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GRADIENT ENTROPY ESTIMATE AND CONVERGENCE OF A SEMI-EXPLICIT SCHEME FOR DIAGONAL HYPERBOLIC SYSTEMS*

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Abstract. In this paper, we consider diagonal hyperbolic systems with monotone continuous initial data. We propose a natural semi-explicit and upwind first-order scheme. Under a certain nonnegativity condition on the Jacobian matrix of the velocities of the system, there is a gradient entropy estimate for the hyperbolic system. We show that our scheme enjoys a similar gradient entropy estimate at the discrete level. This property allows us to prove the convergence of the scheme.

Key words. semi-explicit upwind scheme, diagonal hyperbolic systems, gradient entropy estimate, monotone initial data

AMS subject classifications. 35L45, 65M06, 65M12

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1. Introduction.

In this paper, we are interested in diagonal hyperbolic systems with monotone continuous initial data, and in their discretization. In this introduction, we first present our framework for such hyperbolic systems. We then propose a natural semiexplicit scheme. We give our main results, including the convergence of the scheme. Subsequently, we recall the related literature. Finally, we give the organization of this paper.

1.1. The continuous problem.

Let us consider the following diagonal hyperbolic system (in nonconservative form). Let $\mathbf{u} : \mathbb{R} \times [0,T] \to \mathbb{R}^d$ be a solution of

$$\frac{\partial u^\alpha}{\partial t} + \lambda^\alpha(\mathbf{u})\frac{\partial u^\alpha}{\partial x} = 0 \text{ in } \mathcal{D}'((0,+\infty)\times\mathbb{R}) \tag{1.1}$$

with initial data

$$u^\alpha(0,\cdot) = u_0^\alpha \quad \text{for} \quad \alpha = 1,\ldots,d. \tag{1.2}$$

In order to specify our conditions on the initial data, it will be useful to recall the definition of the Zygmund space:

$$L\log L(\mathbb{R}) = \left\{ w \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} |w|\ln(e+|w|) < +\infty \right\},$$

which is a Banach space with the norm

$$|w|_{L\log L(\mathbb{R})} = \inf\left\{ \mu > 0, \quad \int_{\mathbb{R}} \frac{|w|}{\mu}\ln\left(e + \frac{|w|}{\mu}\right) \le 1 \right\}.$$

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Then we will assume that the initial data satisfy

$$\left\{ \begin{array}{l} u_0^\alpha \quad \text{is bounded and nondecreasing,} \\ (u_0^\alpha)_x \in L\log L(\mathbb{R}) \end{array} \right| \quad \text{for} \quad \alpha = 1, \ldots, d. \tag{1.3}$$

In particular such initial data are continuous.

From now on, we equip the vector space $\mathbb{R}^d$ with the 1-norm $|\mathbf{u}| = \sum_{\alpha=1}^d |u^\alpha|$. We assume that

$$\boldsymbol{\lambda} \in C^1(\mathbb{R}^d;\mathbb{R}^d) \quad \text{with } \boldsymbol{\lambda} \text{ globally Lipschitz continuous} \tag{1.4}$$

with a Lipschitz constant $\mathrm{Lip}(\boldsymbol{\lambda})$. In addition, the symmetric part of the Jacobian matrix of $\boldsymbol{\lambda}$ is supposed to be nonnegative in the following sense:

$$\sum_{\alpha,\beta=1,\ldots,d} \xi_\alpha \xi_\beta \frac{\partial \lambda^\alpha}{\partial u^\beta}(u) \geq 0 \quad \text{for every} \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in [0,+\infty)^d, \quad u \in \mathbb{R}^d, \tag{1.5}$$

where we notice that this inequality is required only for a subset of vectors $\boldsymbol{\xi} \in \mathbb{R}^d$ with nonnegative coordinates. When $d = 1$, this condition for Burgers-type equations ensures that solutions associated to nondecreasing and continuous initial data will stay continuous for all positive times. Under assumption (1.5) for $d \geq 1$, we can recover a similar property: it is possible to show that the solutions formally satisfy the inequality

$$\frac{d}{dt} \int_{\mathbb{R}} \sum_{\alpha=1}^d u_x^\alpha \ln(u_x^\alpha) \leq 0. \tag{1.6}$$

Indeed, we refer the reader to Theorem 1.1 and Remark 1.4 in [10] for a precise statement. In some cases (in particular to ensure the uniqueness of the solution), we will also assume that the system is strictly hyperbolic, i.e., λ satisfies

$$\lambda^\alpha(\mathbf{u}) < \lambda^{\alpha+1}(\mathbf{u}) \quad \text{for} \quad \alpha = 1, \ldots, d-1. \tag{1.7}$$

We also define the total variation of $\mathbf{u}$ at time τ on the open interval (a,b):

$$TV[\mathbf{u}(\tau);(a,b)] = \sup \left\{ \sum_{\alpha=1,\ldots,d} \int_a^b -u^\alpha(\tau,x)\varphi_x^\alpha(x)dx \right\},$$

where the supremum is taken over the set of functions $\varphi^\alpha \in C_c^1(a,b)$ satisfying $|\varphi^\alpha(x)| \leq 1$ for $x \in (a,b)$ and $\alpha = 1, \ldots, d$. In the particular case where for each $\alpha = 1, \ldots, d$, the function $u^\alpha(\tau, \cdot)$ belongs to $W^{1,1}_{loc}(\mathbb{R})$ and is nondecreasing in space, then we simply have

$$TV[\mathbf{u}(\tau);(a,b)] = \int_a^b |\mathbf{u}_x(\tau,x)|dx = |\mathbf{u}(\tau,b) - \mathbf{u}(\tau,a)|,$$

where we set $\mathbf{u}(\tau, \pm\infty) = \lim_{x\to\pm\infty} \mathbf{u}(\tau,x)$ if $a = -\infty$ or $b = +\infty$. In the special case where $(a,b) = \mathbb{R}$, we will simply write $TV[\mathbf{u}(\tau)]$.

Definition 1.1 (continuous vanishing viscosity solutions). *A function* $\mathbf{u} \in [C([0,+\infty) \times \mathbb{R})]^d$ *is a continuous vanishing viscosity solution of system* (1.1)–(1.2) *if* $\mathbf{u}$ *solves* (1.1)–(1.2) *and if the following integral estimate holds.*

There exist constants C, γ, $\eta > 0$ such that, for every $\tau \geq 0$ and $a < \xi < b$, with $b - a \leq \eta$, one has the tame estimate

$$\limsup_{h\to 0^+} \frac{1}{h}\int_{a+\gamma h}^{b-\gamma h} |\mathbf{u}(\tau+h,x) - \mathbf{U}_{(\mathbf{u}(\tau);\tau,\xi)}(h,x)|dx \leq C\left(TV[\mathbf{u}(\tau);(a,b)]\right)^2, \tag{1.8}$$

where $\mathbf{U}_{\mathbf{u}(\tau);\tau,\xi}$ is the solution of the linear hyperbolic Cauchy problem with frozen constant coefficients:

$$\frac{\partial w^\alpha}{\partial t} + \lambda^\alpha(\mathbf{u}(\tau,\xi))\frac{\partial w^\alpha}{\partial x} = 0 \quad \text{with } w^\alpha(0,x) = u^\alpha(\tau,x).$$

This definition is in El Hajj and Monneau [11] and is an adaptation of the definition of Bianchini and Bressan [3] (see also in the book of Dafermos [8] the tame oscillation estimate for solutions constructed with the front tracking method; this last estimate is related to but less precise than the tame estimate (1.8)).

We then recall the following result (see Theorem 1.1 and Remark 1.4 in [10, 11]).

THEOREM 1.2 (existence, uniqueness). *Assume that initial data satisfy* (1.3), *and that $\boldsymbol{\lambda}$ satisfies* (1.4) *and* (1.5).

(i) (Existence) *Then there exists a function $\mathbf{u} \in (C([0,+\infty)\times\mathbb{R}))^d$ with $\mathbf{u}_x \in (L^\infty((0,+\infty); L\log L(\mathbb{R})))^d$, which is a continuous vanishing viscosity solution of* (1.1)–(1.2) *in the sense of Definition* 1.1.

(ii) (Uniqueness) *If, moreover, the system is strictly hyperbolic, i.e., $\boldsymbol{\lambda}$ safisties* (1.7)*, then there is uniqueness of the continuous vanishing viscosity solution $\mathbf{u}$ of* (1.1)–(1.2) *in the sense of Definition* 1.1.

1.2. The semiexplicit discretization. To recover these properties on the discrete level, we consider a time-step $\Delta t > 0$ and a space-step $\Delta x > 0$ and consider $u_i^{\alpha,n}$ as an approximation of $u^\alpha(n\Delta t, i\Delta x)$. We propose the following semiexplicit discretization of the system:

$$\forall\, \alpha \in \{1,\dots,d\},\ \left\{\begin{array}{l} \dfrac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} + \lambda^\alpha(\mathbf{u}_i^{n+1})\left(\dfrac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}\right) = 0 \text{ if } \lambda^\alpha(\mathbf{u}_i^{n+1}) \leq 0;\\ \dfrac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} + \lambda^\alpha(\mathbf{u}_i^{n+1})\left(\dfrac{u_{i}^{\alpha,n} - u_{i-1}^{\alpha,n}}{\Delta x}\right) = 0 \text{ if } \lambda^\alpha(\mathbf{u}_i^{n+1}) \geq 0. \end{array}\right.$$

It is a first-order upwind formulation, with the velocity $\boldsymbol{\lambda}(\mathbf{u})$ being implicit in time. We denote

$$\lambda_i^{\alpha,n+1} = \lambda^\alpha(\mathbf{u}_i^{n+1}),$$

and we define its positive and negative parts $(\lambda_i^{\alpha,n+1})_+$ and $(\lambda_i^{\alpha,n+1})_-$ as follows:

$$(\lambda_i^{\alpha,n+1})_+ = \frac{1}{2}(\lambda_i^{\alpha,n+1} + |\lambda_i^{\alpha,n+1}|), \quad (\lambda_i^{\alpha,n+1})_- = \frac{1}{2}(|\lambda_i^{\alpha,n+1}| - \lambda_i^{\alpha,n+1}).$$

Both $(\lambda_i^{\alpha,n+1})_+$ and $(\lambda_i^{\alpha,n+1})_-$ are positive real numbers. We can write the scheme in a more compact form:

$$\frac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} - (\lambda_i^{\alpha,n+1})_-\left(\frac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}\right) + (\lambda_i^{\alpha,n+1})_+\left(\frac{u_i^{\alpha,n} - u_{i-1}^{\alpha,n}}{\Delta x}\right) = 0. \tag{1.9}$$

In what follows, we set

$$\theta^{\alpha,n}_{i+\frac{1}{2}} = \frac{u^{\alpha,n}_{i+1} - u^{\alpha,n}_{i}}{\Delta x}, \tag{1.10}$$

which is a discrete equivalent of u^α_x.

For a fixed index i_0 and $N \in \mathbb{N}$, we denote

$$I_N(i_0) = \{i_0 - N, \dots, i_0 + N\}, \tag{1.11}$$

and we define $TV[\mathbf{u}^n; I_N(i_0)]$ to be the total variation of $\mathbf{u}^n$ on the set of indices $I_N(i_0)$:

$$TV[\mathbf{u}^n; I_N(i_0)] = \sum_{\alpha=1}^{d} \sum_{i \in I_N(i_0)} |u^{\alpha,n}_{i+1} - u^{\alpha,n}_{i}|.$$

The total variation of $\mathbf{u}^n$ on $\mathbb{Z}$ is simply noted $TV(\mathbf{u}^n)$.

1.3. Main results. We suppose that $u^{\alpha,0}$ is bounded in space by $m^\alpha > -\infty$ and $M^\alpha < +\infty$, and we denote

$$\mathcal{U} = \prod_{\alpha=1}^{d} [m^\alpha, M^\alpha] \quad \text{and} \quad \Lambda^\alpha = \sup_{\mathbf{u} \in \mathcal{U}} |\lambda^\alpha(\mathbf{u})|.$$

For $n \in \mathbb{N}$, we say that $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$ if $\mathbf{u}^n_i \in \mathcal{U}$ for all $i \in \mathbb{Z}$. We now introduce two CFL conditions:

$$\frac{\Delta t}{\Delta x} \mathrm{Lip}(\boldsymbol{\lambda}) TV(\mathbf{u}^0) < 1 \tag{1.12}$$

and

$$\frac{\Delta t}{\Delta x} \sum_{\alpha=1}^{d} \Lambda^\alpha < \frac{1}{2}. \tag{1.13}$$

We first prove that the semiexplicit scheme has a unique bounded solution at each time-step.

THEOREM 1.3 (resolution of the semiexplicit scheme on one time-step). *Assume that $\boldsymbol{\lambda}$ satisfies* (1.4). *Let $\mathbf{u}^0 \in \mathcal{U}^{\mathbb{Z}}$, and assume that the two CFL conditions* (1.12) *and* (1.13) *are satisfied.*

(i) (Existence) *Then for all $n \in \mathbb{N}$, there exists a unique solution $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$ to the semiexplicit scheme* (1.9).

(ii) (Monotonicity) *Moreover, if $\mathbf{u}^0$ is nondecreasing, i.e., satisfies*

$$u^{0,\alpha}_{i+1} \geq u^{0,\alpha}_{i} \quad \textit{for all} \quad i \in \mathbb{Z}, \quad \alpha = 1, \dots, d,$$

then for all $n \in \mathbb{N}$, $\mathbf{u}^n$ is also nondecreasing.

Remark 1.4. The resolution of the nonlinear problem boils down to the resolution of a local fixed point problem at each point $x_i = i\Delta x$. Note also that condition (1.12) is satisfied for $\mathbf{u}^0 \in \mathcal{U}^{\mathbb{Z}}$ nondecreasing if we have

$$\frac{\Delta t}{\Delta x} \mathrm{Lip}(\boldsymbol{\lambda}) \left(\sum_{\alpha=1,\dots,d} |M^\alpha - m^\alpha| \right) < 1.$$

Denoting $f(x) = x\ln(x)$, we then prove the following gradient entropy decay.

THEOREM 1.5 (gradient entropy decay). *Assume that $\boldsymbol{\lambda}$ satisfies assumptions* (1.4) *and* (1.5). *Let us consider initial data $\mathbf{u}^0 \in \mathcal{U}$ which are assumed to be nondecreasing, i.e.,*

$$u_{i+1}^{0,\alpha} \geq u_i^{0,\alpha} \quad \text{for all} \quad i \in \mathbb{Z}, \quad \alpha = 1, \ldots, d.$$

and let us consider the solution $\mathbf{u}^n$ of scheme (1.9), *assuming the CFL conditions* (1.12) *and* (1.13). *Then $\theta_{i+\frac{1}{2}}^{\alpha,n}$ (defined in* (1.10)*) is nonnegative for all $n \in \mathbb{N}$ and satisfies the following gradient entropy inequality for all $i_0 \in \mathbb{Z}$ and $N \in \mathbb{N}$:*

$$\sum_{\alpha=1}^{d} \sum_{i \in I_N(i_0)} f(\theta_{i+\frac{1}{2}}^{\alpha,n+1}) \leq \sum_{\alpha=1}^{d} \sum_{i \in I_N(i_0)} f(\theta_{i+\frac{1}{2}}^{\alpha,n}) - \frac{\Delta t}{\Delta x} \sum_{\alpha=1}^{d} \left(F_{i_0+N+1}^{\alpha,n} - F_{i_0-N}^{\alpha,n} \right), \tag{1.14}$$

where $I_N(i_0)$ is defined in (1.11) *and with the entropy flux*

$$F_i^{\alpha,n} = (\lambda_i^{\alpha,n+1})_+ f(\theta_{i-\frac{1}{2}}^{\alpha,n}) - (\lambda_i^{\alpha,n+1})_- f(\theta_{i+\frac{1}{2}}^{\alpha,n}). \tag{1.15}$$

In particular, formally for $N = +\infty$, the flux terms on the boundary disappear on the right-hand side of (1.14), and we recover (1.6).

We remark that f can become negative. In order to ensure that every term of the sum is nonnegative, we define $\tilde{f}$ as follows:

$$\text{for all } \theta \geq 0, \quad \tilde{f}(\theta) = \left(f(\theta) + \frac{1}{e} \right) \mathbb{1}_{\{\theta > \frac{1}{e}\}}(\theta). \tag{1.16}$$

$\tilde{f}$ is continuous, convex, and nonnegative. A technical entropy estimate similar to (1.14) is obtained on $\tilde{f}$ in Proposition 3.2, and will be used to estimate the $L\log L$ norm of $\mathbf{u}_x$ at a discrete level.

Then we have the following theorem.

THEOREM 1.6 (convergence of the solution of the scheme). *Assume that initial data $\mathbf{u}_0$ satisfy* (1.3), *and that $\boldsymbol{\lambda}$ satisfies* (1.4) *and* (1.5).

Then there exists a bounded set $\mathcal{U}$ such that $\mathbf{u}_0(x) \in \mathcal{U}$ for all $x \in \mathbb{R}$. Let us set the initial condition for the scheme

$$\mathbf{u}_i^0 = \mathbf{u}_0(i\Delta x),$$

and let us consider the solution $(\mathbf{u}^n)_{n\geq 0}$ of the scheme (1.9) *for time-step $\Delta t > 0$ and space-step $\Delta x > 0$ such that the CFL conditions* (1.12) *and* (1.13) *are satisfied for all $n \geq 0$. Let us call $\varepsilon = (\Delta t, \Delta x)$ and $\mathbf{u}^\varepsilon$ the function defined by*

$$\mathbf{u}^\varepsilon(n\Delta t, i\Delta x) = \mathbf{u}_i^n \quad \text{for} \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}.$$

Then as ε goes to zero, we have the following.

(i) (Convergence for a subsequence) *Up to extraction of a subsequence, there exists a continuous vanishing viscosity solution u of* (1.1)–(1.2) *such that for any compact $K \subset [0, +\infty) \times \mathbb{R}$, we have*

$$|\mathbf{u}^\varepsilon - \mathbf{u}|_{L^\infty(K \cap (\Delta t\mathbb{N}) \times (\Delta x\mathbb{Z}), \mathbb{R}^d)} \to 0 \quad \text{as} \quad \varepsilon \to (0,0).$$

(ii) (Convergence of the whole sequence) *If we assume, moreover, that* $\boldsymbol{\lambda}$ *satisfies the strict hyperbolicity condition* (1.7)*, then the whole sequence* $\mathbf{u}^\varepsilon$ *converges to the unique continuous vanishing viscosity solution* $\mathbf{u}$ *of* (1.1)–(1.2)*, as* ε *goes to zero.*

Remark 1.7. It would be interesting to adapt and extend the theory to the case where $\boldsymbol{\lambda}$ also depends on (t, x). At least for the scheme, this is an easy adaptation to write. It would also be interesting to extend the convergence of the solution of the scheme under the assumption of strict hyperbolicity (1.7) without assuming (1.5) as a discrete analogue of Theorem 1.2 in [11].

Remark 1.8. In fact, we show a slightly better estimate than (1.8) without the "limsup," with explicit constants.

1.4. Literature. For references on hyperbolic systems in nonconservative form, we refer the reader to the references cited in [10, 11]. Numerical schemes for hyperbolic systems are mainly written for systems in conservative form which enable recovering the correct Rankine–Hugoniot shock relations. We refer the reader to [14] for a review of the main classes of existing schemes. Among these schemes, convergence results are seldom found for hyperbolic systems.

The Lax–Wendroff theorem [13] shows that if a consistent and conservative numerical scheme converges (in L^1 with bounded total variation), its limit is a weak solution to the hyperbolic system. However, in order to obtain convergence of the scheme, stability is needed, in general in the form of TV-stability. For the scalar Godunov scheme, convergence is obtained due to its total variation diminishing (TVD) property. This is no longer the case for systems [14]. Stability can still be proved for certain special systems of two equations, for instance, in [17, 18, 15]. Similar results can be obtained for a class of nonlinear systems with straight-line fields [5, pp. 102–103]. Nonlinear stability can also be assessed through the use of invariant domains and entropy inequalities [4], for HLL, HLLC, and kinetic solvers for Euler equations of gas dynamics.

In the case of conservative systems where the initial data have sufficiently small total variation, Glimm's random choice method [12] is provably convergent. A deterministic variant (replacing random with equidistributed sampling) has also been proved to converge under the same assumptions [16]. We are not aware of convergence results of numerical schemes for nonconservative hyperbolic systems with large initial data.

1.5. Outline of this paper. This paper is organized as follows. In section 2, we prove some preliminary results on the existence of a solution to the scheme (Theorem 1.3), on the monotonicity and boundedness of the solution, and a discrete analogue of the tame estimate given in Definition 1.1. We then prove the decrease of the discrete entropy (Theorem 1.5) in section 3. In addition, we establish a similar entropic estimate for the scheme. Finally, in section 4, we sum up all the results and prove the convergence of the scheme (Theorem 1.6).

2. Preliminary results on the scheme. We begin by proving Lemmas 2.1, 2.3, and 2.4 at time $n\Delta t$, before applying a recursion on n to conclude the proof of Theorem 1.3.

2.1. Existence and uniqueness of the solution of the semiexplicit scheme.

Lemma 2.1 (existence and uniqueness at each time-step). *Assume that* $\boldsymbol{\lambda}$ *satisfies* (1.4)*. For a given* $n \in \mathbb{N}$*, let* $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$*, and assume that the CFL condition* (1.13) *is* *satisfied as well as CFL condition* (1.12) *at time* $n\Delta t$*:*

$$\frac{\Delta t}{\Delta x}\mathrm{Lip}(\boldsymbol{\lambda})TV(\mathbf{u}^n) < 1. \tag{2.1}$$

Then there exists a unique solution $\mathbf{u}^{n+1} \in \mathcal{U}^{\mathbb{Z}}$ *to the semiexplicit scheme* (1.9)*.*

Proof. We define the truncature $T\lambda^\alpha$ of λ^α by Λ^α:

$$T\lambda^\alpha(\mathbf{u}) = \begin{cases} \lambda^\alpha(\mathbf{u}) & \text{if } |\lambda^\alpha(\mathbf{u})| \le \Lambda^\alpha, \\ \Lambda^\alpha & \text{if } \lambda^\alpha(\mathbf{u}) > \Lambda^\alpha, \\ -\Lambda^\alpha & \text{if } \lambda^\alpha(\mathbf{u}) < -\Lambda^\alpha. \end{cases}$$

$\mathbf{T}\boldsymbol{\lambda}$ is also Lipschitz and $\mathrm{Lip}(\mathbf{T}\boldsymbol{\lambda}) \le \mathrm{Lip}(\boldsymbol{\lambda})$. For $\mathbf{v} \in \mathbb{R}^d$, let us define the function $\mathbf{F}_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}$ such that, for all $\alpha \in \{1, \dots, d\}$,

$$F^\alpha_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}(\mathbf{v}) = u_i^{\alpha,n} + \frac{\Delta t}{\Delta x}\left((T\lambda^\alpha(\mathbf{v}))_-(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}) - (T\lambda^\alpha(\mathbf{v}))_+(u_i^{\alpha,n} - u_{i-1}^{\alpha,n})\right).$$

Then the scheme (1.9) can be written (if $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$) as

$$\mathbf{u}_i^{n+1} = \mathbf{F}_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}(\mathbf{u}_i^{n+1}). \tag{2.2}$$

We observe that, for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^d$, for all $\alpha \in \{1, \dots, d\}$,

$$\begin{aligned} &|F^\alpha_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}(\mathbf{u}) - F^\alpha_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}(\mathbf{v})| \\ &\qquad \le \frac{\Delta t}{\Delta x}|T\lambda^\alpha(\mathbf{u}) - T\lambda^\alpha(\mathbf{v})|\left(|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}| + |u_i^{\alpha,n} - u_{i-1}^{\alpha,n}|\right) \\ &\qquad \le \frac{\Delta t}{\Delta x}\mathrm{Lip}(\boldsymbol{\lambda})TV(\mathbf{u}^n)|\mathbf{u} - \mathbf{v}|. \end{aligned}$$

Thus, $\mathbf{F}_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}$ is contractive on $\mathbb{R}^d$, thanks to CFL condition (2.1), and the Banach fixed point theorem yields the existence and uniqueness of a solution $\mathbf{u}$ of (2.2) on $\mathbb{R}^d$.

In addition, due to CFL condition (1.13), $F^\alpha_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}(\mathbf{u})$ is a convex combination of the $\mathbf{u}_i^n$, $\mathbf{u}_{i-1}^n$, and $\mathbf{u}_{i+1}^n$ contained in the convex $\mathcal{U}$, so that $\mathbf{u} = F^\alpha_{\mathbf{u}_i^n,\mathbf{u}_{i-1}^n,\mathbf{u}_{i+1}^n}(\mathbf{u})$ is also in $\mathcal{U}$. As $\mathbf{T}\boldsymbol{\lambda} = \boldsymbol{\lambda}$ on $\mathcal{U}$, we can conclude that the unique fixed point of (2.2) is the solution $\mathbf{u}_i^{n+1}$ of the scheme (1.9). ∎

2.2. Expression of $\boldsymbol{\theta}^{n+1}$. We derive an equation for the evolution of $\theta^{\alpha,n}_{i+\frac{1}{2}}$ in time.

LEMMA 2.2 (evolution of θ). *Let* $u_i^{\alpha,n}$ *be the solution of the semiexplicit scheme* (1.9)*. Then* $\theta^{\alpha,n+1}_{i+\frac{1}{2}}$ *satisfies the following relation:*

$$\begin{aligned} \theta^{\alpha,n+1}_{i+\frac{1}{2}} = &\left(1 - \frac{\Delta t}{\Delta x}((\lambda_{i+1}^{\alpha,n+1})_+ + (\lambda_i^{\alpha,n+1})_-)\right)\theta^{\alpha,n}_{i+\frac{1}{2}} \\ &+ \frac{\Delta t}{\Delta x}(\lambda_{i+1}^{\alpha,n+1})_-\theta^{\alpha,n}_{i+\frac{3}{2}} + \frac{\Delta t}{\Delta x}(\lambda_i^{\alpha,n+1})_+\theta^{\alpha,n}_{i-\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Proof. With the definition of $\theta^{\alpha,n}_{i+\frac{1}{2}}$, we observe

$$\theta^{\alpha,n+1}_{i+\frac{1}{2}} = \theta^{\alpha,n}_{i+\frac{1}{2}} + \frac{\Delta t}{\Delta x}\frac{u_{i+1}^{\alpha,n+1} - u_{i+1}^{\alpha,n}}{\Delta t} - \frac{\Delta t}{\Delta x}\frac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t}.$$

Inserting (1.9) at points x_i and x_{i+1}, we get (2.3). ∎

2.3. The discrete solution $\mathbf{u}^n$ is nondecreasing if $\mathbf{u}^0$ is nondecreasing.

LEMMA 2.3 (monotonicity). *For a given $n \in \mathbb{N}$, let $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$ be nondecreasing. Assume that $\boldsymbol{\lambda}$ satisfies* (1.4) *and assume the CFL condition* (1.13)*. Then the discrete solution $\mathbf{u}^{n+1}$ of* (2.2) *is nondecreasing.*

Proof. In (2.3), the coefficients $\frac{\Delta t}{\Delta x}(\lambda_{i+1}^{\alpha,n+1})_-$ and $\frac{\Delta t}{\Delta x}(\lambda_i^{\alpha,n+1})_+$ are positive by definition, Theorem 1.3, part (i) yields that $\mathbf{u}^{n+1}$ is in $\mathcal{U}$, and using the CFL condition (1.13), we obtain that

$$\left(1 - \frac{\Delta t}{\Delta x}((\lambda_{i+1}^{\alpha,n+1})_+ + (\lambda_i^{\alpha,n+1})_-)\right) \geq 0.$$

As $u_i^{\alpha,n}$ is nondecreasing, for all $i \in \mathbb{N}$ and $1 \leq \alpha \leq d$, $\theta_{i+\frac{1}{2}}^{\alpha,n} \geq 0$, and therefore, $\theta_{i+\frac{1}{2}}^{\alpha,n+1}$ is nonnegative, too. This is equivalent to $u_i^{\alpha,n+1}$ nondecreasing. ∎

2.4. The map $n \mapsto \mathbf{u}^n$ has a nonincreasing total variation.

LEMMA 2.4 (total variation decay). *Let $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$. Assume that $\boldsymbol{\lambda}$ satisfies* (1.4)*, and assume the CFL condition* (1.13)*. Let $i_0 \in \mathbb{Z}$ be a fixed index and $N \in \mathbb{N}\backslash\{0\}$. Then*

$$TV[\mathbf{u}^{n+1}; I_{N-1}(i_0)] \leq TV[\mathbf{u}^n; I_N(i_0)]$$

and

$$TV(\mathbf{u}^{n+1}) \leq TV(\mathbf{u}^n) \text{ if } TV(\mathbf{u}^n) < +\infty. \tag{2.4}$$

Proof. CFL condition (1.13) allows us to write $u_i^{\alpha,n+1}$ as a convex sum of $u_{i-1}^{\alpha,n}$, $u_i^{\alpha,n}$, and $u_{i+1}^{\alpha,n}$, so that

$$\begin{aligned}|u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}| \leq &\left(1 - \frac{\Delta t}{\Delta x}[(\lambda_i^{\alpha,n+1})_- + (\lambda_{i+1}^{\alpha,n+1})_+]\right)|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}| \\ &+ \frac{\Delta t}{\Delta x}(\lambda_{i+1}^{\alpha,n+1})_-|u_{i+2}^{\alpha,n} - u_{i+1}^{\alpha,n}| + \frac{\Delta t}{\Delta x}(\lambda_i^{\alpha,n+1})_+|u_i^{\alpha,n} - u_{i-1}^{\alpha,n}|.\end{aligned}$$

Summing these terms for $i \in I_{N-1}(i_0)$ gives a sum for $i \in I_{N-1}(i_0)$ of $|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}|$, and the remaining terms are for $i \in I_N(i_0)\backslash I_{N-1}(i_0)$ with coefficients inferior to 1 due to CFL condition (1.13). ∎

Proof of Theorem 1.3. We simply apply a recursion on $n \geq 0$, using Lemmas 2.1, 2.3, and 2.4, and check that the CFL condition (1.12) implies that the CFL condition (2.1) is verified for all $n \in \mathbb{N}$. ∎

2.5. A tame estimate for the scheme. In this subsection, we prove a discrete analogue to the continuous vanishing viscosity solution given in Definition 1.1 for the discrete solution u_i^n.

PROPOSITION 2.5 (discrete tame estimate). *Let $\mathbf{u}^n \in \mathcal{U}^{\mathbb{Z}}$. Assume that $\boldsymbol{\lambda}$ satisfies* (1.4) *and assume the two CFL conditions* (1.12) *and* (1.13)*. Then the following holds. Let (i_0, n_0) $\mathbb{Z} \times \mathbb{N}$ be fixed. Let $(\mathbf{v}^n)_{n \geq n_0}$ be the solution of the explicit discretization of the linear hyperbolic Cauchy problem with frozen constant coefficients for $n \geq n_0$:*

$$\frac{v_i^{\alpha,n+1} - v_i^{\alpha,n}}{\Delta t} - (\lambda^\alpha(\mathbf{u}_{i_0}^{n_0}))_-\left(\frac{v_{i+1}^{\alpha,n} - v_i^{\alpha,n}}{\Delta x}\right) + (\lambda^\alpha(\mathbf{u}_{i_0}^{n_0}))_+\left(\frac{v_i^{\alpha,n} - v_{i-1}^{\alpha,n}}{\Delta x}\right) = 0 \tag{2.5}$$

with $\mathbf{v}^{n_0} = \mathbf{u}^{n_0}$*. Then, for all* $k \in \mathbb{N}\backslash\{0\}$ *such that* $k \leq N$*,*

$$\frac{1}{k\Delta t}\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k}(i_0)}|u_i^{\alpha,n_0+k}-v_i^{\alpha,n_0+k}|\Delta x \leq 2\mathrm{Lip}(\boldsymbol{\lambda})\left(TV[u^{n_0};I_N(i_0)]\right)^2. \tag{2.6}$$

Proof. Let

$$\mathcal{I}_N^k = \sum_{\alpha=1}^{d}\sum_{i\in I_N(i_0)}|u_i^{\alpha,n_0+k}-v_i^{\alpha,n_0+k}|.$$

Using the schemes (1.9) and (2.5), we obtain

$$\begin{aligned}
\mathcal{I}_{N-k-1}^{k+1} \leq &\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k-1}(i_0)}\left|\left(1-\frac{\Delta t}{\Delta x}|\lambda^\alpha(\mathbf{u}_{i_0}^{n_0})|\right)(u_i^{\alpha,n_0+k}-v_i^{\alpha,n_0+k})\right|\\
&+\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k-1}(i_0)}\frac{\Delta t}{\Delta x}(\lambda^\alpha(\mathbf{u}_{i_0}^{n_0}))_-\left|u_{i+1}^{\alpha,n_0+k}-v_{i+1}^{\alpha,n_0+k}\right|\\
&+\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k-1}(i_0)}\frac{\Delta t}{\Delta x}(\lambda^\alpha(\mathbf{u}_{i_0}^{n_0}))_+\left|u_{i-1}^{\alpha,n_0+k}-v_{i-1}^{\alpha,n_0+k}\right|\\
&+\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k-1}(i_0)}\left|\frac{\Delta t}{\Delta x}((\lambda^\alpha(\mathbf{u}_i^{n_0+k+1}))_- -(\lambda^\alpha(\mathbf{u}_{i_0}^{n_0}))_-)(u_{i+1}^{\alpha,n_0+k}-u_i^{\alpha,n_0+k})\right|\\
&+\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k-1}(i_0)}\left|\frac{\Delta t}{\Delta x}((\lambda^\alpha(\mathbf{u}_i^{n_0+k+1}))_+ -(\lambda^\alpha(\mathbf{u}_{i_0}^{n_0}))_+)(u_i^{\alpha,n_0+k}-u_{i-1}^{\alpha,n_0+k})\right|.
\end{aligned}$$

CFL condition (1.13) gives us that $1-\frac{\Delta t}{\Delta x}|\lambda^\alpha(\mathbf{u}_{i_0}^{n_0})|$ is positive. In the right-hand side of the inequality, the first three terms can then be controlled by

$$\sum_{\alpha=1}^{d}\sum_{i\in I_{N-k}(i_0)}\left|u_i^{\alpha,n_0+k}-v_i^{\alpha,n_0+k}\right| = \mathcal{I}_{N-k}^k.$$

We note that, for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^d$, $|(\lambda^\alpha(\mathbf{u}))_- - (\lambda^\alpha(\mathbf{v}))_-| \leq |\lambda^\alpha(\mathbf{u})-\lambda^\alpha(\mathbf{v})|$ and $|(\lambda^\alpha(\mathbf{u}))_+ - (\lambda^\alpha(\mathbf{v}))_+| \leq |\lambda^\alpha(\mathbf{u})-\lambda^\alpha(\mathbf{v})|$, and we recall that

$$|\lambda^\alpha(\mathbf{u}_i^{n_0+k+1})-\lambda^\alpha(\mathbf{u}_{i_0}^{n_0})| \leq \mathrm{Lip}(\boldsymbol{\lambda})|\mathbf{u}_i^{n_0+k+1}-\mathbf{u}_{i_0}^{n_0}|.$$

Using the same convexity argument as in Lemma 2.3, it is easy to see that if, for some $K\in\mathbb{N}\backslash\{0\}$, we have

$$m_n^\alpha(I_K(i_0)) \leq u^{\alpha,n} \leq M_n^\alpha(I_K(i_0)) \quad \text{for all} \quad i\in I_K(i_0),$$

then we have

$$m_n^\alpha(I_K(i_0)) \leq u^{\alpha,n+1} \leq M_n^\alpha(I_K(i_0)) \quad \text{for all} \quad i\in I_{K-1}(i_0).$$

A straightforward recursion yields that u^{α,n_0+k+1} is bounded on $I_{N-k-1}(i_0)$ by the bounds of u^{α,n_0} on $I_N(i_0)$, namely $m_{n_0}^\alpha(I_N(i_0))$ and $M_{n_0}^\alpha(I_N(i_0))$. As a result, for all

$i \in I_{N-k-1}(i_0)$,

$$\begin{aligned}|\lambda^\alpha(\mathbf{u}_i^{n_0+k+1}) - \lambda^\alpha(\mathbf{u}_{i_0}^{n_0})| &\leq \mathrm{Lip}(\boldsymbol{\lambda})|\mathbf{M}_{n_0}(I_N(i_0)) - \mathbf{m}_{n_0}(I_N(i_0))| \\ &\leq \mathrm{Lip}(\boldsymbol{\lambda})TV[\mathbf{u}^{n_0}; I_N(i_0)].\end{aligned}$$

In the end, using Lemma 2.4, we deduce that

$$\mathcal{I}_{N-k-1}^{k+1} \leq \mathcal{I}_{N-k}^{k} + 2\Delta t \mathrm{Lip}(\boldsymbol{\lambda})\left(TV[\mathbf{u}^{n_0}; I_N(i_0)]\right)^2.$$

The result is then obtained through a straightforward recursion on k using the fact that $\mathcal{I}_N^0 = 0$ by definition. ∎

3. The gradient entropy. In this section, we define $f : \mathbb{R}^+ \to \mathbb{R}$ as the convex function $f(x) = x\ln(x)$.

3.1. Preparatory lemma.

LEMMA 3.1 (convexity inequality for f). *Let a_k and θ_k be two finite sequences of nonnegative real numbers such that $0 < \sum_k a_k < +\infty$. Define*

$$\theta = \sum_k a_k \theta_k.$$

Then the following inequality holds:

$$f(\theta) \leq \sum_k a_k f(\theta_k) + \theta \ln\left(\sum_k a_k\right).$$

Proof. As $\sum_k a_k > 0$,

$$\frac{1}{\sum_k a_k}\theta = \sum_k \frac{a_k}{\sum_l a_l}\theta_k$$

is a convex sum of the $\theta_k \geq 0$. Using the convexity of f on $\mathbb{R}^+$,

$$f\left(\frac{1}{\sum_k a_k}\theta\right) \leq \sum_k \frac{a_k}{\sum_l a_l} f(\theta_k).$$

Using the expression of $f(x) = x\ln(x)$,

$$f\left(\frac{1}{\sum_k a_k}\theta\right) = \frac{1}{\sum_k a_k}\left(f(\theta) - \theta\ln\left(\sum_k a_k\right)\right),$$

which proves the result. ∎

3.2. Proof of Theorem 1.5.

Proof. For i, α, and n fixed, Lemma 2.2 gives us the expression (2.3) for $\theta_{i+\frac{1}{2}}^{\alpha,n+1}$. Let us remark that the coefficients $a_1 = \frac{\Delta t}{\Delta x}(\lambda_{i+1}^{\alpha,n+1})_-$ and $a_2 = \frac{\Delta t}{\Delta x}(\lambda_i^{\alpha,n+1})_+$ are nonnegative by definition, and that CFL condition (1.13) yields

$$a_3 = \left(1 - \frac{\Delta t}{\Delta x}((\lambda_{i+1}^{\alpha,n+1})_+ + (\lambda_i^{\alpha,n+1})_-)\right) \geq 0.$$

Defining $\mu^{\alpha,n+1}_{i+\frac{1}{2}} = 1-(a_1+a_2+a_3)$, let us note that the CFL condition (1.12) joined to (2.4) also gives

$$1-\mu^{\alpha,n+1}_{i+\frac{1}{2}} = a_1+a_2+a_3 = \left(1-\frac{\Delta t}{\Delta x}(\lambda^{\alpha,n+1}_{i+1}-\lambda^{\alpha,n+1}_{i})\right) > 0. \tag{3.1}$$

Using Lemma 3.1 on the convex sum, we obtain

$$\begin{aligned} f(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le{}& \left(1-\frac{\Delta t}{\Delta x}((\lambda^{\alpha,n+1}_{i+1})_+ + (\lambda^{\alpha,n+1}_{i})_-)\right) f(\theta^{\alpha,n}_{i+\frac{1}{2}}) \\ &+ \frac{\Delta t}{\Delta x}(\lambda^{\alpha,n+1}_{i+1})_- f(\theta^{\alpha,n}_{i+\frac{3}{2}}) + \frac{\Delta t}{\Delta x}(\lambda^{\alpha,n+1}_{i})_+ f(\theta^{\alpha,n}_{i-\frac{1}{2}}) \\ &+ \theta^{\alpha,n+1}_{i+\frac{1}{2}} \ln(1-\mu^{\alpha,n+1}_{i+\frac{1}{2}}). \end{aligned}$$

Arranging terms, the expression exhibits a discrete divergence form:

$$\begin{aligned} f(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le{}& f(\theta^{\alpha,n}_{i+\frac{1}{2}}) + \frac{\Delta t}{\Delta x}\left((\lambda^{\alpha,n+1}_{i+1})_- f(\theta^{\alpha,n}_{i+\frac{3}{2}}) - (\lambda^{\alpha,n+1}_{i})_- f(\theta^{\alpha,n}_{i+\frac{1}{2}})\right) \\ &- \frac{\Delta t}{\Delta x}\left((\lambda^{\alpha,n+1}_{i+1})_+ f(\theta^{\alpha,n}_{i+\frac{1}{2}}) - (\lambda^{\alpha,n+1}_{i})_+ f(\theta^{\alpha,n}_{i-\frac{1}{2}})\right) \\ &+ \theta^{\alpha,n+1}_{i+\frac{1}{2}} \ln(1-\mu^{\alpha,n+1}_{i+\frac{1}{2}}). \end{aligned} \tag{3.2}$$

Summing (3.2) over $i \in I_N(i_0)$ and over α, the second and third terms cancel and we obtain

$$\begin{aligned} \sum_{\alpha=1}^{d} \sum_{i\in I_N(i_0)} f(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le{}& \sum_{\alpha=1}^{d} \sum_{i\in I_N(i_0)} f(\theta^{\alpha,n}_{i+\frac{1}{2}}) + \sum_{\alpha=1}^{d} \sum_{i\in I_N(i_0)} \theta^{\alpha,n+1}_{i+\frac{1}{2}} \ln(1-\mu^{\alpha,n+1}_{i+\frac{1}{2}}) \\ &- \frac{\Delta t}{\Delta x}\sum_{\alpha=1}^{d}(F^{\alpha,n}_{i_0+N+1}-F^{\alpha,n}_{i_0-N}) \end{aligned} \tag{3.3}$$

with $F^{\alpha,n}_{i}$ defined in (1.15). We observe that $\ln(1-\mu) \le -\mu$ for all $\mu < 1$, we note that $\mu^{\alpha,n+1}_{i+\frac{1}{2}} < 1$ due to (3.1), and we recall that $\theta^{\alpha,n+1}_{i+\frac{1}{2}}$ is nonnegative, so that

$$\begin{aligned} \sum_{\alpha=1}^{d} \sum_{i\in I_N(i_0)} f(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le{}& \sum_{\alpha=1}^{d} \sum_{i\in I_N(i_0)} f(\theta^{\alpha,n}_{i+\frac{1}{2}}) - \sum_{\alpha=1}^{d} \sum_{i\in I_N(i_0)} \theta^{\alpha,n+1}_{i+\frac{1}{2}} \mu^{\alpha,n+1}_{i+\frac{1}{2}} \\ &- \frac{\Delta t}{\Delta x}\sum_{\alpha=1}^{d}(F^{\alpha,n}_{i_0+N+1}-F^{\alpha,n}_{i_0-N}). \end{aligned}$$

Now, by definition,

$$\begin{aligned} \mu^{\alpha,n+1}_{i+\frac{1}{2}} &= \frac{\Delta t}{\Delta x}\left(\lambda^{\alpha}(\mathbf{u}^{n+1}_{i+1}) - \lambda^{\alpha}(\mathbf{u}^{n+1}_{i})\right) \\ &= \frac{\Delta t}{\Delta x}\sum_{\beta=1}^{d}\int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\beta}}\left(\mathbf{u}^{n+1}_{i} + \tau(\mathbf{u}^{n+1}_{i+1}-\mathbf{u}^{n+1}_{i})\right)\cdot(u^{\beta,n+1}_{i+1}-u^{\beta,n+1}_{i})d\tau. \end{aligned}$$

Thus, summing over α and using the definition of $\theta^{\alpha,n}_{i+\frac{1}{2}}$,

$$\sum_{\alpha=1}^{d} \mu^{\alpha,n+1}_{i+\frac{1}{2}} \theta^{\alpha,n+1}_{i+\frac{1}{2}} = \Delta t \int_0^1 \left(\nabla \boldsymbol{\lambda}(\mathbf{u}_i^{n+1} + \tau \Delta x \boldsymbol{\theta}^{n+1}_{i+\frac{1}{2}}) \cdot \boldsymbol{\theta}^{n+1}_{i+\frac{1}{2}} \right) \cdot \boldsymbol{\theta}^{n+1}_{i+\frac{1}{2}} d\tau \geq 0, \tag{3.4}$$

where we have used assumption (1.5). In the end, we obtain the gradient entropy decay:

$$\sum_{\alpha=1}^{d} \sum_{i \in I_N(i_0)} f(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \leq \sum_{\alpha=1}^{d} \sum_{i \in I_N(i_0)} f(\theta^{\alpha,n}_{i+\frac{1}{2}}) - \frac{\Delta t}{\Delta x} \sum_{\alpha=1}^{d} (F^{\alpha,n}_{i_0+N+1} - F^{\alpha,n}_{i_0-N}). \qquad \square$$

3.3. Gradient entropy estimate. As f is negative for $\theta \in (0, \frac{1}{e})$, we use the following similar result on $\tilde{f}$ as defined in (1.16) in order to have a discrete estimate on $\mathbf{u}_x$ in the $L \log L$ norm.

PROPOSITION 3.2 (gradient entropy estimate for the scheme). *Under the assumptions of Theorem* 1.5*, we have*

$$\sum_{\alpha=1}^{d} \sum_{i \in \mathbb{Z}} \tilde{f}(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \Delta x \leq \sum_{\alpha=1}^{d} \sum_{i \in \mathbb{Z}} \tilde{f}(\theta^{\alpha,n}_{i+\frac{1}{2}}) \Delta x + C \Delta t$$

if the right-hand side is finite, with $C = C_2 d \mathrm{Lip}(\boldsymbol{\lambda}) TV(\mathbf{u}^0)$*, where* $C_2 = \frac{1}{e \ln 2}$.

In order to prove this result, we first need two technical lemmas on $\tilde{f}$, analogous to Lemma 3.1.

LEMMA 3.3 (technical estimate). *Let* $\gamma_m > 1$*. There exist a nonnegative function* $g(\theta, \gamma)$ *and a constant* $C_{\gamma_m} > 0$ *(depending only on* γ_m*) such that, for all* $\theta > 0$ *and* $\gamma \in (0, \gamma_m)$,

$$\tilde{f}\left(\frac{\theta}{\gamma}\right) \geq \frac{1}{\gamma} \tilde{f}(\theta) - \frac{1}{\gamma} g(\theta, \gamma) \ln(\gamma) \tag{3.5}$$

and

$$|\theta - g(\theta, \gamma)| \leq C_{\gamma_m} = \frac{\gamma_m - 1}{e \ln(\gamma_m)}.$$

Proof. We detail the four cases.

Case A. $\frac{\theta}{\gamma} \geq \frac{1}{e}$ and $\theta \geq \frac{1}{e}$.

We have for $\gamma \neq 1$ that

$$\frac{1}{\ln(\gamma)} \left(\tilde{f}(\theta) - \gamma \tilde{f}\left(\frac{\theta}{\gamma}\right) \right) = \theta - \frac{1}{e} \frac{\gamma - 1}{\ln(\gamma)}.$$

We then set for any $\gamma > 0$ that

$$g(\theta, \gamma) = \left(\theta - \frac{1}{e} \frac{\gamma - 1}{\ln(\gamma)} \right)_+ . \tag{3.6}$$

This implies (3.5) for $\gamma \geq 1$. As $\gamma \in (0, \gamma_m)$, and $\frac{\gamma - 1}{\ln(\gamma)}$ is nonnegative increasing, we get

$$|\theta - g(\theta, \gamma)| \leq \frac{1}{e} \frac{\gamma_m - 1}{\ln(\gamma_m)}.$$

Now for $\gamma \le 1$, we have

$$g(\theta,\gamma) = \theta - \frac{1}{e}\frac{\gamma-1}{\ln(\gamma)} \ge g(\theta,1) = \theta - \frac{1}{e} \ge 0.$$

This shows that (3.5) still holds for $0 < \gamma \le 1$.

Case B. $\frac{\theta}{\gamma} \ge \frac{1}{e}$ and $\theta < \frac{1}{e}$.

Then we have $0 < \gamma < 1$ and

$$\begin{aligned}\tilde{f}\left(\frac{\theta}{\gamma}\right) - \frac{1}{\gamma}\tilde{f}(\theta) &= \frac{1}{\gamma}\theta\ln(\theta) - \frac{1}{\gamma}\theta\ln(\gamma) + \frac{1}{e}\\ &\ge -\frac{1}{\gamma}\theta\ln(\gamma) + \frac{1}{e}\frac{\gamma-1}{\gamma} = -\frac{1}{\gamma}g(\theta,\gamma)\ln(\gamma)\end{aligned}$$

for $g(\theta,\gamma)$ defined in (3.6).

Case C. $\frac{\theta}{\gamma} < \frac{1}{e}$ and $\theta \ge \frac{1}{e}$.

$$\tilde{f}\left(\frac{\theta}{\gamma}\right) - \frac{1}{\gamma}\tilde{f}(\theta) \ge -\frac{1}{\gamma}\theta\ln(\gamma).$$

We take $g(\theta,\gamma) = \theta \ge 0$ in this case.

Case D. $\frac{\theta}{\gamma} < \frac{1}{e}$ and $\theta < \frac{1}{e}$.

$$\tilde{f}\left(\frac{\theta}{\gamma}\right) - \frac{1}{\gamma}\tilde{f}(\theta) \ge 0.$$

We take $g(\theta,\gamma) = 0$ in this case, and we check that

$$|\theta - g(\theta,\gamma)| = \theta \le \frac{1}{e} \le \frac{\gamma_m - 1}{e\ln(\gamma_m)}. \qquad \square$$

LEMMA 3.4 (convexity inequality for $\tilde{f}$). *Let a_k and θ_k be two finite sequences of nonnegative real numbers such that $0 < \sum_k a_k < 2$. Define*

$$\theta = \sum_k a_k\theta_k.$$

Then the following inequality holds:

$$\tilde{f}(\theta) \le \sum_k a_k\tilde{f}(\theta_k) + g\left(\theta, \sum_k a_k\right)\ln\left(\sum_k a_k\right),$$

where $g(\theta,\gamma)$ is given by Lemma 3.3 for $\gamma_m = 2$.

Proof. The proof is an adaptation of the proof of Lemma 3.1 with $\gamma = \sum_k a_k$, using Lemma 3.3 in the convexity inequality for $\tilde{f}$. $\square$

Proof of Proposition 3.2. The proof can be directly adapted from the proof of Theorem 1.5. We observe that due to CFL condition (1.12), $1 - \mu^{\alpha,n+1}_{i+\frac{1}{2}} \in (0,2)$. We define the entropy flux associated with $\tilde{f}$:

$$\tilde{F}^{\alpha,n}_i = (\lambda^{\alpha,n+1}_i)_+\tilde{f}(\theta^{\alpha,n}_{i-\frac{1}{2}}) - (\lambda^{\alpha,n+1}_i)_-\tilde{f}(\theta^{\alpha,n}_{i+\frac{1}{2}}).$$

Since, for all $n \in \mathbb{N}$, $\mathbf{u}^n$ is bounded nondecreasing by Theorem 1.3, $\lim_{N\to\pm\infty} \theta^{\alpha,n}_{N-\frac{1}{2}} = 0$. Since $\tilde{f}$ is continuous, $\tilde{f}(0) = 0$ and $\boldsymbol{\lambda}$ is bounded on $\mathcal{U}$, $\lim_{N\to\pm\infty} \tilde{F}^{\alpha,n}_N = 0$, and the flux terms on the boundaries vanish. Using Lemma 3.3, we obtain the analogue of (3.3):

$$\sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n}_{i+\frac{1}{2}}) + \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} g(\theta^{\alpha,n+1}_{i+\frac{1}{2}}, 1-\mu^{\alpha,n+1}_{i+\frac{1}{2}}) \ln(1-\mu^{\alpha,n+1}_{i+\frac{1}{2}}).$$

As g is nonnegative, and $\ln(1-\mu) \le -\mu$ for all $\mu < 1$,

$$\sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n}_{i+\frac{1}{2}}) - \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} g(\theta^{\alpha,n+1}_{i+\frac{1}{2}}, 1-\mu^{\alpha,n+1}_{i+\frac{1}{2}}) \mu^{\alpha,n+1}_{i+\frac{1}{2}}.$$

Using Lemma 3.3, we have $|\theta - g(\theta,\gamma)| \le C_2$ for all $\theta \ge 0$ and $\gamma \in (0,2)$ with $\gamma_m = 2$. Using (3.4),

$$\sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n}_{i+\frac{1}{2}}) + C_2 \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} |\mu^{\alpha,n+1}_{i+\frac{1}{2}}|.$$

As $\mu^{\alpha,n+1}_{i+\frac{1}{2}} = \frac{\Delta t}{\Delta x}(\lambda^{\alpha,n+1}_{i+1} - \lambda^{\alpha,n+1}_{i})$, we get

$$\sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} |\mu^{\alpha,n+1}_{i+\frac{1}{2}}| \le d\frac{\Delta t}{\Delta x}\mathrm{Lip}(\boldsymbol{\lambda}) TV(\mathbf{u}^{n+1}).$$

Using Lemma 2.4, we deduce

$$\sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n+1}_{i+\frac{1}{2}}) \le \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{\alpha,n}_{i+\frac{1}{2}}) + C_2 d\frac{\Delta t}{\Delta x}\mathrm{Lip}(\boldsymbol{\lambda}) TV(\mathbf{u}^{0}). \qquad \square$$

4. Convergence.

4.1. Preliminaries. We recall the following result (see Lemma 3.2 in [10]).

LEMMA 4.1 ($L\log L$ estimate). *If $w \in L^1(\mathbb{R})$ is a nonnegative function, then $\int_{\mathbb{R}} \tilde{f}(w) < +\infty$ if and only if $w \in L\log L(\mathbb{R})$. Moreover, we have the following estimates:*

$$\int_{\mathbb{R}} \tilde{f}(w) \le 1 + |w|_{L\log L(\mathbb{R})} + |w|_{L^1(\mathbb{R})} \ln\left(1 + |w|_{L\log L(\mathbb{R})}\right) \tag{4.1}$$

and

$$|w|_{L\log L(\mathbb{R})} \le 1 + |w|_{L^1(\mathbb{R})} \ln(1+e^2) + \int_{\mathbb{R}} \tilde{f}(w). \tag{4.2}$$

Remark 4.2 (idea of the proof of Lemma 4.1). Recall that the proof follows from the following inequalities (for $\mu \in (0,1]$ and $w \ge 0$):

$$\tilde{f}(w) \le w\ln(e+\mu w) + w|\ln\mu| \quad \text{with} \quad 1/\mu = 1 + ||w||_{L\log L}$$

and

$$w\ln(e+w) \le 1 + w\ln(1+e^2) + \tilde{f}(w).$$

We can easily check the following result.

LEMMA 4.3 (trivial estimate). *For any $\gamma \geq 1$ and $\theta \geq 0$, we have*

$$\tilde{f}(\gamma\theta) \leq \gamma\tilde{f}(\theta) + \theta\tilde{f}(\gamma). \tag{4.3}$$

We recall the following result (see Lemma 4.3 in [10]).

LEMMA 4.4 (modulus of continuity). *Let $T > 0$. Assume that $v \in L^\infty((0,+\infty)\times\mathbb{R})$ such that*

$$|v_x|_{L^\infty((0,T);L\log L(\mathbb{R}))} + |v_t|_{L^\infty((0,T);L\log L(\mathbb{R}))} \leq C_1.$$

Then for all $\delta, h \geq 0$, and all $(t,x) \in (0,T-h)\times\mathbb{R}$, we have

$$|v(t+h,x+\delta) - v(t,x)| \leq 6C_1\left(\frac{1}{\ln(1+\frac{1}{h})} + \frac{1}{\ln(1+\frac{1}{\delta})}\right).$$

We will state a convergence result in the linear case which will be used later to establish the tame estimate (1.8) (see Step 4 of the proof of Theorem 1.6). We consider a scalar function v solution of a linear transport equation

$$v_t + \lambda^0 v_x = 0 \quad \text{on} \quad (0,+\infty)\times\mathbb{R}, \tag{4.4}$$

where λ^0 is a real constant with initial data

$$v(0,\cdot) = v_0. \tag{4.5}$$

We then consider a solution v^n of an upwind scheme

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} - (\lambda^\varepsilon)_-\left(\frac{v_{i+1}^n - v_i^n}{\Delta x}\right) + (\lambda^\varepsilon)_+\left(\frac{v_i^n - v_{i-1}^n}{\Delta x}\right) = 0 \qquad \text{for} \quad i\in\mathbb{Z}, \quad n\geq 0, \tag{4.6}$$

where λ^ε is a real constant with initial data

$$v_i^0 = v_i^\varepsilon. \tag{4.7}$$

PROPOSITION 4.5 (convergence for the linear scheme). *We consider a solution v of (4.4)–(4.5) with $v_0 \in BUC(\mathbb{R})$ (the space of bounded and uniformly continuous functions). We set $\varepsilon = (\Delta t, \Delta x)$ and consider the solution v^n to the scheme (4.6)–(4.7) for the CFL condition*

$$\frac{\Delta x}{\Delta t} \geq |\lambda^\varepsilon|.$$

We set $t_n = n\Delta t$, $x_i = i\Delta x$ and assume that

$$|\lambda^\varepsilon - \lambda^0| \to 0 \quad \textit{and} \quad \sup_{i\in\mathbb{Z}} |v_i^\varepsilon - v_0(x_i)| \to 0 \quad \textit{as} \quad \varepsilon\to 0.$$

Then for any compact set $K \subset [0,+\infty)\times\mathbb{R}$, we have, with $v^\varepsilon(t_n,x_i) = v_i^n$,

$$|v^\varepsilon - v|_{L^\infty(K\cap((\Delta t\mathbb{N})\times(\Delta x\mathbb{Z})))} \to 0 \quad \textit{as} \quad \varepsilon\to 0.$$

Proof of Proposition 4.5. For a viscosity solution v, this is an easy adaptation of the general convergence result of Barles and Souganidis [2]. It is also easy to check that the limit of the scheme (or directly that v) is also a solution in the sense of distributions. ◻

PROPOSITION 4.6 (weak-$*$ compactness). *We consider a sequence of functions θ^ε satisfying for some $T > 0$*

$$|\theta^\varepsilon(t,\cdot)|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} \tilde{f}(\theta^\varepsilon(t,\cdot)) \leq M_T \quad \text{for a.e.} \quad t \in (0,T)$$

with M_T a constant independent of ε. Then there exist a function θ and a constant $C_T = C(M_T)$ such that

$$|\theta(t,\cdot)|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} \tilde{f}(\theta(t,\cdot)) \leq C_T \quad \text{for a.e.} \quad t \in (0,T) \tag{4.8}$$

such that for any function $\varphi \in C_c((0,+\infty)\times\mathbb{R})$ (space of continuous functions with compact support), we have

$$\int_{(0,+\infty)\times\mathbb{R}} \theta^\varepsilon \, \varphi \to \int_{(0,+\infty)\times\mathbb{R}} \theta \, \varphi \quad \text{as} \quad \varepsilon \to 0. \tag{4.9}$$

First proof of Proposition 4.6. We follow here the lines of the proofs given in [9]. We consider φ with support in $(0,T)\times I$ with I a bounded interval. We recall that $L\log L(I)$ is defined as $L\log L(\mathbb{R})$ with $\mathbb{R}$ replaced by the interval I. It is known that $L\log L(I)$ is the dual of $E_{exp}(I) \subset L^\infty(I)$ (see Theorems 8.16, 8.18, 8.20 in Adams [1]). Therefore, $L^\infty((0,T); L\log L(I))$ is the dual of $L^1((0,T); E_{exp}(I))$ (see Theorem 1.4.19, page 17 in Cazenave and Haraux [7]). Moreover, $L^1((0,T); E_{exp}(I)) \subset L^1((0,T); L^\infty(I))$. From (4.2), we deduce that

$$|\theta^\varepsilon|_{L^\infty((0,T);L\log L(I))} \leq C_{T,I}.$$

By general weak-$*$ compactness (see Brezis [6]), we deduce that for a subsequence, there exists a limit θ (which a priori depends on the compact $[0,T]\times I$, but can be chosen independent a posteriori by a classical diagonal extraction argument) such that (4.9) holds. Finally, (4.8) follows from Lemma 4.1. ◻

Second proof of Proposition 4.6. We recall that from (4.2), we have

$$|\theta^\varepsilon|_{L^\infty((0,T);L\log L(I))} \leq C_{T,I}.$$

We set $A := (0,T)\times I$. We check that $\tilde{f} : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $\lim_{t\to+\infty} \frac{\tilde{f}(t)}{t} = +\infty$, and $\int_A \tilde{f}(\theta^\varepsilon) \leq M_T T$. The De la Vallée Poussin equi-integrability criterion [6] yields that $(\theta^\varepsilon)_\varepsilon$ is equi-integrable. We can then apply the Dunford–Pettis theorem (see Brezis [6]), which shows that $(\theta^\varepsilon)_\varepsilon$ is weakly compact in $L^1(A)$, i.e., for any $\varphi \in L^\infty(A)$, we have

$$\int_A \theta^\varepsilon \, \varphi \to \int_A \theta \, \varphi$$

for some function $\theta \in L^1(A)$. In particular this proves Proposition 4.6. ◻

Remark 4.7. Another way to obtain the equi-integrability of $(\theta^\varepsilon)_\varepsilon$ is through the use of an analogue of Lemma 4.1 on A (see Remark 4.2 for its justification), following the lines of the proofs given in [10]. We then have

$$|\theta^\varepsilon|_{L\log L(A)} \leq C'_{T,I}$$

for some new constant $C'_{T,I} > 0$. It is known (see page 234 in Adams [1]) that there is a Hölder inequality for the Orlicz space $L\log L(A)$ (with a constant C independent on A):

$$||uv||_{L^1(A)} \le C||u||_{L\log L(A)}||v||_{EXP(A)}$$

with

$$||v||_{EXP(A)} = \inf\left\{\lambda > 0, \quad \int_A (e^{\frac{|v|}{\lambda}} - 1) \le 1\right\}.$$

Applying this to $u = \theta^\varepsilon$ and $v = 1$, we get that for any measurable set $B \subset A$

$$||\theta^\varepsilon||_{L^1(B)} \le \frac{C''}{\ln(1 + 1/|B|)} \quad \text{with} \quad C'' = CC'_{T,I}.$$

This shows that the sequence θ^ε is uniformly integrable on A.

4.2. Proof of Theorem 1.6.

Proof. We define $\tilde{S}^n$ to be the discrete entropy estimate:

$$\tilde{S}^n = \sum_{\alpha=1}^{d}\sum_{i\in\mathbb{Z}} \tilde{f}(\theta^{n,\alpha}_{i+\frac{1}{2}})\Delta x.$$

Step 1: *Estimate on* $\tilde{S}^0$. Using the convexity of $\tilde{f}$, we have with $x_i = i\Delta x$

$$\tilde{f}\left(\theta^{0,\alpha}_{i+\frac{1}{2}}\right) = \tilde{f}\left(\frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} (u_0^\alpha)_x(y)dy\right) \le \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \tilde{f}((u_0^\alpha)_x(y))dy.$$

This implies that

$$\tilde{S}^0 \le \sum_{\alpha=1,\dots,d}\int_{\mathbb{R}} \tilde{f}((u_0^\alpha)_x(y))dy \le C_0,$$

where we have used (4.1) to estimate

$$C_0 := \sum_{\alpha=1,\dots,d}\left\{1 + |(u_0^\alpha)_x|_{L\log L(\mathbb{R})} + |(u_0^\alpha)_x|_{L^1(\mathbb{R})}\ln\left(1 + |(u_0^\alpha)_x|_{L\log L(\mathbb{R})}\right)\right\}.$$

Step 2: *Estimates on the* Q^1 *extension* $\mathbf{u}^\varepsilon$. We set $x_i = i\Delta x$ and $t_n = n\Delta t$. Now for $\varepsilon = (\Delta t, \Delta x)$, we define the Q^1 extension of the function defined on the grid, for any $(t,x) \in [t_n, t_{n+1}] \times [x_i, x_{i+1}]$, by

$$\begin{aligned}\mathbf{u}^\varepsilon(t,x) = {} & \left(\frac{t-t_n}{\Delta t}\right)\left\{\left(\frac{x-x_i}{\Delta x}\right)\mathbf{u}^{n+1}_{i+1} + \left(1 - \frac{x-x_i}{\Delta x}\right)\mathbf{u}^{n+1}_i\right\} \\ & + \left(1 - \frac{t-t_n}{\Delta t}\right)\left\{\left(\frac{x-x_i}{\Delta x}\right)\mathbf{u}^n_{i+1} + \left(1 - \frac{x-x_i}{\Delta x}\right)\mathbf{u}^n_i\right\}.\end{aligned} \tag{4.10}$$

Step 2.1: *Estimate on* $\mathbf{u}^\varepsilon_x$. We have for $(t,x) \in [t_n, t_{n+1}] \times (x_i, x_{i+1})$

$$\mathbf{u}^\varepsilon_x(t,x) = \left(\frac{t-t_n}{\Delta t}\right)\boldsymbol{\theta}^{n+1}_{i+\frac{1}{2}} + \left(1 - \frac{t-t_n}{\Delta t}\right)\boldsymbol{\theta}^n_{i+\frac{1}{2}}, \tag{4.11}$$

and then using the convexity of $\tilde{f}$,

$$\tilde{f}(u_x^{\varepsilon,\alpha}) \leq \left(\frac{t-t_n}{\Delta t}\right)\tilde{f}\left(\theta_{i+\frac{1}{2}}^{n+1,\alpha}\right) + \left(1-\frac{t-t_n}{\Delta t}\right)\tilde{f}(\theta_{i+\frac{1}{2}}^{n,\alpha}),$$

and then for $t \in [t_n, t_{n+1}]$, we get

$$\sum_{\alpha=1,\dots,d}\int_{\mathbb{R}}\tilde{f}(u_x^{\varepsilon,\alpha}) \leq \left(\frac{t-t_n}{\Delta t}\right)\tilde{S}^{n+1} + \left(1-\frac{t-t_n}{\Delta t}\right)\tilde{S}^n \leq \tilde{S}^0 + Ct, \tag{4.12}$$

where we have used Proposition 3.2 for the last inequality.

Step 2.2: *Estimate on* $\mathbf{u}_t^\varepsilon$. Let us define

$$\boldsymbol{\tau}_i^{n+\frac{1}{2}} = \frac{\mathbf{u}_i^{n+1} - \mathbf{u}_i^n}{\Delta t}.$$

We have for $(t,x) \in (t_n, t_{n+1}) \times [x_i, x_{i+1}]$

$$\begin{aligned} |\mathbf{u}_t^\varepsilon| &= \left|\left(\frac{x-x_i}{\Delta x}\right)\boldsymbol{\tau}_{i+1}^{n+\frac{1}{2}} + \left(1-\frac{x-x_i}{\Delta x}\right)\boldsymbol{\tau}_i^{n+\frac{1}{2}}\right| \\ &\leq \left(\frac{x-x_i}{\Delta x}\right)\left|\boldsymbol{\tau}_{i+1}^{n+\frac{1}{2}}\right| + \left(1-\frac{x-x_i}{\Delta x}\right)\left|\boldsymbol{\tau}_i^{n+\frac{1}{2}}\right|. \end{aligned} \tag{4.13}$$

Then integrating on $[x_i, x_{i+1})$,

$$\int_{x_i}^{x_{i+1}} |\mathbf{u}_t^\varepsilon| \leq \frac{\Delta x}{2}\left(|\boldsymbol{\tau}_{i+1}^{n+\frac{1}{2}}| + |\boldsymbol{\tau}_i^{n+\frac{1}{2}}|\right). \tag{4.14}$$

We recall that from the scheme we have with $\boldsymbol{\lambda}_i^{n+1} = \boldsymbol{\lambda}(\mathbf{u}_i^{n+1})$,

$$\tau_i^{n+\frac{1}{2},\alpha} = -(\lambda_i^{n+1,\alpha})_+\theta_{i-\frac{1}{2}}^{n,\alpha} + (\lambda_i^{n+1,\alpha})_-\theta_{i+\frac{1}{2}}^{n,\alpha} \tag{4.15}$$

and also recall the bound (Theorem 1.3 shows that $\mathbf{u}_i^{n+1} \in \mathcal{U}$)

$$|\lambda_i^{n+1,\alpha}| \leq M \quad \text{with} \quad M = \max(\Lambda^\alpha, 1).$$

We use the monotonicity of $u^{n,\alpha}$ and its bound by m^α and M^α to assess that

$$0 \leq \Delta x \sum_{i\in\mathbb{Z}} \theta_{i-\frac{1}{2}}^{n,\alpha} \leq |M^\alpha - m^\alpha|. \tag{4.16}$$

Summing (4.14) on all $i \in \mathbb{Z}$ and using (4.15) and (4.16), we obtain

$$\sum_{\alpha=1,\dots,d} |u_t^{\varepsilon,\alpha}|_{L^\infty((0,T),L^1(\mathbb{R}))} \leq 2M \sum_{\alpha=1,\dots,d} |M^\alpha - m^\alpha|. \tag{4.17}$$

Step 3: *Extraction of a convergent subsequence of* $\mathbf{u}^\varepsilon$. We recall the bound ($\mathbf{u}_i^n \in \mathcal{U}$)

$$|\mathbf{u}_i^n| \leq M_0/(2d) \quad \text{with} \quad M_0 = 2d\sum_{\alpha=1}^{d} \max(|m^\alpha|, |M^\alpha|), \tag{4.18}$$

which implies (using the monotonicity in x of u^ε)

$$\sum_{\alpha=1,\dots,d} |u_x^{\varepsilon,\alpha}|_{L^\infty((0,T);L^1(\mathbb{R}))} \le M_0 \tag{4.19}$$

and

$$\int_{\mathbb{R}} \sum_{\alpha=1,\dots,d} \tilde f(u_t^{\varepsilon,\alpha}) \le 4M(\tilde S^0 + Ct) + 4\tilde f(M)M_0. \tag{4.20}$$

From (4.19), (4.17), (4.12), (4.20), and the bound on $\tilde S^0$ given in Step 1, we see that for any $T>0$, we get the existence of a constant C_T such that

$$\sum_{\alpha=1,\dots,d} \left\{ |u_x^{\varepsilon,\alpha}|_{L^\infty((0,T);L\log L(\mathbb{R}))} + |u_t^{\varepsilon,\alpha}|_{L^\infty((0,T);L\log L(\mathbb{R}))} \right\} \le C_T,$$

where we have used (4.2) to estimate the $L\log L$ norm with, moreover, (4.19). We also notice that

$$\sum_{\alpha=1,\dots,d} |u^{\varepsilon,\alpha}| \le M_0/2.$$

We can then apply Lemma 4.4 to get that for any $(t,x)\in(0,T-h)\times\mathbb{R}$, we have

$$\sum_{\alpha=1,\dots,d} |u^{\varepsilon,\alpha}(t+h,x+\delta) - u^{\varepsilon,\alpha}(t,x)| \le 6C_T\left(\frac{1}{\ln(1+\frac{1}{h})} + \frac{1}{\ln(1+\frac{1}{\delta})}\right).$$

Therefore, by the Ascoli–Arzela theorem, we can extract a subsequence (still denoted by $\mathbf{u}^\varepsilon$) which converges to a limit function u on every compact set K of $[0,+\infty)\times\mathbb{R}$. In particular, we see that the limit function $\mathbf{u}$ satisfies the initial condition

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0.$$

Moreover, the limit $\mathbf{u}$ still satisfies

$$\sum_{\alpha=1,\dots,d} |u^{\alpha}(t+h,x+\delta) - u^{\alpha}(t,x)| \le 6C_T\left(\frac{1}{\ln(1+\frac{1}{h})} + \frac{1}{\ln(1+\frac{1}{\delta})}\right).$$

Step 4: *Tame estimate for* $\mathbf{u}$. We want to prove (1.8). To this end, we consider a big compact K such that the set

$$\mathcal{T} := \{x \ge a + \gamma(t-\tau)\} \cap \{x \le b - \gamma(t-\tau)\} \cap \{t \ge \tau\}$$

is in the interior of K. For any $\varepsilon = (\Delta t, \Delta x)$, we consider $i_0\in\mathbb{Z}$ and $N\in\mathbb{N}$ such that

$$[x_{i_0-(N-2)}, x_{i_0+(N-2)}] \subset [a,b] \subset [x_{i_0-N}, x_{i_0+N}].$$

We consider $n_0\in\mathbb{N}$ and $k\in\mathbb{N}\setminus\{0\}$ such that for $\tau_h = \tau + h$ we have

$$\tau \in [t_{n_0}, t_{n_0+1}) \quad \text{and} \quad \tau_h \in [t_{n_0+k}, t_{n_0+k+1}].$$

We recall from (2.6) that we have

$$\frac{1}{k\Delta t}\sum_{\alpha=1,\dots,d}\sum_{i\in I_{N-k}(i_0)}|u_i^{n_0+k,\alpha}-v_i^{n_0+k,\alpha}|\Delta x\le 2\mathrm{Lip}(\boldsymbol{\lambda})\left(TV[\mathbf{u}^{n_0};I_N(i_0)]\right)^2.$$

We recall that from Step 3 we have for $(t,x)\in[t_{n_0+k},t_{n_0+k+1}]\times[x_i,x_{i+1}]$

$$\sum_{i=1,\dots,d}|u_i^{n_0+k,\alpha}-u^\alpha(t,x)|\le 6C_T\left(\frac{1}{\ln(1+\frac{1}{\Delta t})}+\frac{1}{\ln(1+\frac{1}{\Delta x})}\right)$$
$$+\sum_{i=1,\dots,d}|u^{\varepsilon,\alpha}-u^\alpha|_{L^\infty(K\cap((\Delta t\mathbb{N})\times(\Delta x\mathbb{Z})))}.$$

Using Proposition 4.5, this implies, in particular, that as $\varepsilon\to(0,0)$,

$$\frac{1}{k\Delta t}\sum_{\alpha=1,\dots,d}\sum_{i\in I_{N-k}(i_0)}|u_i^{n_0+k,\alpha}-v_i^{n_0+k,\alpha}|\Delta x$$
$$\to\frac{1}{h}\sum_{\alpha=1,\dots,d}\int_{a+\gamma h}^{b-\gamma h}|u^\alpha(\tau+h,x)-v^\alpha(\tau+h,x)|dx$$

with (at least for a subsequence)

$$k\Delta t\to h,\quad k\Delta x\to\gamma h,$$

where γ can be chosen bounded in order to satisfy the CFL conditions (notice that γ is also bounded from below, also because of the CFL conditions). On the other hand, we have

$$TV[\mathbf{u}^{n_0};I_N(i_0)]=\sum_{\alpha=1,\dots,d}|u_{i_0+N+1}^{n_0,\alpha}-u_{i_0-N}^{n_0,\alpha}|,$$

and for the same reasons as previously, we get, in particular, that

$$TV[\mathbf{u}^{n_0};I_N(i_0)]\to\sum_{\alpha=1,\dots,d}|u^\alpha(\tau,b)-u^\alpha(\tau,a)|=TV[\mathbf{u}(\tau,\cdot);(a,b)].$$

Finally, we get

$$\frac{1}{h}\sum_{\alpha=1,\dots,d}\int_{a+\gamma h}^{b-\gamma h}|u^\alpha(\tau+h,x)-v^\alpha(\tau+h,x)|dx\le 2\mathrm{Lip}(\boldsymbol{\lambda})(TV[\mathbf{u}(\tau,\cdot);(a,b)])^2,$$

which implies (1.8).

Step 5: *Passing to the limit in the PDE.*

Step 5.1: *Preliminaries.*

From (4.10) and (4.15), we have, for $(t,x)\in(t_n,t_{n+1})\times(x_i,x_{i+1})$ with $\boldsymbol{\lambda}=\boldsymbol{\lambda}(\mathbf{u}(t,x))$ and $a_x\left(\frac{x-x_i}{\Delta x}\right)$, $b_x=\left(1-\frac{x-x_i}{\Delta x}\right)$,

$$\begin{aligned}u_t^{\varepsilon,\alpha}=&a_x\left\{-(\lambda_{i+1}^{n+1,\alpha})_+\theta_{i+1-\frac12}^{n,\alpha}+(\lambda_{i+1}^{n+1,\alpha})_-\theta_{i+1+\frac12}^{n,\alpha}\right\}\\&+b_x\left\{-(\lambda_i^{n+1,\alpha})_+\theta_{i-\frac12}^{n,\alpha}+(\lambda_i^{n+1,\alpha})_-\theta_{i+\frac12}^{n,\alpha}\right\}\\=&-\lambda_+^\alpha\left\{a_x\theta_{i+\frac12}^{n,\alpha}+b_x\theta_{i-\frac12}^{n,\alpha}\right\}\\&+\lambda_-^\alpha\left\{a_x\theta_{i+\frac32}^{n,\alpha}+b_x\theta_{i+\frac12}^{n,\alpha}\right\}+e^{\varepsilon,\alpha}(t,x)\end{aligned}\tag{4.21}$$

with

$$e^{\varepsilon,\alpha}(t,x) = a_x \left\{ -\left[(\lambda_{i+1}^{n+1,\alpha})_+ - \lambda_+^\alpha \right] \theta^{n,\alpha}_{i+1-\frac{1}{2}} + \left[(\lambda_{i+1}^{n+1,\alpha})_- - \lambda_-^\alpha \right] \theta^{n,\alpha}_{i+1+\frac{1}{2}} \right\} + b_x \left\{ -\left[(\lambda_{i}^{n+1,\alpha})_+ - \lambda_+^\alpha \right] \theta^{n,\alpha}_{i-\frac{1}{2}} + \left[(\lambda_{i}^{n+1,\alpha})_- - \lambda_-^\alpha \right] \theta^{n,\alpha}_{i+\frac{1}{2}} \right\}.$$

In particular, for any test function φ with compact support in $K := [0,T] \times \overline{B_R(0)}$, we have

$$\sum_{i=1,\dots,\alpha} \left| \int_{[0,+\infty)\times\mathbb{R}} \varphi \, e^{\varepsilon,\alpha} \right| \leq 4|\varphi|_\infty T M_0 \sup_{(\tau,y)\in K} \left(\sup_{|t_{n+1}-\tau|\leq \Delta t, \quad |x_i - y| \leq \Delta x} |\boldsymbol{\lambda}(\mathbf{u}_i^{n+1}) - \boldsymbol{\lambda}(\mathbf{u}(\tau,y))| \right),$$

where we have used (4.18). From the uniform convergence of u^ε on compact sets, we deduce, in particular, that

$$e^{\varepsilon,\alpha} \to 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty)\times\mathbb{R}).$$

Step 5.2: *Introduction of* $\boldsymbol{\theta}^\varepsilon$. We define the function $\boldsymbol{\theta}^\varepsilon$ as

$$\boldsymbol{\theta}^\varepsilon(t,x) = \boldsymbol{\theta}^n_{i+\frac{1}{2}} \quad \text{for} \quad (t,x) \in [t_n, t_{n+1}) \times [x_i, x_{i+1}).$$

Using (4.12) and (4.19) and applying Proposition 4.6, we know that there exists a limit $\boldsymbol{\theta}$ such that for any test function $\boldsymbol{\varphi}$ (smooth with compact support in $(0,T)\times I$), we have

$$\int_{(0,+\infty)\times\mathbb{R}} \boldsymbol{\theta}^\varepsilon \cdot \boldsymbol{\varphi} \to \int_{(0,+\infty)\times\mathbb{R}} \boldsymbol{\theta} \cdot \boldsymbol{\varphi}.$$

From (4.21), we also have, with $a_x = \dfrac{x}{\Delta x} - \left\lfloor \dfrac{x}{\Delta x} \right\rfloor$, $b_x = 1 - a_x$,

$$u_t^{\varepsilon,\alpha} - e^{\varepsilon,\alpha} = -\lambda_+^\alpha \left\{ a_x \theta^{\varepsilon,\alpha} + b_x \theta^{\varepsilon,\alpha}(\cdot, \cdot - \Delta x) \right\} + \lambda_-^\alpha \left\{ a_x \theta^{\varepsilon,\alpha}(\cdot, \cdot + \Delta x) + b_x \theta^{\varepsilon,\alpha} \right\}.$$

Then

$$A^{\varepsilon,\alpha} = \int_{(0,+\infty)\times\mathbb{R}} (u_t^{\varepsilon,\alpha} - e^{\varepsilon,\alpha}) \, \varphi$$

can be computed as follows:

$$A^{\varepsilon,\alpha} = \int_{(0,+\infty)\times\mathbb{R}} \theta^{\varepsilon,\alpha} \left\{ -a_x(\lambda_+^\alpha \varphi) - b_x(\lambda_+^\alpha\varphi)(\cdot, \cdot + \Delta x) + a_x(\lambda_-^\alpha\varphi)(\cdot, \cdot - \Delta x) + b_x(\lambda_-^\alpha \varphi) \right\}.$$

Let us define

$$B^{\varepsilon,\alpha} = \int_{(0,+\infty)\times\mathbb{R}} \theta^{\varepsilon,\alpha} \left\{ -a_x(\lambda_+^\alpha \varphi) - b_x(\lambda_+^\alpha\varphi) + a_x(\lambda_-^\alpha\varphi) + b_x(\lambda_-^\alpha \varphi) \right\} = \int_{(0,+\infty)\times\mathbb{R}} -\lambda^\alpha \varphi \, \theta^{\varepsilon,\alpha}.$$

As $\|\theta^\varepsilon\|_{L^\infty((0,T),L^1(\mathbb{R}))} < +\infty$, $\mathbf{u}$ is continuous from Theorem 1.2, and $\lambda^\alpha_\pm \phi = (\lambda^\alpha_\pm \circ \mathbf{u})\phi$ is uniform continuous as a continuous function with compact support, we then have

$$|A^{\varepsilon,\alpha} - B^{\varepsilon,\alpha}| \leq \left(\sup_{\pm} |(\lambda^\alpha_\pm \varphi)(\cdot, \cdot + \Delta x) - (\lambda^\alpha_\pm \varphi)|_{L^\infty((0,T)\times\mathbb{R})} \right) \int_{(0,T)\times\mathbb{R}} |\theta^{\varepsilon,\alpha}| \to 0.$$

On the other hand, we have

$$B^{\varepsilon,\alpha} \to \int_{(0,+\infty)\times\mathbb{R}} -\lambda^\alpha \varphi \, \theta^\alpha.$$

This finally shows that

$$\text{for all } \alpha \in \{1, \ldots, d\}, \quad u^\alpha_t + \lambda^\alpha \theta^\alpha = 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty)\times\mathbb{R}). \tag{4.22}$$

Step 5.3: *Consequence.* Starting from (4.11), we deduce similarly (as in Step 5.2) that

$$u^{\varepsilon,\alpha}_x \to \theta^\alpha \quad \text{in} \quad \mathcal{D}'((0,+\infty)\times\mathbb{R}).$$

Therefore,

$$\boldsymbol{\theta} = \mathbf{u}_x,$$

and from (4.22), we deduce that

$$\text{for all } \alpha \in \{1, \ldots, d\}, \quad u^\alpha_t + \lambda^\alpha u^\alpha_x = 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty)\times\mathbb{R}),$$

with $\mathbf{u}_x \in L^\infty_{loc}([0,+\infty); L\log L(\mathbb{R}))^d$.

Step 6: *Convergence of the whole sequence when the limit is unique.* When we have, moreover, condition (1.7) for strictly hyperbolic systems, we know that the solution $\mathbf{u}$ is unique (among continuous vanishing viscosity solutions). Therefore, the whole sequence $\mathbf{u}^\varepsilon$ converges locally uniformly to its unique limit $\mathbf{u}$. This ends the proof of the theorem. ◻

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