M2 MPA - Computational Algebraic Geometry

Mid-term exam - duration: one hour

October 21st, 2021

Exercise 1 Let I be an ideal in $k[x, y, z]$, where k is an algebraically closed field. A primary decomposition of I is given by

$$I = (x) \cap (x, y - z + 1) \cap (x, y, z - 1).$$

What are the associated primes of I? what are the embedded primes and the minimal primes? Give a geometric description (possibly with a drawing).

Exercise 2 Let I be a homogeneous ideal in $R = k[x, y, z]$, where k is a field. Assume that R/I admits the following graded finite free resolution:

$$0 \to R(-3)^2 \to R(-2)^3 \to R.$$

1. How many generators, and of which degree, define the ideal I.
2. Compute the Hilbert series of R/I.
3. Deduce the dimension and degree of $V(I)$.
4. What is the Hilbert polynomial of R/I?

Exercise 3 Let $R = k[x_1, \ldots, x_n]$, where k is a field, and I be an ideal in R.

- Prove that the radical of a principal ideal is principal.
- Show that the following inclusions hold but are strict in general:

$$\sqrt{I}\sqrt{J} \subseteq \sqrt{IJ} \text{ and } \mathrm{LT}(\sqrt{I}) \subseteq \sqrt{\mathrm{LT}(I)},$$

(where a monomial order has been chosen for the second inclusion).

Exercise 4 Let I and J be two ideals in $R = k[x_1, \ldots, x_n]$, where k is a field.

1. Let t be a new indeterminate and consider the ideal $L = t \times I + (1 - t) \times J$ of $R[t]$ (where the products are formed by multiplying generators by t and $(1 - t)$ respectively).
   i) Show that $I \cap J = L \cap R$.
   ii) Deduce how the intersection of two ideals can be computed by means of Gröbner basis.
2. Let us write $J = (f_1, \ldots, f_m)$ and let $f \in R$.
   i) Show that $(I : J) = \cap_{i=1}^{m}(I : (f_i))$ and that $(I : (f)) = (I \cap (f))f^{-1}$, which means the ideal of the elements in $I \cap (f)$, divided by f.
   ii) Explain how quotient ideals can be computed by means of Gröbner basis computations.
3. The saturation of I by J is defined as the ideal $(I : J^\infty) = \cup_{i \in \mathbb{N}}(I : J^i)$.
   i) Prove that there exists an integer s such that $(I : J^\infty) = (I : J^s)$.
   ii) Write $J = (f_1, \ldots, f_m)$, then prove that $(I : J^\infty) = \cap_{i=1}^{n}(I : (f_i)^\infty)$.
   iii) Let $f \in k[x_1, \ldots, x_n]$, t be a new indeterminate and consider the ideal $I_t := I + (tf - 1) \subseteq R[t]$. Prove that $(I : (f)^\infty) = I_t \cap R$.
   iv) From the previous results, explain how the saturation of I by J can be computed by means of Gröbner basis computations.