M2 MPA - Computational Algebraic Geometry

Final exam - duration: 3 hours

November 30, 2021

Exercise 1 Let $R = k[x, y]$ where k is an algebraically closed field.

1. Give a primary decomposition of the ideal $(x, y(y-1))$. What is the corresponding variety in $\mathbb{A}^2_k$?
2. Using the above example, show that a radical ideal is not always a prime ideal.
3. Show that a radical and primary ideal is prime.

Exercise 2 Let I be a homogeneous ideal in $R = k[x, y, z, w]$, where k is a field. Assume that R/I admits the following graded finite free resolution:

$$0 \to R(-5) \oplus R(-4) \to R(-3)^3 \to R.$$

1. How many generators, and of which degree, define the ideal I.
2. Compute the Hilbert series of R/I.
3. Deduce the dimension and degree of $V(I)$.
4. What is the Hilbert polynomial of R/I?

Exercise 3 Let I be an ideal in $R = k[x_1, \dots, x_n]$, k an algebraically closed field, $f \in I$, and G be a Gröbner basis of the ideal $I + (1 - x_{n+1}f)$ of $R[x_{n+1}]$. Then, show that $f \in \sqrt{I}$ if and only if G contains a constant.

Exercise 4 Let $V = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\} \subset \mathbb{A}^2_{\mathbb{C}}$ be n distinct points in the affine plane such that $a_i \neq a_j$ for all i, j. We denote by x, y the coordinates of $\mathbb{A}^2_{\mathbb{C}}$ and set $R = \mathbb{C}[x, y]$. We also define the Lagrange interpolation polynomial

$$h(x) = \sum_{i=1}^{n} b_i \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \in \mathbb{C}[x].$$

1. Show that $h(a_i) = b_i$ for all $i = 1, \dots, n$ and $\deg(h) \leq n - 1$.
2. Prove that $h(x)$ is the unique polynomial of degree $\leq n-1$ such that $h(a_i) = b_i$ for all $i = 1, \dots, n$.
3. Prove that $I(V) = (f(x), y - h(x))$, where $f(x) = \prod_{i=1}^{n}(x - a_i)$ (hint: divide $g \in I(V)$ by $\{f(x), y - h(x)\}$ using the lexicographic order $y > x$).
4. Prove that $\{f(x), y - h(x)\}$ is the reduced Gröbner basis of $I(V) \subset R$ for the lexicographic order $y > x$.

Exercise 5 Let $f(x) = a_0 x^m + \cdots + a_{m-1}x + a_m$ and $g(x) = b_0 x^n + \cdots + b_{n-1}x + b_n$ be two univariate polynomials in x with coefficients in a field k. We assume that $a_0 \neq 0$, $b_0 \neq 0$ and $n \geq m \geq 1$. If we divide g by f we get $g = qf + r$, where $\deg(r) < \deg(f) = m$. Then, assuming that r is not a constant, show that

$$\mathrm{Res}_{m,n}(f, g) = a_0^{n-\deg(r)} \mathrm{Res}_{m,n-\deg(r)}(f, r).$$

Exercise 6 Let $f_1, \dots, f_{n-1}$ be linear forms and f_n be a homogeneous polynomial of degree $d \geq 1$ in the variables $x_1, \dots, x_n$. We denote by $M = (u_{i,j})_{1 \leq i \leq n-1, 1 \leq j \leq n}$ the matrix of coefficients of $f_1, \dots, f_{n-1}$ and for all $i = 1, \dots, n$ we define $\Delta_i = (-1)^{n+i} \det(M_i)$, where M_i is the submatrix of M obtained by removing the i^{th} column.

1. Show that for all $i, j \in \{1, \dots, n\}$, $\Delta_i x_j - \Delta_j x_i \in (f_1, \dots, f_{n-1})$.

2. If g is a homogeneous polynomial of degree D, deduce that for all $i = 1, \ldots, n$,

$$g(\Delta_1, \ldots, \Delta_n)x_i^D - \Delta_i^D g(x_1, \ldots, x_n) \in (f_1, \ldots, f_{n-1}).$$

3. Deduce that $f_n(\Delta_1, \ldots, \Delta_n)$ is an inertia form of $f_1, \ldots, f_n$.
4. Prove that $\mathrm{Res}(f_1, \ldots, f_n) = f_n(\Delta_1, \ldots, \Delta_n)$.
5. Compute the resultant of $f_1 = u_1x_1 + u_2x_2 + u_3x_3$, $f_2 = v_1x_1 + v_2x_2 + v_3x_3$ and $f_3 = x_1^2 + x_2^2 - x_3^2$.

Exercise 7 Let $I = (f_1, \ldots, f_s)$ be a proper ideal in $R = \mathbb{C}[x_1, \ldots, x_n]$.

1. If $n = 1$, show that I is a principal ideal, say $I = (r(x_1))$, and that $\mathbb{C}[x_1]/(r(x_1))$ is a $\mathbb{C}$-vector space of finite dimension. What is this dimension?
2. Prove that if $V(I)$ is a finite set of points, then $\mathbb{C}[x_i] \cap I \neq \{0\}$ for all $i = 1, \ldots, n$.
3. Prove that if $\mathbb{C}[x_i] \cap I \neq \{0\}$ for all $i = 1, \ldots, n$, then R/I is a finite dimensional $\mathbb{C}$-vector space.
4. Deduce that $V(I)$ is a finite set of points if and only if R/I is a finite dimensional $\mathbb{C}$-vector space.
5. From now on, we assume that $V(I)$ is a finite set of distinct points $\{p_1, \ldots, p_r\}$ and we consider the map of $\mathbb{C}$-vector spaces

$$\begin{array}{rcl} \varphi : R/I & \to & \mathbb{C}^r \\ f & \mapsto & (f(p_1), \ldots, f(p_r)). \end{array}$$

   Explain why this map is well defined.
6. Prove that $\dim_{\mathbb{C}}(R/I) \geq r$, where $\dim_{\mathbb{C}}(R/I)$ denotes the dimension of R/I as a $\mathbb{C}$-vector space (hint: prove that φ is surjective).
7. Prove that $\dim_{\mathbb{C}}(R/I) = r$ if and only if $I = \sqrt{I}$.