Syzygies and distance functions to parameterized curves and surfaces

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Overall motivation

Compute, or estimate, the distance of a point to a parameterized curve or surface in $\mathbb{R}^3$.

- This is a famous problem which is **important in practice**, and there is an important existing literature on this topic.
- **A new approach** : explore the use of syzygies to tackle this distance problem.
- **Accuracy and robustness** are important in practice; our approach relies on numerical linear algebra tools.

Content of the talk

1. Overview of matrix representation (M-Rep) of parameterized curves and surfaces
2. Mrep, a distance-like function and an implicitization formula
3. Computation of closest point(s) – euclidean distance function
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Compute, or estimate, the **distance of a point to a parameterized curve or surface in** $\mathbb{R}^3$

- This is a famous problem which is **important in practice**, and there is an important existing literature on this topic.
- **A new approach**: explore the use of syzygies to tackle this distance problem
- **Accuracy and robustness** are important in practice; our approach relies on **numerical linear algebra tools**.

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1. Overview of matrix representation (M-Rep) of parameterized curves and surfaces
2. Mrep, a distance-like function and an implicitization formula
3. Computation of closest point(s) – euclidian distance function
1. Matrix representations (M-Rep)

**Given:** A parameterization $\phi$ of a curve or surface

$$
\begin{align*}
    t & \in \mathbb{R}^1 \\
    \text{or} \\
    (s, t) & \in \mathbb{R}^2
\end{align*}
\right\} \xrightarrow{\phi} \left( \frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_0} \right) \in \mathbb{R}^3
$$

where the $f_i$'s are polynomials of degree $d$ (without common gcd).

It encapsulates the following cases:

- A space curve of degree $d$.
- A triangular surface.
- A tensor-product surface with bi-degree $d := (d_1, d_2)$.

**Notation:** The $f_i$'s are supposed to be given in a polynomial basis $\mathcal{B}_d$ (e.g. monomial, Bernstein, ...).
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\phi \rightarrow \left( \frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_0} \right) \in \mathbb{R}^3
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M-Rep are built from syzygies

For all (bi-)degree $\nu \geq 0$, build the matrix $\mathbb{M}_\nu(\phi)$ as follows:

1. Compute a basis $L_1, \ldots, L_r$ of the vector space of 4-uples of polynomials $(g_0, g_1, g_2, g_3)$ such that

$$\deg(g_i) = \nu \text{ and } \sum_{i=0}^{3} g_if_i \equiv 0.$$  

$\Rightarrow$ solve a single linear system

2. Identify each basis element $L_j = (g_0, g_1, g_2, g_3)$ with the linear equation

$$L_j := g_0 + Xg_1 + Yg_2 + Zg_3.$$  

3. Build the matrix of coefficients:

$$\mathbb{M}_\nu(\phi) := \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ L_1 & L_2 & \cdots & L_r \end{pmatrix} B_\nu \in \text{Mat} (\mathbb{R}[X,Y,Z]_{\leq 1})$$
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Theorem (drop-of-rank property)

For all (bi-)degree \( \nu \geq \nu_0 \)

- \( \mathbb{M}_\nu(\phi) \) is generically full rank (= number of rows)
- the rank of \( \mathbb{M}_\nu(\phi) \) drops exactly on the closure of \( \text{Im}(\phi) \).

The matrix \( \mathbb{M}_\nu(\phi) \) is called a **matrix representation (M-Rep)**.

The (bi-)degree \( \nu_0 \) is defined as follows:

- If \( \phi \) defines a curve then \( \nu_0 := d - 1 \) (can be lower) \cite{LuuBa10}
- If \( \phi \) defines a triangular surface, set \( \nu_0 := 2(d - 1) \)
  (or \( \nu_0 := 2(d - 1) - 1 \) if there are base points) \cite{ChardinJouanolou05}
- If \( \phi \) defines a tensor-product surface, take \( \nu_0 := (d_1 - 1, 2d_2 - 1) \)
  (or \( \nu_0 = (2d_1 - 1, d_2 - 1) \)) \cite{Botbol11}.

**Remark:** There is a technical hypothesis on the nature of the base points: they need to be local complete intersections (not restrictive).
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Example: the sphere

A parameterization of the sphere:

\[ \mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^3 \]

\[ (s, t) \mapsto \left( \frac{1-s^2-t^2}{1+s^2+2t^2}, \frac{2s}{1+s^2+2t^2}, \frac{2t}{1+s^2+2t^2} \right) \]

A matrix representation of the sphere

\[ M_1(\phi) = \begin{pmatrix}
-Z & -Y & 1-X & 0 \\
0 & 1+X & -Y & -Z \\
1+X & 0 & -Z & Y
\end{pmatrix} \begin{pmatrix}
1 \\
s \\
t
\end{pmatrix} \]

In a computer, the sphere is given by 4 matrices with coefficients in \( \mathbb{R} \):

\[ M_1(\phi) = M_0 + M_1X + M_2Y + M_3Z. \]
Example: a space curve

Let $C$ be the rational space curve parameterized by

$$t \in \mathbb{R}^1 \xrightarrow{\phi} \left( \frac{f_1}{f_0}, \frac{f_2}{f_0}, \frac{f_3}{f_0} \right) \in \mathbb{R}^3$$

where

\[
\begin{align*}
    f_0(t) &= 6t^6 + 12t^5 - 3t^4 - 9t^3 + 3t^2, \\
    f_1(t) &= -6t^6 + 6t^5 + 33t^4 - 12t^3 - 27t^2 + 18t - 3, \\
    f_2(t) &= -6t^6 + 14t^5 + 9t^4 - 16t^3 + 13t^2 - 6t + 1, \\
    f_3(t) &= -6t^6 + 20t^5 - 14t^4 + 8t^3 - 2t^2.
\end{align*}
\]

We can take $\nu = 3$ and get the following matrix representation

\[
M_3(\phi) = \begin{pmatrix}
1 & 0 & X + 3Y & 0 & Z & 0 \\
-3 & 1 & -X - 3Y & X + 3Y & -2Z & Z \\
X + 1 & -3 & 3X - 3Y & -X - 3Y & 2Y - 2Z & -2Z \\
0 & X + 1 & 0 & 3X - 3Y & 0 & 2Y - 2Z
\end{pmatrix}
\]
M-Rep and numerical computations

The matrices $M_0, M_1, M_2, M_3$ can be computed numerically

- Computation of a numerical kernel via a SVD (within a tolerance)
- Distance between kernel and approx. kernel is controlled (theory)

The drop-of-rank property has good numerical behaviors (via SVD)

- Given $P \in \mathbb{R}^3$ and a tolerance $\varepsilon$,
  \[ \|P_\varepsilon - P\|_2 \leq \varepsilon \Rightarrow \text{rank}_\varepsilon(M(P_\varepsilon)) \leq \text{rank}(M(P)). \]

- Roughly, ($S$ stands for the curve or surface)
  \[ \text{dist}(P, S) \leq a \varepsilon^{(m-r_\varepsilon)b} \]

$a, b$ constants, $r_\varepsilon := \text{rank}_\varepsilon(M(P))$, $m =$ number of rows of $M(P)$. 
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Come back to the sphere

A (triangular Bézier) parameterization of the sphere:

\[
\phi(s, t) = \left( \frac{1 - s^2 - t^2}{1 + s^2 + t^2}, \frac{2s}{1 + s^2 + t^2}, \frac{2t}{1 + s^2 + t^2} \right)
\]

▶ Exact computation of matrix representation

\[
M = \begin{bmatrix}
-1 & -Y & 1 - X & 0 \\
0 & 1 + X & -Y & -Z \\
1 + X & 0 & -Z & Y
\end{bmatrix}
\]

▶ Approximate computation of matrix representation by SVD

\[
\begin{bmatrix}
-0.000 + 0.000X - 0.354Y + 0.354Z & 0.0811 - 0.0811X + 0.324Y + 0.324Z \\
-0.354 - 0.354X + 0.000Y + 0.354Z & -0.243 - 0.406X + 0.324Y + 0.243Z \\
0.354 + 0.354X - 0.354Y + 0.000Z & -0.243 - 0.406X + 0.243Y + 0.324Z \\
-0.354 + 0.354X + 0.000Y - 0.000Z & -0.005 + 0.005X - 0.000Y + 0.000Z \\
-0.354 + 0.354X + 0.0107Y + 0.354Z & -0.005 + 0.005X - 0.707Y + 0.005Z \\
-0.354 + 0.354X + 0.354Y - 0.0107Z & -0.005 + 0.005X + 0.005Y + 0.707Z
\end{bmatrix}
\]
2. A distance-like function from an M-Rep

Suppose given

- A parameterization $\phi$ of a curve or surface $S$
- An M-Rep $M := M_\nu(\phi)$, of size $m \times n$
- A point $P \in \mathbb{R}^3$

Evaluate $M(X, Y, Z)$ at the point $P$. From the SVD of $M(P)$

$$
M(P) \sim \begin{bmatrix}
\sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \sigma_2 & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \sigma_m & 0 & \cdots & 0 \\
\end{bmatrix} \in \mathbb{R}^{m \times n}
$$

we deduce that

$$
\text{rank}(M(P)) < m \iff P \text{ belongs to } S \\
\iff \sigma_m(M(P)) = 0 \\
\iff \prod_{i=1}^{m} \sigma_i(M(P)) = 0
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Evaluate $\mathbb{M}(X, Y, Z)$ at the point $P$. From the SVD of $\mathbb{M}(P)$

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A distance-like function

Recall that the M-Rep $\mathbb{M}(X, Y, Z)$ is of size $m \times n$.

**Definition**

$$\delta_{\mathbb{M}} : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$$

$$P \mapsto \delta_{\mathbb{M}}(P) := \prod_{i=1}^{m} \sigma_i(\mathbb{M}(P))$$

where $\sigma_i(\mathbb{M}(P))$ are the singular values of $\mathbb{M}(P)$.

- $\delta_{\mathbb{M}}$ vanishes exactly on the curve or surface $S$
- $\delta_{\mathbb{M}}$ is similar to a distance-like function obtained by means of an implicit equation
- The growth of $\delta_{\mathbb{M}}$ can be compared to the usual Euclidean distance, denoted $\text{dist}(-, S)$ (classical Lojasiewicz inequality)

$$\exists c_1, n_1, c_2, n_2 \text{ such that}$$

$$\text{dist}(P, S)^{n_1} \leq c_1 \delta_{\mathbb{M}}(P), \quad \delta_{\mathbb{M}}(P)^{n_2} \leq c_2 \text{dist}(P, S).$$
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Come back again to the sphere

A (triangular Bézier) parameterization of the sphere:

\[
\phi(s, t) = \left( \frac{1 - s^2 - t^2}{1 + s^2 + t^2}, \frac{2s}{1 + s^2 + t^2}, \frac{2t}{1 + s^2 + t^2} \right)
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By-product: an implicitization formula

- As the euclidian distance, the square of $\delta_M$ is algebraic:

$$\delta_M(P)^2 = \prod_{i=1}^{m} \sigma_i(M(P))^2 = \det(M(P).M(P)^T).$$

So we define the polynomial

$$\Delta_M(X, Y, Z) := \det(M.M^T) \in \mathbb{R}[X, Y, Z]$$

and we get an implicization formula in the presence of base points, over the real numbers:

**Theorem ([–'13])**

Assuming the base points of $\phi$ are finitely many and locally complete intersection, then $\Delta_M(X, Y, Z)$ is a real implicit equation of the curve or surface $S$ parameterized by $\phi$:

$$\forall P \in \mathbb{R}^3 \ \Delta_M(P) = 0 \iff P \in S \subset \mathbb{R}^3.$$
An implicitization matrix

- It is a square matrix of size $m$
  - curve of degree $d$: size $d$
  - triangular surface of degree $d$: size $d(2d - 1)$
  - tensor product surface of degree $(d_1, d_2)$: size $2d_1d_2$

- Its entries are degree 2 polynomials in $X, Y, Z$; its determinant is of degree $2m$

- It is a symmetric matrix

- Its columns (and rows) are filled by moving quadrics

- Its determinant is equal to $\sum_{i=1}^{r} (L_j)^2$, i.e. the sum of square of the moving planes chosen to build the M-Rep.

- The implicit equation which is obtained is not of minimal degree:
  - The implicit equation appears with power $2\deg(\phi)$
  - There is an extraneous factor that does not vanish over $\mathbb{R}$

Here is the determinant of $\Delta_M$ for the example of the sphere:

$$\left((X - Y + 1)^2 + 2Y^2 + Z^2\right) \times (X^2 + Y^2 + Z^2 - 1)^2.$$
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- Its columns (and rows) are filled by moving quadrics
- Its determinant is equal to $\sum_{i=1}^{r} (L_j)^2$, i.e. the sum of square of the moving planes chosen to built the M-Rep.
- The implicit equation which is obtained is not of minimal degree:
  - The implicit equation appears with power $2 \deg(\phi)$
  - There is an extraneous factor that does not vanish over $\mathbb{R}$

Here is the determinant of $\Delta_M$ for the example of the sphere:

$$\left((X - Y + 1)^2 + 2Y^2 + Z^2\right) \times (X^2 + Y^2 + Z^2 - 1)^2.$$
3. Euclidian distance by means of syzygies

[Joint work with N. Botbol and M. Chardin]

▷ A classical approach to compute the closest point(s) of \( P \) to a parameterized curve/surface is to compute its orthogonal projections on this curve/surface.

▷ Several methods are known in the literature
  - Iterative methods, typically Newton’s method.
  - Subdivision methods, based on convex hull properties.

▷ Our syzyzy-based approach is the following
  - define a trivariate parameterization of normal planes to a curve, or the congruence of normal lines to a surface
  - Compute orthogonal projections of \( P \) by inversion of these parameterizations
  - define M-Rep for these trivariate parameterizations, and apply the inversion property of M-Rep
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Suppose given a point \( P \in \text{Im}(\phi) \subset \mathbb{R}^3 \) having a unique pre-image

\[
P = (x, y, z) = \phi(s_0, t_0).
\]

Let \( M_\nu \) be a M-Rep of \( \phi \). Then, by definition

\[
\left[ B_\nu(s_0, t_0) \right] \times M_\nu(P) = \begin{bmatrix} 0, \cdots, \sum_{i=0}^{3} g_i f_i(s_0, t_0) = 0, \cdots, 0 \end{bmatrix}.
\]

⇒ The inverse of \( P \) is computed directly from the kernel of \( M_\nu(P)^T \).

This is more general:

**Inversion Property ([– ’13], [Shen,–,Alliez,Dodgson’16])**

If \( P \) has a finite number of pre-images through \( \phi \), then all these pre-images can be computed from the nullspace of \( M_\nu(P)^T \).

⇒ application to meshing NURBS models by Delaunay refinement

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Suppose given a point $P \in \text{Im}(\phi) \subset \mathbb{R}^3$ having a unique pre-image

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$\Rightarrow$ application to meshing NURBS models by Delaunay refinement [Shen,–,Alliez,Dodgson’16]
Parameterization of the normal planes to a space curve

Let \( \phi(t) = (f_1/f_0, f_2/f_0, f_3/f_0) \) be parameterized space curve.

A tangent vector at the point at \( \phi(t) \) is given by

\[
\tau(t) = \left( \frac{f_0(t)f'_1(t) - f_1(t)f'_0(t)}{f_0(t)^2}, \frac{f_0(t)f'_2(t) - f_2(t)f'_0(t)}{f_0(t)^2}, \frac{f_0(t)f'_3(t) - f_3(t)f'_0(t)}{f_0(t)^2} \right)
\]

Choose two vectors \( \eta_1(t) \) and \( \eta_2(t) \) that generate the normal plane to the curve at \( \phi(t) \).

the family of normal planes is then parameterized by

\[
\psi : [0; 1] \times \mathbb{R}^2 \to \mathbb{R}^3
\]

\[
(t, u, v) \mapsto \phi(t) + u.\eta_1(t) + v.\eta_2(t).
\]

After homogeneization, we get a the rational map

\[
\Psi : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^3
\]

\[
(\overline{t} : t) \times (w : u : v) \mapsto (\Psi_0 : \Psi_1 : \Psi_2 : \Psi_3)
\]

where \( \Psi_i \) are bi-homogeneous polynomials of bi-degree \((2d, 1)\), and of bi-degree \((d, 1)\) in case of a non-rational curve.
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Let $\phi(s, t) = (f_1/f_0, f_2/f_0, f_3/f_0)$ be tensor-product surface. A normal vector to the surface at $\phi(t)$ is computed as

$$\nabla(s, t) := \frac{\partial \phi(s, t)}{\partial s} \wedge \frac{\partial \phi(s, t)}{\partial t}.$$ 

The congruence of normal lines to the surface is given by

$$\psi : [0; 1] \times [0; 1] \times \mathbb{R} \rightarrow \mathbb{R}^3$$

$$(s, t, u) \mapsto \phi(s, t) + u.\nabla(s, t).$$

After homogeneization, in the case of a tensor-product surface:

$$\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$$

$$(\bar{s} : s) \times (\bar{t} : t) \times (\bar{u} : u) \mapsto (\Psi_0 : \Psi_1 : \Psi_2 : \Psi_3)$$

where the $\Psi_i$‘s are tri-homogeneous of tri-degree $(3d_1, 3d_2, 1)$, and of tri-degree $(2d_1 - 1, 2d_2 - 1, 1)$ if the surface is non-rational.
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M-Rep for trivariate parameterizations

It turns out that \textbf{M-Rep also applied for trivariate parameterizations}:

- Fix a (bi/tri-)degree $\nu \geq 0$ and build the matrix $\mathbb{M}_\nu(\Psi)$
- Compute a basis $L_1, \ldots, L_r$ of the syzygies of degree $\nu$

$$\begin{align*}
(g_0, g_1, g_2, g_3) & : \deg(g_i) = \nu \text{ and } \sum_{i=0}^{3} g_i f_i \equiv 0. \\

\end{align*}$$

- Identify each $L_j$ with a \textit{moving plane} and build the matrix

$$\mathbb{M}_\nu(\Psi) := \begin{pmatrix}
\uparrow & \uparrow & \cdots & \uparrow \\
L_1 & L_2 & \cdots & L_r \\
\downarrow & \downarrow & \cdots & \downarrow
\end{pmatrix} \begin{array}{c}
\mathcal{B}_\nu \\
\in \text{Mat}(\mathbb{R}[X, Y, Z]_{\leq 1})
\end{array}$$

\textbf{The key facts}

- M-Rep still have the same properties, \textit{under suitable assumptions}
- The threshold degree allows to eliminate the variables of the normal lines/planes:
  - case of curves : $\nu_0 = (6d - 1, 0)$ ($\nu_0 = (3d - 1, 0)$ for non-rational)
  - case of tensor-product surfaces : $\nu_0 = (6d_1 - 1, 9d_2 - 1, 0)$ ($\nu_0 = (4d_1 - 3, 6d_2 - 4, 0)$ for non-rational)

$\Rightarrow$ use syzygies that only depend on the param. of the curve/surface.
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- Identify each $L_j$ with a *moving plane* and build the matrix

$$\mathbb{M}_\nu(\Psi) := \left( \begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
L_1 & L_2 & \cdots \\
\downarrow & \downarrow & \downarrow 
\end{array} \right) \left\{ B_\nu \in \text{Mat}(\mathbb{R}[X, Y, Z]_{\leq 1}) \right\}$$

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    ($\nu_0 = (4d_1 - 3, 6d_2 - 4, 0)$ for non-rational)

$\Rightarrow$ use syzygies that only depend on the param. of the curve/surface.
The case of space curves

- Assumptions on the base locus seem to fit into the existing theory
- The method essentially amounts to solve the classical equation

\[(\phi(t) - P) \cdot \partial_t \phi(t) = 0\]

by means of its companion matrix (eigenvalues computation).

![Figure: 3 orth. projections on a cubic curve, two being at equal distance.](image)

<table>
<thead>
<tr>
<th>Degree of the rational curve</th>
<th>3</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation time of M-Rep (ms)</td>
<td>52</td>
<td>155</td>
<td>888</td>
</tr>
<tr>
<td>size of M-Rep</td>
<td>$18 \times 11$</td>
<td>$30 \times 17$</td>
<td>$60 \times 42$</td>
</tr>
<tr>
<td>Computation time of inversion (ms)</td>
<td>8</td>
<td>15</td>
<td>39</td>
</tr>
</tbody>
</table>
Assumptions on the base locus are more intricate: there is a positive dimensional base locus.

We are at work for extending the theory to this situation.

Experiments show good properties in low bi-degrees.

More work are needed to deal with numerical stability, typically for rational bi-cubics.

Main gain: identify points with several closest points in a single computation.
Non-rational tensor-product surfaces

<table>
<thead>
<tr>
<th>Bidegree of ( S )</th>
<th>(1,1)</th>
<th>(2,1)</th>
<th>(2,2)</th>
<th>(3,2)</th>
<th>(3,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_0 )</td>
<td>(2,2)</td>
<td>(5,2)</td>
<td>(5,8)</td>
<td>(9,8)</td>
<td>(9,14)</td>
</tr>
<tr>
<td>Size of M-Rep</td>
<td>9x5</td>
<td>18x8</td>
<td>54x33</td>
<td>90x66</td>
<td>150x130</td>
</tr>
<tr>
<td>M-Rep comp. time (s)</td>
<td>0.015</td>
<td>0.04</td>
<td>0.4</td>
<td>1.25</td>
<td>3.9</td>
</tr>
<tr>
<td>Inversion comp. time (s)</td>
<td>( 2.10^{-3} )</td>
<td>( 3.10^{-3} )</td>
<td>( 5.10^{-3} )</td>
<td>( 6.10^{-3} )</td>
<td>( 10^{-2} )</td>
</tr>
</tbody>
</table>

Figure: Computation (MAPLE) of an M-Rep for non-rational Bézier patches.

Figure: Three orthogonal projection point on a biquadratic non-rational Bézier patch.
Rational tensor-product surfaces

<table>
<thead>
<tr>
<th>Bidegree of $S$</th>
<th>(1,1)</th>
<th>(2,1)</th>
<th>(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_0$</td>
<td>(5,8)</td>
<td>(11,8)</td>
<td>(11,17)</td>
</tr>
<tr>
<td>Size of M-Rep</td>
<td>54x76</td>
<td>108x136</td>
<td>216x253</td>
</tr>
<tr>
<td>M-Rep comp. time (s)</td>
<td>0.36</td>
<td>1.95</td>
<td>13</td>
</tr>
<tr>
<td>Inversion comp. time (s)</td>
<td>$3.10^{-3}$</td>
<td>$7.10^{-3}$</td>
<td>$7.10^{-2}$</td>
</tr>
</tbody>
</table>

**Figure:** Computation (MAPLE) of an M-Rep for rational Bézier patches.

**Figure:** Three orthogonal projection point on a biquadratic rational Bézier patch.
Available PhD positions in the Arcades european project

Algebraic Representations in Computer-Aided Design for comolEx Shapes

ARCADES

Marie Skłodowska-Curie European Training Network,

Members: ATHENA Research & Innovation Center (Greece, coordinator),
U. Barcelona (Spain), INRIA (France),
J. Kepler U. Linz (Austria), SINTEF (Norway),
U. Strathclyde (UK), T.U. Wien (Austria),
Evolute GmbH (Austria),
Hellenic Register of Shipping S.A. (Greece),
Hue AS (Norway), Missler Software (France),
RISC-Software (Austria), ITI TranscenData (UK).

13 Open Phd Positions
http://erga.di.uoa.gr/projects/main.html#arcades
(emiris@athena-innovation.gr)

ARCADES aims at disrupting the traditional paradigm in Computer-Aided Design (CAD) by exploiting cutting-edge research in mathematics and algorithm design. Geometry is now a critical tool in a large number of key applications; somewhat surprisingly, however, several approaches of the CAD industry are outdated, and 3D geometry processing is becoming increasingly the weak link. This is alarming in sectors where CAD faces new challenges arising from fast point acquisition, big data, and mobile computing, but also in robotics, simulation, animation, fabrication and manufacturing, where CAD strives to address crucial societal and market needs. The challenge taken up by ARCADES is to invert the trend of CAD industry lagging behind mathematical breakthroughs and to build the next generation of CAD software based on strong foundations from algebraic geometry, differential geometry, scientific computing, and algorithm design. Our game-changing methods lead to real-time modelers for architectural geometry and visualisation, to isogeometric and design-through-analysis software for shape optimisation, and marine design & hydrodynamics, and to tools for motion design, robot kinematics, path planning, and control of machining tools.