for which the parameterization is not well defined (typically division by zero).

- **Resultant-based methods**:
  - parameterization without base point over a toric variety (A. Khetan)
  - parameterization with prescribed base points (Busé, Eisenbud/Schreyer)
- **Square matrices filled with syzygies**:
  - Sederberg-Chen: the case of plane curves
  - Cox, Goldman and al.: moving planes and quadratics without base point
  - Cox, D'Andrea, Busé : moving planes and quadratics with base points
- **Non square matrices filled with approximation complexes**:
  - Busé, Chardin and Jouanolou: approximation complexes with less hypothesis, more general.

$$\text{Parameterized form} \xrightarrow{\text{Matrix representation}} \text{Implicit equation}$$

Matrix-based methods for implicitization

The **main difficulty** comes from the **base points**: points of the parameter space for which the parameterization is not well defined (typically division by zero).

- **Resultant-based methods**:
  - parameterization without base point over a toric variety (A. Khetan)
  - parameterization with prescribed base points (Busé, Eisenbud/Schreyer)
- **Square matrices filled with syzygies**:
  - Sederberg-Chen: the case of plane curves
  - Cox, Goldman and al.: moving planes and quadratics without base point
  - Cox, D'Andrea, Busé : moving planes and quadratics with base points
- **Non square matrices filled with approximation complexes**:
  - Busé, Chardin and Jouanolou: approximation complexes with less hypothesis, more general.

$$\text{Parameterized form} \xrightarrow{\text{Matrix representation}} \text{Implicit equation}$$

Matrix-based methods for implicitization

The **main difficulty** comes from the **base points**: points of the parameter space for which the parameterization is not well defined (typically division by zero).

- **Resultant-based methods**:
  - parameterization without base point over a toric variety (A. Khetan)
  - parameterization with prescribed base points (Busé, Eisenbud/Schreyer)
- **Square matrices filled with syzygies**:
  - Sederberg-Chen: the case of plane curves
  - Cox, Goldman and al.: moving planes and quadratics without base point
  - Cox, D'Andrea, Busé : moving planes and quadratics with base points
- **Non square matrices filled with approximation complexes**:
  - Busé, Chardin and Jouanolou: approximation complexes with less hypothesis, more general.

$$\textbf{Parameterized form} \xrightarrow{\textbf{Matrix representation}} \textbf{Implicit equation}$$

Syzygies of a parameterized surface

Consider a surface $\mathcal{S}$ parameterized by: ($\mathbb{K}$ algebraically closed field)

$$\begin{array}{ccc} \mathbb{P}^2_{\mathbb{K}} & \xrightarrow{\phi} & \mathbb{P}^3_{\mathbb{K}} \\ (X_1 : X_2 : X_3) & \mapsto & (f_1 : f_2 : f_3 : f_4)(X_1, X_2, X_3) \end{array}$$

with $d := \deg(f_i) \geq 1$. Assume that $\gcd(f_1, \ldots, f_4) \in \mathbb{K} \setminus \{0\}$.

The **graded** $\mathbb{K}[X_1, X_2, X_3]$-module of syzygies is

$$\mathscr{L}(\mathbf{f}) := \left\{ \sum_{i=1}^{4} g_i(X_1, X_2, X_3) T_i \in \mathbb{K}[X_1, X_2, X_3][T_1, T_2, T_3, T_4] \right.$$

$$\left. \text{such that } \sum_{i=1}^{4} g_i(X_1, X_2, X_3) f_i(X_1, X_2, X_3) \equiv 0 \right\}$$

Syzygies of a parameterized surface

Consider a surface $\mathcal{S}$ parameterized by: (𝕂 algebraically closed field)

$$\begin{array}{ccc} \mathbb{P}^2_{\mathbb{K}} & \xrightarrow{\phi} & \mathbb{P}^3_{\mathbb{K}} \\ (X_1 : X_2 : X_3) & \mapsto & (f_1 : f_2 : f_3 : f_4)(X_1, X_2, X_3) \end{array}$$

with $d := \deg(f_i) \geq 1$. Assume that $\gcd(f_1, \ldots, f_4) \in \mathbb{K} \setminus \{0\}$.

The **graded** $\mathbb{K}[X_1, X_2, X_3]$-module of syzygies is

$$\mathscr{L}(\mathbf{f}) := \left\{ \sum_{i=1}^{4} g_i(X_1, X_2, X_3) T_i \in \mathbb{K}[X_1, X_2, X_3][T_1, T_2, T_3, T_4] \right.$$

$$\left. \text{such that } \sum_{i=1}^{4} g_i(X_1, X_2, X_3) f_i(X_1, X_2, X_3) \equiv 0 \right\}$$

Matrices of syzygies

For all integer $\nu \geq 0$, build the matrix $\mathbf{L}(\phi)_\nu$ as follows:

1. Compute a basis $L^{(1)}, \ldots, L^{(n_\nu)}$ of $\mathscr{L}(\mathbf{f})_\nu$
2. $\mathbf{L}(\phi)_\nu$ is the matrix of coefficients of this basis, i.e.

$$\begin{pmatrix} X_1^\nu & X_1^{\nu-1}X_2 & \cdots & X_3^\nu \end{pmatrix} \mathbf{L}(\phi)_\nu = \begin{pmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{pmatrix}$$

Theorem (Busé, Chardin, Jouanolou)

Assume that the base points are all locally complete intersections. For all integer

$$\nu \geq \mathbf{2(d-1)} - \operatorname{indeg}((f_1, \ldots, f_4) : (X_1, X_2, X_3)^\infty)$$

the matrix $\mathbf{L}(\phi)_\nu$ is said to be a **representation matrix** *of ϕ because:*

- *$\mathbf{L}(\phi)_\nu$ is generically full rank*
- *the rank of $\mathbf{L}(\phi)_\nu$ drops exactly on the surface $\mathcal{S} = \overline{\operatorname{Im}}(\phi)$*
- *the GCD of the maximal minors of $\mathbf{L}(\phi)_\nu$ is equal to $S(T_1, \ldots, T_4)^{\deg(\phi)}$ where S is an implicit equation of $\mathcal{S}$.*

Matrices of syzygies

For all integer $\nu \geq 0$, build the matrix $\mathbf{L}(\phi)_\nu$ as follows:

1. Compute a basis $L^{(1)}, \ldots, L^{(n_\nu)}$ of $\mathscr{L}(\mathbf{f})_\nu$
2. $\mathbf{L}(\phi)_\nu$ is the matrix of coefficients of this basis, i.e.

$$\begin{pmatrix} X_1^\nu & X_1^{\nu-1}X_2 & \cdots & X_3^\nu \end{pmatrix} \mathbf{L}(\phi)_\nu = \begin{pmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{pmatrix}$$

Theorem (Busé, Chardin, Jouanolou)

Assume that the base points are all locally complete intersections.
For all integer

$$\nu \geq \mathbf{2}(\mathbf{d}-\mathbf{1}) - \mathrm{indeg}((f_1, \ldots, f_4) : (X_1, X_2, X_3)^\infty)$$

the matrix $\mathbf{L}(\phi)_\nu$ is said to be a **representation matrix** *of ϕ because:*

- *$\mathbf{L}(\phi)_\nu$ is generically full rank*
- *the rank of $\mathbf{L}(\phi)_\nu$ drops exactly on the surface $\mathcal{S} = \overline{\mathrm{Im}}(\phi)$*
- *the GCD of the maximal minors of $\mathbf{L}(\phi)_\nu$ is equal to $S(T_1, \ldots, T_4)^{\deg(\phi)}$ where S is an implicit equation of $\mathcal{S}$.*

Matrices of syzygies

For all integer $\nu \geq 0$, build the matrix $\mathbf{L}(\phi)_\nu$ as follows:

1. Compute a basis $L^{(1)}, \ldots, L^{(n_\nu)}$ of $\mathscr{L}(\mathbf{f})_\nu$
2. $\mathbf{L}(\phi)_\nu$ is the matrix of coefficients of this basis, i.e.

$$\begin{pmatrix} X_1^\nu & X_1^{\nu-1}X_2 & \cdots & X_3^\nu \end{pmatrix} \mathbf{L}(\phi)_\nu = \begin{pmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{pmatrix}$$

Theorem (Busé, Chardin, Jouanolou)

Assume that the base points are all locally complete intersections.
For all integer

$$\nu \geq \mathbf{2}(\mathbf{d}-\mathbf{1}) - \operatorname{indeg}((f_1, \ldots, f_4) : (X_1, X_2, X_3)^\infty)$$

the matrix $\mathbf{L}(\phi)_\nu$ is said to be a **representation matrix** *of ϕ because:*

- *$\mathbf{L}(\phi)_\nu$ is generically full rank*
- *the rank of $\mathbf{L}(\phi)_\nu$ drops exactly on the surface $\mathcal{S} = \overline{\operatorname{Im}}(\phi)$*
- *the GCD of the maximal minors of $\mathbf{L}(\phi)_\nu$ is equal to $S(T_1, \ldots, T_4)^{\deg(\phi)}$ where S is an implicit equation of $\mathcal{S}$.*

Matrices of syzygies

For all integer $\nu \geq 0$, build the matrix $\mathbf{L}(\phi)_\nu$ as follows:

1. Compute a basis $L^{(1)}, \ldots, L^{(n_\nu)}$ of $\mathscr{L}(\mathbf{f})_\nu$
2. $\mathbf{L}(\phi)_\nu$ is the matrix of coefficients of this basis, i.e.

$$\begin{pmatrix} X_1^\nu & X_1^{\nu-1}X_2 & \cdots & X_3^\nu \end{pmatrix} \mathbf{L}(\phi)_\nu = \begin{pmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{pmatrix}$$

Theorem (Busé, Chardin, Jouanolou)

Assume that the base points are all locally complete intersections. For all integer

$$\nu \geq \mathbf{2(d-1)} - \mathrm{indeg}((f_1, \ldots, f_4) : (X_1, X_2, X_3)^\infty)$$

the matrix $\mathbf{L}(\phi)_\nu$ *is said to be a* **representation matrix** *of* ϕ *because:*

- $\mathbf{L}(\phi)_\nu$ *is generically full rank*
- *the rank of* $\mathbf{L}(\phi)_\nu$ *drops exactly on the surface* $\mathcal{S} = \overline{\mathrm{Im}}(\phi)$
- *the GCD of the maximal minors of* $\mathbf{L}(\phi)_\nu$ *is equal to* $S(T_1, \ldots, T_4)^{\deg(\phi)}$ *where* S *is an implicit equation of* $\mathcal{S}$.

Matrices of syzygies

For all integer $\nu \geq 0$, build the matrix $\mathbf{L}(\phi)_\nu$ as follows:

1. Compute a basis $L^{(1)}, \ldots, L^{(n_\nu)}$ of $\mathscr{L}(\mathbf{f})_\nu$
2. $\mathbf{L}(\phi)_\nu$ is the matrix of coefficients of this basis, i.e.

$$\begin{pmatrix} X_1^\nu & X_1^{\nu-1}X_2 & \cdots & X_3^\nu \end{pmatrix} \mathbf{L}(\phi)_\nu = \begin{pmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{pmatrix}$$

Theorem (Busé, Chardin, Jouanolou)

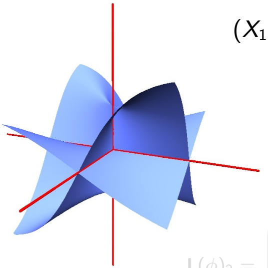
Assume that the base points are all locally complete intersections. For all integer

$$\nu \geq \mathbf{2}(\mathbf{d}-\mathbf{1}) - \mathrm{indeg}((f_1, \ldots, f_4) : (X_1, X_2, X_3)^\infty)$$

the matrix $\mathbf{L}(\phi)_\nu$ is said to be a **representation matrix** *of ϕ because:*

- $\mathbf{L}(\phi)_\nu$ *is generically full rank*
- *the rank of $\mathbf{L}(\phi)_\nu$ drops exactly on the surface $\mathcal{S} = \overline{\mathrm{Im}}(\phi)$*
- *the GCD of the maximal minors of $\mathbf{L}(\phi)_\nu$ is equal to $S(T_1, \ldots, T_4)^{\deg(\phi)}$ where S is an implicit equation of $\mathcal{S}$.*

Example of a Steiner surface

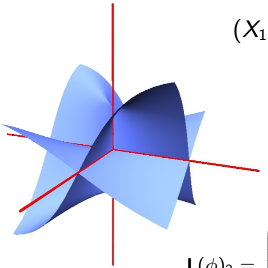


$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\phi} & \mathbb{P}^3 \\ (X_1 : X_2 : X_3) & \mapsto & (X_2X_3 : X_1X_3 : X_1X_2 : X_1^2 - X_2^2 + X_3^2) \end{array}$$

$$T_1^2 T_2^2 - T_1^2 T_3^2 + T_2^2 T_3^2 - T_1 T_2 T_3 T_4 = 0$$

$$\mathbf{L}(\phi)_2 = \begin{pmatrix} 0 & T_4 & 0 & T_1 & 0 & -T_1 & -T_1 & 0 & T_2 \\ T_4 & 0 & 0 & 0 & 0 & T_3 & 0 & T_2 & 0 \\ -T_1 & -T_3 & -T_1 & -T_2 & 0 & 0 & 0 & 0 & -T_1 \\ T_1 & T_3 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 \\ -T_1 & 0 & 0 & -T_2 & T_3 & 0 & 0 & 0 & 0 \\ 0 & -T_1 & T_3 & T_4 & -T_1 & 0 & 0 & -T_1 & 0 \end{pmatrix}$$

Example of a Steiner surface



$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\phi} & \mathbb{P}^3 \\ (X_1 : X_2 : X_3) & \mapsto & (X_2X_3 : X_1X_3 : X_1X_2 : X_1^2 - X_2^2 + X_3^2) \end{array}$$

$$T_1^2T_2^2 - T_1^2T_3^2 + T_2^2T_3^2 - T_1T_2T_3T_4 = 0$$

$$\mathbf{L}(\phi)_2 = \begin{pmatrix} 0 & T_4 & 0 & T_1 & 0 & -T_1 & -T_1 & 0 & T_2 \\ T_4 & 0 & 0 & 0 & 0 & T_3 & 0 & T_2 & 0 \\ -T_1 & -T_3 & -T_1 & -T_2 & 0 & 0 & 0 & 0 & -T_1 \\ T_1 & T_3 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 \\ -T_1 & 0 & 0 & -T_2 & T_3 & 0 & 0 & 0 & 0 \\ 0 & -T_1 & T_3 & T_4 & -T_1 & 0 & 0 & -T_1 & 0 \end{pmatrix}$$

Part I.2: Representation matrix of the intersection curve

Representation matrix of the intersection curve

Suppose given two parameterized surface $\mathcal{S}_1$ and $\mathcal{S}_2$.

$$\mathcal{C} := \mathcal{S}_1 \cap \mathcal{S}_2$$

$M(x, y, z, w)$: Representation matrix of $\mathcal{S}_1$ and let

$$\Psi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}} : (s : t : u) \mapsto (a(s, t, u) : b(s, t, u) : c(s, t, u) : d(s, t, u))$$

be a parameterization of $\mathcal{S}_2$.
In $M(x, y, z, w)$, substituting

$$x = a(s, t, u), y = b(s, t, u), z = c(s, t, u), w = d(s, t, u),$$

we get the matrix

$$\mathbb{M}(s, t, u) := M(\Psi(s, t, u)) = M(a(s, t, u), b(s, t, u), c(s, t, u), d(s, t, u)).$$

$\Longrightarrow \mathbb{M}(s, t, u)$: Representation matrix of the intersection curve $\mathcal{C}$.

Representation matrix of the intersection curve

Suppose given two parameterized surface $\mathcal{S}_1$ and $\mathcal{S}_2$.

$$\mathcal{C} := \mathcal{S}_1 \cap \mathcal{S}_2$$

$M(x, y, z, w)$: Representation matrix of $\mathcal{S}_1$ and let

$$\Psi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}} : (s : t : u) \mapsto (a(s, t, u) : b(s, t, u) : c(s, t, u) : d(s, t, u))$$

be a parameterization of $\mathcal{S}_2$.

In $M(x, y, z, w)$, substituting

$$x = a(s, t, u), y = b(s, t, u), z = c(s, t, u), w = d(s, t, u),$$

we get the matrix

$$\mathbb{M}(s, t, u) := M(\Psi(s, t, u)) = M(a(s, t, u), b(s, t, u), c(s, t, u), d(s, t, u)).$$

$\Longrightarrow \mathbb{M}(s, t, u)$: Representation matrix of the intersection curve $\mathcal{C}$.

Representation matrix of the intersection curve

Suppose given two parameterized surface $\mathcal{S}_1$ and $\mathcal{S}_2$.

$$\mathcal{C} := \mathcal{S}_1 \cap \mathcal{S}_2$$

$M(x, y, z, w)$: Representation matrix of $\mathcal{S}_1$ and let

$$\Psi : \mathbb{P}^2_{\mathbb{C}} \rightarrow \mathbb{P}^3_{\mathbb{C}} : (s : t : u) \mapsto (a(s, t, u) : b(s, t, u) : c(s, t, u) : d(s, t, u))$$

be a parameterization of $\mathcal{S}_2$.
In $M(x, y, z, w)$, substituting

$$x = a(s, t, u), y = b(s, t, u), z = c(s, t, u), w = d(s, t, u),$$

we get the matrix

$$\mathbb{M}(s, t, u) := M(\Psi(s, t, u)) = M(a(s, t, u), b(s, t, u), c(s, t, u), d(s, t, u)).$$

$\Longrightarrow \mathbb{M}(s, t, u)$: Representation matrix of the intersection curve $\mathcal{C}$.

Representation matrix of the intersection curve

Suppose given two parameterized surface $\mathcal{S}_1$ and $\mathcal{S}_2$.

$$\mathcal{C} := \mathcal{S}_1 \cap \mathcal{S}_2$$

$M(x, y, z, w)$: Representation matrix of $\mathcal{S}_1$ and let

$$\Psi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}} : (s : t : u) \mapsto (a(s,t,u) : b(s,t,u) : c(s,t,u) : d(s,t,u))$$

be a parameterization of $\mathcal{S}_2$.
In $M(x, y, z, w)$, substituting

$$x = a(s,t,u), y = b(s,t,u), z = c(s,t,u), w = d(s,t,u),$$

we get the matrix

$$\mathbb{M}(s,t,u) := M(\Psi(s,t,u)) = M(a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)).$$

$\Longrightarrow \mathbb{M}(s,t,u)$: Representation matrix of the intersection curve $\mathcal{C}$.

Representation matrix of the intersection curve

- For all point $(s_0 : t_0 : u_0) \in \mathbb{P}^2$ we have

$$\text{rank}(\mathbb{M}(s_0, t_0, u_0)) < \rho \text{ iff } \begin{cases} \Psi(s_0, t_0, u_0) \in \mathcal{S}_1 \cap \mathcal{S}_2 \\ \text{or} \\ (s_0 : t_0 : u_0) \text{ is a base point of } \Psi. \end{cases} \quad (1)$$

where $\rho := \text{rank } \mathbb{M}(s, t, u)$.

-

$$\text{Spectrum of } \mathbb{M}(s, t, u) := \{(s_0 : t_0 : u_0) \in \mathbb{P}^2 : \text{rank}\mathbb{M}(s_0, t_0, u_0) < \rho\}$$

$\Longrightarrow$ Spectrum of $\mathbb{M}(s, t, u) \equiv$ the intersection locus of $\mathcal{S}_1 \cap \mathcal{S}_2$ + the base points of the parameterization Ψ of $\mathcal{S}_2$

Representation matrix of the intersection curve

- For all point $(s_0 : t_0 : u_0) \in \mathbb{P}^2$ we have

$$\text{rank}(\mathbb{M}(s_0, t_0, u_0)) < \rho \text{ iff } \begin{cases} \Psi(s_0, t_0, u_0) \in \mathcal{S}_1 \cap \mathcal{S}_2 \\ \text{or} \\ (s_0 : t_0 : u_0) \text{ is a base point of } \Psi. \end{cases} \quad (1)$$

  where $\rho := \text{rank } \mathbb{M}(s, t, u)$.

- $$\text{Spectrum of } \mathbb{M}(s, t, u) := \left\{ (s_0 : t_0 : u_0) \in \mathbb{P}^2 : \text{rank}\mathbb{M}(s_0, t_0, u_0) < \rho \right\}$$

  $\Longrightarrow$ Spectrum of $\mathbb{M}(s, t, u) \equiv$ the intersection locus of $\mathcal{S}_1 \cap \mathcal{S}_2$ + the base points of the parameterization Ψ of $\mathcal{S}_2$

Spectrum of the intersection matrix

Theorem (Busé,Luu Ba)

The spectrum of the matrix $\mathbb{M}(s, t, u)$ is an algebraic curve in $\mathbb{P}^2$, that is to say is equal to the zero locus of a homogeneous polynomial in $\mathbb{C}[s, t, u]$. In particular, there is no isolated points in the spectrum of $\mathbb{M}(s, t, u)$.

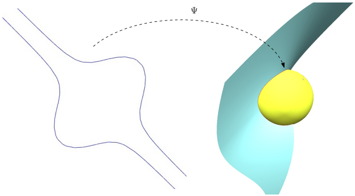


Figure: The plane curve $\mathcal{C}$ corresponding to $\mathbf{S}_1 \cap \mathbf{S}_2$

By dehomogenization ($u = 1$), we obtain a bivariate polynomial matrix $\mathbb{M}(s, t, 1)$

Extract a pencil of $\mathbb{M}(s, t, 1)$ that yields a matrix representation of the intersection curve as a matrix determinant.

Part II.1: Reduction of a bivariate pencil of matrices

Spectrum of a bivariate polynomial matrix

Let $M(s,t)$ be a matrix of size $m \times n$ depending on the two variables s and t.

The spectrum of $M(s,t)$ is defined to be the set

$$\{(s_0, t_0) \in \mathbb{K} \times \mathbb{K} : \mathrm{rank}(M(s_0, t_0)) < \rho\}$$

where $\rho := \mathrm{rank} M(s,t)$.

$$\mathrm{Spectrum}(M(s,t)) := \{(s_0,t_0) : \det M^{j_1,\dots,j_\rho}_{i_1,\dots,i_\rho} = 0, \quad {}^{1\le i_1<\cdots<i_\rho\le m}_{1\le j_1<\cdots<j_\rho\le n}\}. \qquad (2)$$

- The continuous part of the spectrum $\leftrightsquigarrow$ The one-dimensional roots of the system (2) $\leftrightsquigarrow$ The one-dimensional eigenvalues of the matrix $M(s,t)$.
- The discrete part of the spectrum $\leftrightsquigarrow$ The zero-dimensional roots of the system (2) $\leftrightsquigarrow$ The zero-dimensional eigenvalues of the matrix $M(s,t)$.

Spectrum of a bivariate polynomial matrix

Let $M(s,t)$ be a matrix of size $m \times n$ depending on the two variables s and t.

The spectrum of $M(s,t)$ is defined to be the set

$$\{(s_0,t_0) \in \mathbb{K} \times \mathbb{K} : \text{rank}(M(s_0,t_0)) < \rho\}$$

where $\rho := \text{rank} M(s,t)$.

$$\text{Spectrum}(M(s,t)) := \{(s_0,t_0) : \det M^{j_1,\dots,j_\rho}_{i_1,\dots,i_\rho} = 0, \quad {}^{1 \le i_1 < \dots < i_\rho \le m}_{1 \le j_1 < \dots < j_\rho \le n}\}. \qquad (2)$$

- The continuous part of the spectrum ⇜⇝ The one-dimensional roots of the system (2) ⇜⇝ The one-dimensional eigenvalues of the matrix $M(s,t)$.
- The discrete part of the spectrum ⇜⇝ The zero-dimensional roots of the system (2) ⇜⇝ The zero-dimensional eigenvalues of the matrix $M(s,t)$.

Spectrum of a bivariate polynomial matrix

Let $M(s,t)$ be a matrix of size $m \times n$ depending on the two variables s and t.

The spectrum of $M(s,t)$ is defined to be the set

$$\{(s_0,t_0) \in \mathbb{K} \times \mathbb{K} : \operatorname{rank}(M(s_0,t_0)) < \rho\}$$

where $\rho := \operatorname{rank} M(s,t)$.

$$\operatorname{Spectrum}(M(s,t)) := \{(s_0,t_0) : \det M_{i_1,\ldots,i_\rho}^{j_1,\ldots,j_\rho} = 0, \quad {}^{1 \le i_1 < \cdots < i_\rho \le m}_{1 \le j_1 < \cdots < j_\rho \le n}\}. \tag{2}$$

- The continuous part of the spectrum $\leftrightsquigarrow$ The one-dimensional roots of the system (2) $\leftrightsquigarrow$ The one-dimensional eigenvalues of the matrix $M(s,t)$.
- The discrete part of the spectrum $\leftrightsquigarrow$ The zero-dimensional roots of the system (2) $\leftrightsquigarrow$ The zero-dimensional eigenvalues of the matrix $M(s,t)$.

Spectrum of a bivariate polynomial matrix

Let $M(s,t)$ be a matrix of size $m \times n$ depending on the two variables s and t.

The spectrum of $M(s,t)$ is defined to be the set

$$\{(s_0,t_0) \in \mathbb{K} \times \mathbb{K} : \text{rank}(M(s_0,t_0)) < \rho\}$$

where $\rho := \text{rank} M(s,t)$.

$$\text{Spectrum}(M(s,t)) := \{(s_0,t_0) : \det M_{i_1,\dots,i_\rho}^{j_1,\dots,j_\rho} = 0, \quad {}^{1\le i_1<\dots<i_\rho\le m}_{1\le j_1<\dots<j_\rho\le n}\}. \qquad (2)$$

- The continuous part of the spectrum $\leftrightsquigarrow$ The one-dimensional roots of the system (2) $\leftrightsquigarrow$ The one-dimensional eigenvalues of the matrix $M(s,t)$.
- The discrete part of the spectrum $\leftrightsquigarrow$ The zero-dimensional roots of the system (2) $\leftrightsquigarrow$ The zero-dimensional eigenvalues of the matrix $M(s,t)$.

Linearization of a bivariate polynomial matrix

Given an $m \times n$-matrix $M(s,t) = (a_{i,j}(s,t))$ with $a_{i,j}(s,t) \in \mathbb{K}[s,t]$.

$$M(s,t) = M_d(t)s^d + M_{d-1}(t)s^{d-1} + \ldots + M_0(t)$$

where $M_i(t) \in \mathbb{K}[t]^{m\times n}$ and $d = \max_{i,j}\{\deg_s(a_{i,j}(s,t))\}$.

The generalized companion matrices A, B of the matrix $M(s,t)$ are the matrices with coefficients in $\mathbb{K}[t]$ of size $((d-1)m+n) \times dm$ that are given by

$$A = \begin{pmatrix} 0 & I_m & \ldots & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & I_m \\ M_0^t(t) & M_1^t(t) & \ldots & \ldots & M_{d-1}^t(t) \end{pmatrix}, \quad B = \begin{pmatrix} I_m & 0 & \ldots & \ldots & 0 \\ 0 & I_m & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_m & 0 \\ 0 & 0 & \ldots & 0 & -M_d^t(t) \end{pmatrix}$$

Linearization of a bivariate polynomial matrix

Given an $m \times n$-matrix $M(s,t) = (a_{i,j}(s,t))$ with $a_{i,j}(s,t) \in \mathbb{K}[s,t]$.

$$M(s,t) = M_d(t)s^d + M_{d-1}(t)s^{d-1} + \ldots + M_0(t)$$

where $M_i(t) \in \mathbb{K}[t]^{m \times n}$ and $d = \max_{i,j}\{\deg_s(a_{i,j}(s,t))\}$.

The generalized companion matrices A, B of the matrix $M(s,t)$ are the matrices with coefficients in $\mathbb{K}[t]$ of size $((d-1)m+n) \times dm$ that are given by

$$A = \begin{pmatrix} 0 & I_m & \ldots & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & I_m \\ M_0^t(t) & M_1^t(t) & \ldots & \ldots & M_{d-1}^t(t) \end{pmatrix}, \quad B = \begin{pmatrix} I_m & 0 & \ldots & \ldots & 0 \\ 0 & I_m & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_m & 0 \\ 0 & 0 & \ldots & 0 & -M_d^t(t) \end{pmatrix}$$

Property

There exists two unimodular matrices $E(s,t)$ et $F(s,t)$ with coefficients in $\mathbb{C}[s,t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(s,t)\,(A(t)-sB(t))\,F(s,t) = \left(\begin{array}{c|c} {}^{t}M(s,t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \qquad (3)$$

We provide a direct proof of:

Theorem (Kublanovskaya)

$$A(t)-sB(t) \rightsquigarrow \begin{pmatrix} * & * & M_3(s,t) \\ * & M_2(s,t) & 0 \\ M_1(s,t) & 0 & 0 \end{pmatrix}$$

- *$M_2(s,t)$ is a regular pencil and has only continuous spectrum coinciding with the continuous of spectrum of $M(s,t)$.*
- *The union of the discrete spectrum of the pencil $M_1(s,t)$ and $M_3(s,t)$ coincides with the discrete spectrum of $M(s,t)$.*

Property

There exists two unimodular matrices $E(s,t)$ et $F(s,t)$ with coefficients in $\mathbb{C}[s,t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(s,t)\,(A(t)-sB(t))\,F(s,t) = \left(\begin{array}{c|c} {}^tM(s,t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \qquad (3)$$

We provide a direct proof of:

Theorem (Kublanovskaya)

$$A(t)-sB(t) \leftrightsquigarrow \begin{pmatrix} * & * & M_3(s,t) \\ * & M_2(s,t) & 0 \\ M_1(s,t) & 0 & 0 \end{pmatrix}$$

- *$M_2(s,t)$ is a regular pencil and has only continuous spectrum coinciding with the continuous of spectrum of $M(s,t)$.*
- *The union of the discrete spectrum of the pencil $M_1(s,t)$ and $M_3(s,t)$ coincides with the discrete spectrum of $M(s,t)$.*

Property

There exists two unimodular matrices $E(s,t)$ et $F(s,t)$ with coefficients in $\mathbb{C}[s,t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(s,t)\,(A(t)-sB(t))\,F(s,t) = \left(\begin{array}{c|c} {}^{t}M(s,t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \quad (3)$$

We provide a direct proof of:

Theorem (Kublanovskaya)

$$A(t)-sB(t) \leftrightsquigarrow \begin{pmatrix} * & * & M_3(s,t) \\ * & M_2(s,t) & 0 \\ M_1(s,t) & 0 & 0 \end{pmatrix}$$

- *$M_2(s,t)$ is a regular pencil and has only continuous spectrum coinciding with the continuous of spectrum of $M(s,t)$.*
- *The union of the discrete spectrum of the pencil $M_1(s,t)$ and $M_3(s,t)$ coincides with the discrete spectrum of $M(s,t)$.*

Property

There exists two unimodular matrices $E(s,t)$ et $F(s,t)$ with coefficients in $\mathbb{C}[s,t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(s,t)\,(A(t)-sB(t))\,F(s,t) = \left(\begin{array}{c|c} {}^tM(s,t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \quad (3)$$

We provide a direct proof of:

Theorem (Kublanovskaya)

$$A(t)-sB(t) \leftrightsquigarrow \begin{pmatrix} * & * & M_3(s,t) \\ * & M_2(s,t) & 0 \\ M_1(s,t) & 0 & 0 \end{pmatrix}$$

- *$M_2(s,t)$ is a regular pencil and has only continuous spectrum coinciding with the continuous of spectrum of $M(s,t)$.*
- *The union of the discrete spectrum of the pencil $M_1(s,t)$ and $M_3(s,t)$ coincides with the discrete spectrum of $M(s,t)$.*

Extracting the regular part and the discrete part

The Δ W - Decomposition.

The decomposition of an univariate polynomial $M(t)$ of rank ρ under the form

$$M(t)W(t) = [\Delta(t), 0],$$

$W(t)$: an unimodular polynomial matrix,
$\Delta(t)$: a polynomial matrix of full column rank ρ.

- The Δ W - Decomposition is much more complicated than LU (QR)-Decomposition.
- Reason: The operations of the transformation of $M(t)$ have been done over the polynomial ring $\mathbb{K}[t]$, not the field $\mathbb{K}$.

Extracting the regular part and the discrete part

The Δ W - Decomposition.

The decomposition of an univariate polynomial $M(t)$ of rank ρ under the form

$$M(t)W(t) = [\Delta(t), 0],$$

$W(t)$: an unimodular polynomial matrix,
$\Delta(t)$: a polynomial matrix of full column rank ρ.

- The Δ W - Decomposition is much more complicated than LU (QR)-Decomposition.
- Reason: The operations of the transformation of $M(t)$ have been done over the polynomial ring $\mathbb{K}[t]$, not the field $\mathbb{K}$.

Extracting the regular part and the discrete part

The Δ W - Decomposition.

The decomposition of an univariate polynomial $M(t)$ of rank ρ under the form

$$M(t)W(t) = [\Delta(t), 0],$$

$W(t)$: an unimodular polynomial matrix,
$\Delta(t)$: a polynomial matrix of full column rank ρ.

- The Δ W - Decomposition is much more complicated than LU (QR)-Decomposition.
- Reason: The operations of the transformation of $M(t)$ have been done over the polynomial ring $\mathbb{K}[t]$, not the field $\mathbb{K}$.

Extracting the regular part and the discrete part

The Δ W - Decomposition.

The decomposition of an univariate polynomial $M(t)$ of rank ρ under the form

$$M(t)W(t) = [\Delta(t), 0],$$

$W(t)$: an unimodular polynomial matrix,
$\Delta(t)$: a polynomial matrix of full column rank ρ.

- The Δ W - Decomposition is much more complicated than LU (QR)-Decomposition.
- Reason: The operations of the transformation of $M(t)$ have been done over the polynomial ring $\mathbb{K}[t]$, not the field $\mathbb{K}$.

Algorithm for extracting the regular and discrete part

We start with a pencil $A(t) - sB(t)$, $A(t), B(t)$: matrix of size $p \times q$ and $\rho = \text{rank}A(t)$ via the classical Δ W - Decomposition.

- Step 1
  - $A(t)Q_0(t) = [\underbrace{\Delta_0(t)}_{\rho} | \underbrace{0}_{q-\rho}], \quad B(t)Q_0(t) = [\underbrace{B_{1,1}(t)}_{\rho} | \underbrace{B_{1,2}(t)}_{q-\rho}]$
  - $P_0(t)B_{1,2}(t) = \left(\frac{B'_{1,2}(t)}{0}\right)$; $B'_{1,2}(t)$ has full row rank.
  - Matrices $A(t)$ and $B(t)$ are represented under the form

$$P_0(t)A(t)Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) & 0 \\ \hline A_2(t) & 0 \end{array}\right), \quad P_0(t)B(t)Q_0(t) = \left(\begin{array}{c|c} B'_{1,1}(t) & B'_{1,2}(t) \\ \hline B_2(t) & 0 \end{array}\right)$$

Algorithm for extracting the regular and discrete part

We start with a pencil $A(t) - sB(t)$, $A(t), B(t)$: matrix of size $p \times q$ and $\rho = \mathrm{rank} A(t)$ via the classical Δ W - Decomposition.

- Step 1
  - $A(t)Q_0(t) = [\underbrace{\Delta_0(t)}_{\rho} \,|\, \underbrace{0}_{q-\rho}], \quad B(t)Q_0(t) = [\underbrace{B_{1,1}(t)}_{\rho} \,|\, \underbrace{B_{1,2}(t)}_{q-\rho}]$
  - $P_0(t)B_{1,2}(t) = \begin{pmatrix} B'_{1,2}(t) \\ \hline 0 \end{pmatrix}$; $B'_{1,2}(t)$ has full row rank.
  - Matrices $A(t)$ and $B(t)$ are represented under the form

$$P_0(t)A(t)Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) & 0 \\ \hline A_2(t) & 0 \end{array}\right), \quad P_0(t)B(t)Q_0(t) = \left(\begin{array}{c|c} B'_{1,1}(t) & B'_{1,2}(t) \\ \hline B_2(t) & 0 \end{array}\right)$$

Algorithm for extracting the regular and discrete part

We start with a pencil $A(t) - sB(t)$, $A(t), B(t)$: matrix of size $p \times q$ and $\rho = \text{rank}A(t)$ via the classical Δ W - Decomposition.

- Step 1
  - $A(t)Q_0(t) = [\underbrace{\Delta_0(t)}_{\rho} | \underbrace{0}_{q-\rho}]$, $B(t)Q_0(t) = [\underbrace{B_{1,1}(t)}_{\rho} | \underbrace{B_{1,2}(t)}_{q-\rho}]$
  - $P_0(t)B_{1,2}(t) = \left(\frac{B'_{1,2}(t)}{0} \right)$; $B'_{1,2}(t)$ has full row rank.
  - Matrices $A(t)$ and $B(t)$ are represented under the form

$$P_0(t)A(t)Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) & 0 \\ \hline A_2(t) & 0 \end{array} \right), \quad P_0(t)B(t)Q_0(t) = \left(\begin{array}{c|c} B'_{1,1}(t) & B'_{1,2}(t) \\ \hline B_2(t) & 0 \end{array} \right)$$

Algorithm for extracting the regular and discrete part

We start with a pencil $A(t) - sB(t)$, $A(t), B(t)$: matrix of size $p \times q$ and $\rho = \mathrm{rank}A(t)$ via the classical Δ W - Decomposition.

- Step 1
  - $A(t)Q_0(t) = [\underbrace{\Delta_0(t)}_{\rho} | \underbrace{0}_{q-\rho}]$, $B(t)Q_0(t) = [\underbrace{B_{1,1}(t)}_{\rho} | \underbrace{B_{1,2}(t)}_{q-\rho}]$
  - $P_0(t)B_{1,2}(t) = \begin{pmatrix} B'_{1,2}(t) \\ \hline 0 \end{pmatrix}$; $B'_{1,2}(t)$ has full row rank.
  - Matrices $A(t)$ and $B(t)$ are represented under the form

$$P_0(t)A(t)Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) & 0 \\ \hline A_2(t) & 0 \end{array}\right), \quad P_0(t)B(t)Q_0(t) = \left(\begin{array}{c|c} B'_{1,1}(t) & B'_{1,2}(t) \\ \hline B_2(t) & 0 \end{array}\right)$$

At the end of Step 1:

$$P_0(t)(A(t) - sB(t))Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) - sB'_{1,1}(t) & -sB'_{1,2}(t) \\ \hline A_2(t) - sB_2(t) & 0 \end{array}\right)$$

- Step 2
  - Repeat the Step 1 for the pencil $A_2(t) - sB_2(t)$ until the step k where the matrix $A_{k+1}(t)$ is of full column rank.
  - At the step k:

$$P(t)(A(t) - sB(t))Q(t) = \left(\begin{array}{c|c} A_{k+1,k}(t) - sB'_{k+1,k}(t) & M_3(s,t) \\ \hline A_{k+1}(t) - sB_{k+1}(t) & 0 \end{array}\right)$$

- If the pencil $A_{k+1}(t) - sB_{k+1}(t)$ is not a regular pencil, repeat the above procedure to the transposed pencil $A^t_{k+1}(t) - sB^t_{k+1}(t)$.

At the end of Step 1:

$$P_0(t)(A(t) - sB(t))Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) - sB'_{1,1}(t) & -sB'_{1,2}(t) \\ \hline A_2(t) - sB_2(t) & 0 \end{array}\right)$$

- Step 2
  - Repeat the Step 1 for the pencil $A_2(t) - sB_2(t)$ until the step k where the matrix $A_{k+1}(t)$ is of full column rank.
  - At the step k:

$$P(t)(A(t) - sB(t))Q(t) = \left(\begin{array}{c|c} A_{k+1,k}(t) - sB'_{k+1,k}(t) & M_3(s,t) \\ \hline A_{k+1}(t) - sB_{k+1}(t) & 0 \end{array}\right)$$

- If the pencil $A_{k+1}(t) - sB_{k+1}(t)$ is not a regular pencil, repeat the above procedure to the transposed pencil $A^t_{k+1}(t) - sB^t_{k+1}(t)$.

At the end of Step 1:

$$P_0(t)(A(t) - sB(t))Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) - sB'_{1,1}(t) & -sB'_{1,2}(t) \\ \hline A_2(t) - sB_2(t) & 0 \end{array}\right)$$

- Step 2
  - Repeat the Step 1 for the pencil $A_2(t) - sB_2(t)$ until the step k where the matrix $A_{k+1}(t)$ is of full column rank.
  - At the step k:

$$P(t)(A(t) - sB(t))Q(t) = \left(\begin{array}{c|c} A_{k+1,k}(t) - sB'_{k+1,k}(t) & M_3(s,t) \\ \hline A_{k+1}(t) - sB_{k+1}(t) & 0 \end{array}\right)$$

- If the pencil $A_{k+1}(t) - sB_{k+1}(t)$ is not a regular pencil, repeat the above procedure to the transposed pencil $A^t_{k+1}(t) - sB^t_{k+1}(t)$.

At the end of Step 1:

$$P_0(t)(A(t) - sB(t))Q_0(t) = \left(\begin{array}{c|c} A_{1,1}(t) - sB'_{1,1}(t) & -sB'_{1,2}(t) \\ \hline A_2(t) - sB_2(t) & 0 \end{array}\right)$$

- Step 2
  - Repeat the Step 1 for the pencil $A_2(t) - sB_2(t)$ until the step k where the matrix $A_{k+1}(t)$ is of full column rank.
  - At the step k:

$$P(t)(A(t) - sB(t))Q(t) = \left(\begin{array}{c|c} A_{k+1,k}(t) - sB'_{k+1,k}(t) & M_3(s,t) \\ \hline A_{k+1}(t) - sB_{k+1}(t) & 0 \end{array}\right)$$

- If the pencil $A_{k+1}(t) - sB_{k+1}(t)$ is not a regular pencil, repeat the above procedure to the transposed pencil $A^t_{k+1}(t) - sB^t_{k+1}(t)$.

Algorithm for extracting the regular and discrete part

- At the end:

$$A(t) - sB(t) \Longrightarrow \left(\begin{array}{c|c|c} * & * & M_3(s,t) \\ \hline * & M_2(s,t) & 0 \\ \hline M_1(s,t) & 0 & 0 \end{array}\right)$$

Part II.2: Applications to Intersection Problems

The point in the curve intersection problem

Given a point $P \in \mathbb{P}^3$ and two parameterized surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$,
test whether $P \in \mathcal{S}_1 \cap \mathcal{S}_2$ **or not**.

- Compute an implicit equation $S_1(x, y, z, w)$ and $S_2(x, y, z, w)$ $\in \mathbb{K}[x, \dots, w]$
- Evaluate S_1 and S_2 at P
- Check if $\|S_1(P)\| < \epsilon$ and $\|S_2(P)\| < \epsilon$

- Compute two representation matrices $M_1(x, \dots, w)$ and $M_2(x, \dots, w)$ $\in \mathrm{Mat}(\mathbb{K}[x, \dots, w])$
- Evaluate $M_1(P)$ and $M_2(P)$
- Check if the ϵ-rank of $M_1(P)$ and $M_2(P)$ drops.

General philosophy:
replace implicit equations by representation matrices whenever possible

The point in the curve intersection problem

Given a point $P \in \mathbb{P}^3$ and two parameterized surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$,
test whether $P \in \mathcal{S}_1 \cap \mathcal{S}_2$ **or not**.

- Compute an implicit equation $S_1(x, y, z, w)$ and $S_2(x, y, z, w)$ $\in \mathbb{K}[x, \ldots, w]$
- Evaluate S_1 and S_2 at P
- Check if $\|S_1(P)\| < \epsilon$ and $\|S_2(P)\| < \epsilon$

- Compute two representation matrices $M_1(x, \ldots, w)$ and $M_2(x, \ldots, w)$ $\in \mathrm{Mat}(\mathbb{K}[x, \ldots, w])$
- Evaluate $M_1(P)$ and $M_2(P)$
- Check if the ϵ-rank of $M_1(P)$ and $M_2(P)$ drops.

General philosophy:
replace implicit equations by representation matrices whenever possible

The point in the curve intersection problem

Given a point $P \in \mathbb{P}^3$ and two parameterized surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, **test whether** $P \in \mathcal{S}_1 \cap \mathcal{S}_2$ **or not**.

- Compute an implicit equation $S_1(x, y, z, w)$ and $S_2(x, y, z, w)$ $\in \mathbb{K}[x, \dots, w]$
- Evaluate S_1 and S_2 at P
- Check if $\|S_1(P)\| < \epsilon$ and $\|S_2(P)\| < \epsilon$

- Compute two representation matrices $M_1(x, \dots, w)$ and $M_2(x, \dots, w)$ $\in \text{Mat}(\mathbb{K}[x, \dots, w])$
- Evaluate $M_1(P)$ and $M_2(P)$
- Check if the ϵ-rank of $M_1(P)$ and $M_2(P)$ drops.

General philosophy:
replace implicit equations by representation matrices whenever possible

The point in the curve intersection problem

Given a point $P \in \mathbb{P}^3$ and two parameterized surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$,
test whether $P \in \mathcal{S}_1 \cap \mathcal{S}_2$ **or not**.

- Compute an implicit equation $S_1(x, y, z, w)$ and $S_2(x, y, z, w)$ $\in \mathbb{K}[x, \dots, w]$
- Evaluate S_1 and S_2 at P
- Check if $\|S_1(P)\| < \epsilon$ and $\|S_2(P)\| < \epsilon$

- Compute two representation matrices $M_1(x, \dots, w)$ and $M_2(x, \dots, w)$ $\in \mathrm{Mat}(\mathbb{K}[x, \dots, w])$
- Evaluate $M_1(P)$ and $M_2(P)$
- Check if the ϵ-rank of $M_1(P)$ and $M_2(P)$ drops.

General philosophy:
replace implicit equations by representation matrices whenever possible

The point in the curve intersection problem

Given a point $P \in \mathbb{P}^3$ and two parameterized surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$,
test whether $P \in \mathcal{S}_1 \cap \mathcal{S}_2$ **or not**.

- Compute an implicit equation $S_1(x, y, z, w)$ and $S_2(x, y, z, w)$ $\in \mathbb{K}[x, \ldots, w]$
- Evaluate S_1 and S_2 at P
- Check if $\|S_1(P)\| < \epsilon$ and $\|S_2(P)\| < \epsilon$

- Compute two representation matrices $M_1(x, \ldots, w)$ and $M_2(x, \ldots, w)$ $\in \mathrm{Mat}(\mathbb{K}[x, \ldots, w])$
- Evaluate $M_1(P)$ and $M_2(P)$
- Check if the ϵ-rank of $M_1(P)$ and $M_2(P)$ drops.

General philosophy:
replace implicit equations by representation matrices whenever possible

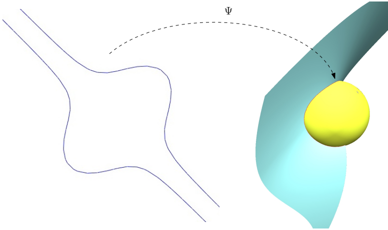
The surface/surface intersection problem

Suppose given two parameterized surfaces:

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^3 : (s : t : u) \mapsto (f_1 : \ldots : f_4)(s, t, u)$$

$$\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^3 : (s : t : u) \mapsto (g_1 : \ldots : g_4)(s, t, u)$$

with ψ *regular*.



Algorithm (Busé, Luu Ba): "Compute" the intersection curve

- Build the representation matrix $\mathbf{M}(\phi)$ of S_1.
- Form the companion matrices $A(t), B(t)$ of the matrix $\mathbf{M}(\phi)(g_1(s,t,1), \ldots, g_4(s,t,1))$.
- Compute the regular part $A'(t) - sB'(t)$ of the pencil $A(t) - sB(t)$.
- Compute the determinant of the pencil $A'(t) - sB'(t) \rightsquigarrow$ Curves C in the plane.
- Return the intersection curves $\{\psi(s,t,1) : (s,t) \in C\}$.

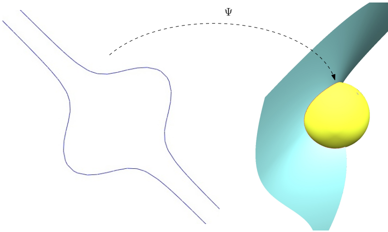
The surface/surface intersection problem

Suppose given two parameterized surfaces:

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^3 : (s:t:u) \mapsto (f_1 : \ldots : f_4)(s,t,u)$$

$$\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^3 : (s:t:u) \mapsto (g_1 : \ldots : g_4)(s,t,u)$$

with ψ *regular*.



Algorithm (Busé, Luu Ba): "Compute" the intersection curve

- Build the representation matrix $\mathbf{M}(\phi)$ of $\mathcal{S}_1$.
- Form the companion matrices $A(t), B(t)$ of the matrix $\mathbf{M}(\phi)(g_1(s,t,1), \ldots, g_4(s,t,1))$.
- Compute the regular part $A'(t) - sB'(t)$ of the pencil $A(t) - sB(t)$.
- Compute the determinant of the pencil $A'(t) - sB'(t) \rightsquigarrow$ Curves $\mathcal{C}$ in the plane.
- Return the intersection curves $\{\psi(s,t,1) : (s,t) \in \mathcal{C}\}$.

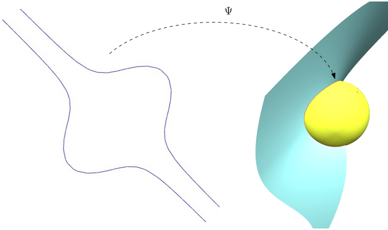
The surface/surface intersection problem

Suppose given two parameterized surfaces:

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^3 : (s : t : u) \mapsto (f_1 : \ldots : f_4)(s, t, u)$$

$$\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^3 : (s : t : u) \mapsto (g_1 : \ldots : g_4)(s, t, u)$$

with ψ *regular*.



Algorithm (Busé, Luu Ba): "Compute" the intersection curve

- Build the representation matrix $\mathbf{M}(\phi)$ of $\mathcal{S}_1$.
- Form the companion matrices $A(t), B(t)$ of the matrix $\mathbf{M}(\phi)(g_1(s,t,1), \ldots, g_4(s,t,1))$.
- Compute the regular part $A'(t) - sB'(t)$ of the pencil $A(t) - sB(t)$.
- Compute the determinant of the pencil $A'(t) - sB'(t) \rightsquigarrow$ Curves $\mathcal{C}$ in the plane.
- Return the intersection curves $\{\psi(s,t,1) : (s,t) \in \mathcal{C}\}$.

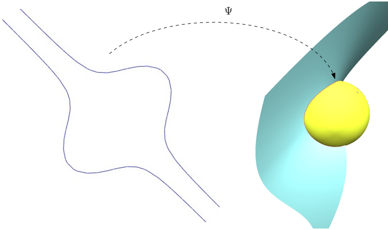
The surface/surface intersection problem

Suppose given two parameterized surfaces:

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^3 : (s:t:u) \mapsto (f_1 : \ldots : f_4)(s,t,u)$$

$$\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^3 : (s:t:u) \mapsto (g_1 : \ldots : g_4)(s,t,u)$$

with ψ *regular*.



Algorithm (Busé, Luu Ba): "Compute" the intersection curve

- Build the representation matrix $\mathbf{M}(\phi)$ of $\mathcal{S}_1$.
- Form the companion matrices $A(t), B(t)$ of the matrix $\mathbf{M}(\phi)(g_1(s,t,1),\ldots,g_4(s,t,1))$.
- Compute the regular part $A'(t) - sB'(t)$ of the pencil $A(t) - sB(t)$.
- Compute the determinant of the pencil $A'(t) - sB'(t) \rightsquigarrow$ Curves $\mathcal{C}$ in the plane.
- Return the intersection curves $\{\psi(s,t,1) : (s,t) \in \mathcal{C}\}$.

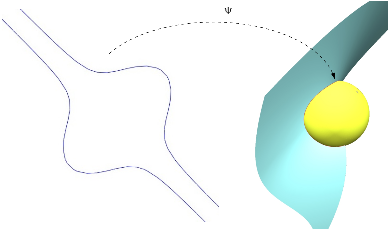
The surface/surface intersection problem

Suppose given two parameterized surfaces:

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^3 : (s:t:u) \mapsto (f_1 : \ldots : f_4)(s,t,u)$$

$$\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^3 : (s:t:u) \mapsto (g_1 : \ldots : g_4)(s,t,u)$$

with ψ *regular*.



Algorithm (Busé, Luu Ba): "Compute" the intersection curve

- Build the representation matrix $\mathbf{M}(\phi)$ of $\mathcal{S}_1$.
- Form the companion matrices $A(t), B(t)$ of the matrix $\mathbf{M}(\phi)(g_1(s,t,1), \ldots, g_4(s,t,1))$.
- Compute the regular part $A'(t) - sB'(t)$ of the pencil $A(t) - sB(t)$.
- Compute the determinant of the pencil $A'(t) - sB'(t) \rightsquigarrow$ Curves $\mathcal{C}$ in the plane.
- Return the intersection curves $\{\psi(s,t,1) : (s,t) \in \mathcal{C}\}$.

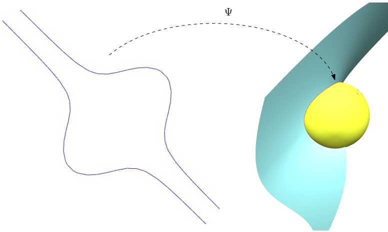
The surface/surface intersection problem

Suppose given two parameterized surfaces:

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^3 : (s:t:u) \mapsto (f_1 : \ldots : f_4)(s,t,u)$$

$$\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^3 : (s:t:u) \mapsto (g_1 : \ldots : g_4)(s,t,u)$$

with ψ *regular*.



Algorithm (Busé, Luu Ba): "Compute" the intersection curve

- Build the representation matrix $\mathbf{M}(\phi)$ of $\mathcal{S}_1$.
- Form the companion matrices $A(t), B(t)$ of the matrix $\mathbf{M}(\phi)(g_1(s,t,1), \ldots, g_4(s,t,1))$.
- Compute the regular part $A'(t) - sB'(t)$ of the pencil $A(t) - sB(t)$.
- Compute the determinant of the pencil $A'(t) - sB'(t) \rightsquigarrow$ Curves $\mathcal{C}$ in the plane.
- Return the intersection curves $\{\psi(s,t,1) : (s,t) \in \mathcal{C}\}$.

An example

- Suppose given two parameterized surfaces:

$$\mathbf{S}_1 : f_1 = s^2 + t^2 + u^2, f_2 = 2su, f_3 = 2st, f_4 = s^2 - t^2 - u^2$$
$$\mathbf{S}_2 : g_1 = s^3 + t^3, g_2 = stu, g_3 = su^2 + tu^2, g_4 = u^3.$$

- The matrix representation of the sphere $\mathbf{S}_1$ gives

$$\begin{pmatrix} -y & 0 & z & x+w \\ 0 & -y & -x+w & -z \\ z & x+w & y & 0 \end{pmatrix}.$$

- $P \in \mathbf{S}_1 \cap \mathbf{S}_2$ iff $P = (s^3 + t^3 : stu : su^2 + tu^2 : u^3)$ and $(s : t : u)$ is one of the generalized eigenvalues of the polynomial matrix

$$M(s,t,u) = \begin{pmatrix} -stu & 0 & su^2 + tu^2 & s^3 + t^3 + u^3 \\ 0 & -stu & -s^3 - t^3 + u^3 & -su^2 - tu^2 \\ su^2 + tu^2 & s^3 + t^3 + u^3 & st & 0 \end{pmatrix}.$$

An example

- Suppose given two parameterized surfaces:

$$\mathbf{S}_1 : f_1 = s^2 + t^2 + u^2, f_2 = 2su, f_3 = 2st, f_4 = s^2 - t^2 - u^2$$
$$\mathbf{S}_2 : g_1 = s^3 + t^3, g_2 = stu, g_3 = su^2 + tu^2, g_4 = u^3.$$

- The matrix representation of the sphere $\mathbf{S}_1$ gives

$$\begin{pmatrix} -y & 0 & z & x+w \\ 0 & -y & -x+w & -z \\ z & x+w & y & 0 \end{pmatrix}.$$

- $P \in \mathbf{S}_1 \cap \mathbf{S}_2$ iff $P = (s^3 + t^3 : stu : su^2 + tu^2 : u^3)$ and $(s : t : u)$ is one of the generalized eigenvalues of the polynomial matrix

$$M(s,t,u) = \begin{pmatrix} -stu & 0 & su^2 + tu^2 & s^3 + t^3 + u^3 \\ 0 & -stu & -s^3 - t^3 + u^3 & -su^2 - tu^2 \\ su^2 + tu^2 & s^3 + t^3 + u^3 & st & 0 \end{pmatrix}.$$

An example

- Suppose given two parameterized surfaces:

$$\mathbf{S}_1 : f_1 = s^2 + t^2 + u^2, f_2 = 2su, f_3 = 2st, f_4 = s^2 - t^2 - u^2$$
$$\mathbf{S}_2 : g_1 = s^3 + t^3, g_2 = stu, g_3 = su^2 + tu^2, g_4 = u^3.$$

- The matrix representation of the sphere $\mathbf{S}_1$ gives

$$\begin{pmatrix} -y & 0 & z & x+w \\ 0 & -y & -x+w & -z \\ z & x+w & y & 0 \end{pmatrix}.$$

- $P \in \mathbf{S}_1 \cap \mathbf{S}_2$ iff $P = (s^3 + t^3 : stu : su^2 + tu^2 : u^3)$ and $(s : t : u)$ is one of the generalized eigenvalues of the polynomial matrix

$$M(s,t,u) = \begin{pmatrix} -stu & 0 & su^2 + tu^2 & s^3 + t^3 + u^3 \\ 0 & -stu & -s^3 - t^3 + u^3 & -su^2 - tu^2 \\ su^2 + tu^2 & s^3 + t^3 + u^3 & st & 0 \end{pmatrix}.$$

An example

- The points $(s:t:u)$, $u \neq 0$, are correspondence to the set of the generalized eigenvalues $(s,t) \in \mathbb{C}^2$ of the bivariate matrix $M(s,t)$

$$M(s,t) = \begin{pmatrix} -st & 0 & s+t & s^3+t^3+1 \\ 0 & -st & -s^3-t^3+1 & -s-t \\ s+t & s^3+t^3+1 & st & 0 \end{pmatrix}.$$

- $M(s,t) = M_3 t^3 + M_2 t^2 + M_1 t + M_0$ and companion matrices

$$A(s) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ M_0^t & M_1^t & M_2^t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & s & -s & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s^3+1 & 0 & -s & 0 & 0 & 0 & 0 \\ s & -s^3+1 & 0 & 1 & 0 & s & 0 & 0 & 0 \\ s^3+1 & -s & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An example

- The points $(s:t:u)$, $u \neq 0$, are correspondence to the set of the generalized eigenvalues $(s,t) \in \mathbb{C}^2$ of the bivariate matrix $M(s,t)$

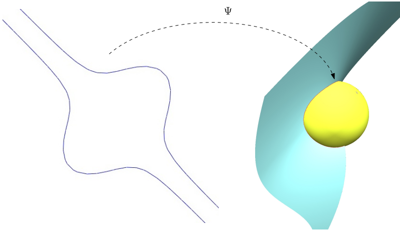
$$M(s,t) = \begin{pmatrix} -st & 0 & s+t & s^3+t^3+1 \\ 0 & -st & -s^3-t^3+1 & -s-t \\ s+t & s^3+t^3+1 & st & 0 \end{pmatrix}.$$

- $M(s,t) = M_3 t^3 + M_2 t^2 + M_1 t + M_0$ and companion matrices

$$A(s) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ M_0^t & M_1^t & M_2^t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & s & -s & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s^3+1 & 0 & -s & 0 & 0 & 0 & 0 \\ s & -s^3+1 & 0 & 1 & 0 & s & 0 & 0 & 0 \\ s^3+1 & -s & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An example

$$B(s)=\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&-1&0&0 \end{pmatrix}$$



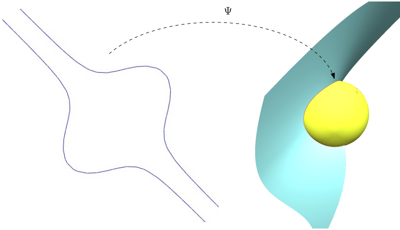
- The regular pencil part $M_1(s,t)=A_1(s)-tB_1(s)$ where

$$A_1(s)=\begin{pmatrix} 1&0&s&0&1&0\\ -s^3+1&0&1&0&0&0\\ -s^3+1&0&0&-s&0&0\\ 2s&0&0&1&s&0\\ 0&0&0&0&0&1\\ 0&1&0&-1&0&0 \end{pmatrix},\ B_1(s)=\begin{pmatrix} 0&0&1&0&0&0\\ 0&1&0&-1&0&0\\ s^3&1&0&0&0&0\\ -s^2&0&s&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix}.$$

- $\det(M_1(s,t))=-s^6-2s^3t^3+t^2s^2+s^2+2st-t^6+t^2+1$ is the equation of the curve $\mathcal{C}$ in the parametric space corresponding to $S_1\cap S_2$ through the regular map Ψ.

An example

$$B(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$



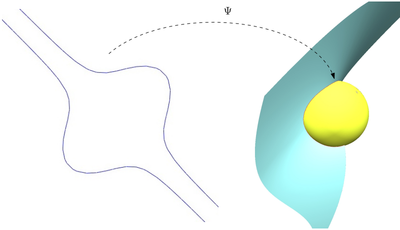
- The regular pencil part $M_1(s,t) = A_1(s) - tB_1(s)$ where

$$A_1(s) = \begin{pmatrix} 1 & 0 & s & 0 & 1 & 0 \\ -s^3+1 & 0 & 1 & 0 & 0 & 0 \\ -s^3+1 & 0 & 0 & -s & 0 & 0 \\ 2s & 0 & 0 & 1 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}, B_1(s) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ s^3 & 1 & 0 & 0 & 0 & 0 \\ -s^2 & 0 & s & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $\det(M_1(s,t)) = -s^6 - 2s^3t^3 + t^2s^2 + s^2 + 2st - t^6 + t^2 + 1$ is the equation of the curve $\mathcal{C}$ in the parametric space corresponding to $S_1 \cap S_2$ through the regular map Ψ.

An example

$$B(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$



- The regular pencil part $M_1(s,t) = A_1(s) - tB_1(s)$ where

$$A_1(s) = \begin{pmatrix} 1 & 0 & s & 0 & 1 & 0 \\ -s^3+1 & 0 & 1 & 0 & 0 & 0 \\ -s^3+1 & 0 & 0 & -s & 0 & 0 \\ 2s & 0 & 0 & 1 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}, B_1(s) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ s^3 & 1 & 0 & 0 & 0 & 0 \\ -s^2 & 0 & s & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $\det(M_1(s,t)) = -s^6 - 2s^3t^3 + t^2s^2 + s^2 + 2st - t^6 + t^2 + 1$ is the equation of the curve $\mathcal{C}$ in the parametric space corresponding to $S_1 \cap S_2$ through the regular map Ψ.

Thank you for your attention