M2 MPA - Computational Algebraic Geometry

Final exam - duration: two hours

November 30th, 2020

Exercise 1 Let $f(x_0, x_1, x_2, x_3)$ and $g(x_0, x_1, x_2, x_3)$ be two homogeneous polynomials in $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ of degree 3 and 2 that define a cubic surface $\mathcal{H}$ and a quadratic surface $\mathcal{Q}$ in $\mathbb{P}^3$, respectively.

1. We assume that $\mathcal{H}$ and $\mathcal{Q}$ intersect in a curve $\mathcal{C}$. Show that this implies that (f, g) is a regular sequence in R.
2. Give a minimal graded finite free resolution of R/I.
3. Compute the Hilbert polynomial of the intersection curve $\mathcal{C}$. What is the degree of this curve?

Exercise 2 Let k be a commutative ring and $f_1, \dots, f_n$ be n homogeneous polynomials in $k[x_1, \dots, x_n]$ of degree $d_1, \dots, d_n \geq 1$ respectively. Moreover, suppose given n homogeneous polynomials $g := (g_1, \dots, g_n)$ in $k[x_1, \dots, x_n]$ of the same degree $d \geq 1$. The goal of this exercise is to prove that the following equality holds in k:

$$\mathrm{Res}(f_1 \circ g, \dots, f_n \circ g) = \mathrm{Res}(g_1, \dots, g_n)^{d_1 d_2 \dots d_n} \mathrm{Res}(f_1, \dots, f_n)^{d^{n-1}}$$

1. Justify that it is enough to prove the above formula over a universal ring of coefficients. Describe this ring.
2. Show that there exists an integer N such that for all $i = 1, \dots, n$
$$g_i^N \mathrm{Res}(f_1, \dots, f_n) \in (f_1 \circ g, \dots, f_n \circ g).$$
3. Deduce that
$$\mathrm{Res}(f_1 \circ g, \dots, f_n \circ g) = \varepsilon \mathrm{Res}(g_1, \dots, g_n)^{\lambda} \mathrm{Res}(f_1, \dots, f_n)^{\mu}$$
with λ, μ positive integers and $\varepsilon = \pm 1$.
4. Conclude with the help of the specialization $f_j \mapsto u_j x_j^{d_j}$, $g_j \mapsto v_j x_j^d$ for all j.
5. Make explicit the behavior of the resultant under a linear change of coordinates.

Exercise 3 We suppose given $n + 1$ homogeneous polynomials $f_0, \dots, f_n$ in $R = \mathbb{C}[s, t]$ of the same degree $d \geq 1$. We denote by I the ideal generated by $f_0, \dots, f_n$ and we assume that $V(f_0, \dots, f_n) = \emptyset$.

1. Show that R/I admits a finite free resolution of the form
$$0 \to \oplus_{i=1}^{n} R(-d-\mu_i) \to R^{n+1}(-d) \to R$$
where the μ_i's are non negative integers such that $\sum_{i=1}^{n} \mu_i = d$.

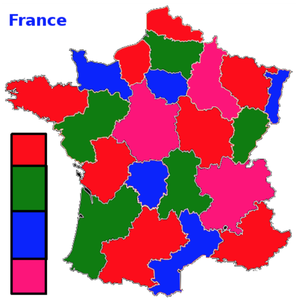
2. We consider the curve $\mathcal{C} \in \mathbb{P}^3$ which is obtained as the image of the parameterization
$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^3 \\ (s:t) & \mapsto & (s^3 : s^2t : st^2 : t^3). \end{array}$$
Define $f_0 = s^3, f_1 = s^2t, f_2 = st^2$ and $f_3 = t^3$. Give the finite free resolution of R/I in this particular case and provide the maps.

3. Denoting by $x_0, \ldots, x_3$ the coordinates in $\mathbb{P}^3$, describe the equations in $R[x_0, \ldots, x_3]$ of the symmetric algebra $\mathrm{Sym}_A(I)$ of I from the syzygies obtained in the previous question.

4. Admitting that the annihilator over $A = k[x_0, \ldots, x_3]$ of the graded component of $\mathrm{Sym}_A(I)$ of degree $\nu \geq 1$ with respect to s, t yields the defining ideal of $\mathcal{C}$, build a 2×3-matrix with entries in A that could serve as an implicit representation of this curve. Explain what this means.

Exercise 4

The famous "Four Color Theorem" shows that only four colors are needed to color planar map so that no bordering regions have the same color. Typical examples are a colored world map, a colored map of the states of the USA, or a colored map of the French regions (see the side picture). In this exercise, we will provide a method to determine if three colors are sufficient for a particular map.



1. Could you provide a simple planar map to illustrate that three colors are not always enough to color it so that no bordering regions have the same color?

2. The three colors are represented by a complex cubic root of the unit and each region is represented by a variable x_i. Justify that for each region we have the polynomial equation
$$x_i^3 - 1 = 0.$$

3. Let x_j and x_k be two neighboring regions. As neighboring regions cannot have the same color, show that x_j and x_k must satisfy a polynomial equation of degree 2.
(Hint: use that $x_j^3 - x_k^3 = 0$).

4. Deduce from the previous questions that there exists a polynomial system such that a map with n regions can be colored with three colors if and only if there exists at least one solution to this polynomial system.

5. Given a particular map, explain how you would use a computer algebra system to determine if it can be colored with three colors.

Exercise 5 Let M be a finitely generated A-module, show that if M is generated by q elements then
$$\mathrm{ann}_A(M)^q \subset \mathcal{F}_0(M) \subset \mathrm{ann}_A(M).$$