M2 MPA - Computational Algebraic Geometry

Final exam - duration: two hours

November 30th, 2020

Exercise 1 Let $f(x_0, x_1, x_2, x_3)$ and $g(x_0, x_1, x_2, x_3)$ be two homogeneous polynomials in $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ of degree 3 and 2 that define a cubic surface $\mathcal{H}$ and a quadratic surface $\mathcal{Q}$ in $\mathbb{P}^3$, respectively.

1. We assume that $\mathcal{H}$ and $\mathcal{Q}$ intersect in a curve $\mathcal{C}$. Show that this implies that (f, g) is a regular sequence in R.
2. Give a minimal graded finite free resolution of R/I.
3. Compute the Hilbert polynomial of the intersection curve $\mathcal{C}$. What is the degree of this curve?

Solution 1

1. The polynomial f being nonzero, it is a nonzero divisor in R. Since $\mathcal{H}$ and $\mathcal{Q}$ cut out a curve then f and g have no common factors (otherwise $\mathcal{H}$ and $\mathcal{Q}$ would have a surface as a common component of dimension 2). Now, if h and k are polynomials in R such that $hf + kg = 0$ we deduce that f divides k, which means that g is not a zero divisor in $R/(f)$.
2. Since (f, g) is a regular sequence in R, its associated Koszul complex is a F.F.R. of R/I:

$$0 \to R(-5) \xrightarrow{\begin{pmatrix} -g \\ f \end{pmatrix}} R(-3) \oplus R(-2) \xrightarrow{(f,g)} R.$$

   It is clearly a minimal resolution.

3. Using the F.F.R. of R/I we get:

$$\begin{aligned} \mathrm{HP}(R/I, t) &= \mathrm{HP}(R, t) - \mathrm{HP}(R, t-2) - \mathrm{HP}(R, t-3) + \mathrm{HP}(R, t-5) \\ &= \binom{t+3}{3} - \binom{t+1}{3} - \binom{t}{3} + \binom{t-2}{3} \\ &= 6t - 3. \end{aligned}$$

   The curve is of degree 6.

Exercise 2 Let k be a commutative ring and $f_1, \ldots, f_n$ be n homogeneous polynomials in $k[x_1, \ldots, x_n]$ of degree $d_1, \ldots, d_n \geq 1$ respectively. Moreover, suppose given n homogeneous polynomials $g := (g_1, \ldots, g_n)$ in $k[x_1, \ldots, x_n]$ of the same degree $d \geq 1$. The goal of this exercise is to prove that the following equality holds in k:

$$\mathrm{Res}(f_1 \circ g, \ldots, f_n \circ g) = \mathrm{Res}(g_1, \ldots, g_n)^{d_1 d_2 \ldots d_n} \mathrm{Res}(f_1, \ldots, f_n)^{d^{n-1}}$$

1. Justify that it is enough to prove the above formula over a universal ring of coefficients. Describe this ring.

2. Show that there exists an integer N such that for all $i = 1, \dots, n$

$$g_i^N \mathrm{Res}(f_1, \dots, f_n) \in (f_1 \circ g, \dots, f_n \circ g).$$

3. Deduce that

$$\mathrm{Res}(f_1 \circ g, \dots, f_n \circ g) = \varepsilon \mathrm{Res}(g_1, \dots, g_n)^\lambda \mathrm{Res}(f_1, \dots, f_n)^\mu$$

with λ, μ positive integers and $\varepsilon = \pm 1$.

4. Conclude with the help of the specialization $f_j \mapsto u_j x_j^{d_j}$, $g_j \mapsto v_j x_j^d$ for all j.

5. Make explicit the behavior of the resultant under a linear change of coordinates.

Solution 2

1. By definition, the resultant is a universal object: it is first defined in the universal setting and then defined over any commutative by specialization (there is always a ring map from $\mathbb{Z}$ to any commutative ring). In our setting, the universal ring A is the polynomial ring built from the coefficients of the f_i's and g_j's over the ring integers.

2. The resultant is an inertia form, so there exists N such that for all $i = 1, \dots, n$ we have

$$x_i^N \mathrm{Res}(f_1, \dots, f_n) \in (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)).$$

Substituting x_i by $g_i(x_1, \dots, x_n)$ in the above equality gives the claimed relation (notice that the resultant belongs to A and hence does not depend on the x_i's).

3. Using the relations obtained in the previous question and the divisibility property of the resultant, we get that

$$\mathrm{Res}(f_1 \circ g, \dots, f_n \circ g) \text{ divides } \mathrm{Res}(g_1^N \mathrm{Res}(f_1, \dots, f_n), \dots, g_n^N \mathrm{Res}(f_1, \dots, f_n))$$

in A. But since $\mathrm{Res}(f_1, \dots, f_n) \in A$, by homogeneity and multiplicativity of the resultant we have

$$\begin{aligned}\mathrm{Res}(g_1^N \mathrm{Res}(f_1, \dots, f_n), \dots, g_n^N \mathrm{Res}(f_1, \dots, f_n)) &= \mathrm{Res}(f_1, \dots, f_n)^{n(Nd)^{n-1}} \mathrm{Res}(g_1^N, \dots, g_n^N) \\ &= \mathrm{Res}(f_1, \dots, f_n)^{n(Nd)^{n-1}} \mathrm{Res}(g_1, \dots, g_n)^{N^n}.\end{aligned}$$

Now, since we are in the universal setting, over A, $\mathrm{Res}(f_1, \dots, f_n)$ and $\mathrm{Res}(g_1, \dots, g_n)$ are both irreducible polynomials that are moreover coprime (they do not depend on the same variables). It follows that

$$\mathrm{Res}(f_1 \circ g, \dots, f_n \circ g) = \varepsilon \mathrm{Res}(g_1, \dots, g_n)^\lambda \mathrm{Res}(f_1, \dots, f_n)^\mu \tag{1}$$

for some non negative integers λ, μ and an invertible element ε in $\mathbb{Z}$.

4. Using this specialization, (1) yields the equality

$$\mathrm{Res}(u_1 v_1^{d_1} x_1^{dd_1}, \dots, u_n v_n^{d_n} x_n^{dd_n}) = \varepsilon \mathrm{Res}(v_1 x_1^{d_1}, \dots, v_n x_n^{d_n})^\lambda \mathrm{Res}(u_1 x_1^{d_1, \dots, u_n x_n^{d_n}})^\mu.$$

Applying the homogeneity and multiplicativity properties of the resultant we get

$$\prod_i \left(u_i v_i^{d_i}\right)^{d^{n-1} \frac{d_1 \dots d_n}{d_i}} = \varepsilon \left(\prod_i v_i^{d^{n-1}}\right)^\lambda \left(\prod_i u_i^{\frac{d_1 \dots d_n}{d_i}}\right)^\mu,$$

so we deduce that $\varepsilon = 1$, $\mu = d^{n-1}$ and $\lambda = d_1 \dots d_n$.

5. If the g_i's are linear forms $g_i = \sum_{i=1}^n a_{i,j} x_j$ then we know that
$$\mathrm{Res}(g_1, \ldots, g_n) = \det(a_{i,j})_{i,j=1,\ldots,n}.$$
In this case, we get
$$\mathrm{Res}(f_1 \circ g, \ldots, f_n \circ g) = \det(a_{i,j})^{d_1 d_2 \ldots d_n} \mathrm{Res}(f_1, \ldots, f_n).$$
The resultant is said to be invariant under a linear change of coordinates.

Exercise 3 We suppose given $n+1$ homogeneous polynomials $f_0, \ldots, f_n$ in $R = \mathbb{C}[s,t]$ of the same degree $d \geq 1$. We denote by I the ideal generated by $f_0, \ldots, f_n$ and we assume that $V(f_0, \ldots, f_n) = \emptyset$.

1. Show that R/I admits a finite free resolution of the form
$$0 \to \oplus_{i=1}^n R(-d-\mu_i) \to R^{n+1}(-d) \to R$$
where the μ_i's are non negative integers such that $\sum_{i=1}^n \mu_i = d$.

2. We consider the curve $\mathcal{C} \in \mathbb{P}^3$ which is obtained as the image of the parameterization
$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^3 \\ (s:t) & \mapsto & (s^3 : s^2t : st^2 : t^3). \end{array}$$
Define $f_0 = s^3, f_1 = s^2t, f_2 = st^2$ and $f_3 = t^3$. Give the finite free resolution of R/I in this particular case and provide the maps.

3. Denoting by $x_0, \ldots, x_3$ the coordinates in $\mathbb{P}^3$, describe the equations in $R[x_0, \ldots, x_3]$ of the symmetric algebra $\mathrm{Sym}_A(I)$ of I from the syzygies obtained in the previous question.

4. Admitting that the annihilator over $A = k[x_0, \ldots, x_3]$ of the graded component of $\mathrm{Sym}_A(I)$ of degree $\nu \geq 1$ with respect to s,t yields the defining ideal of $\mathcal{C}$, build a 2×3-matrix with entries in A that could serve as an implicit representation of this curve. Explain what this means.

Solution 3

1. We proceed as in the lectures where we proved this result for $n = 2$. The syzygy module of I is free because of Hilbert Syzygy Theorem. It follows that R/I has a F.F.R. of the form
$$0 \to \oplus_{i=1}^N R(-d-\mu_i) \to \oplus_{i=0}^n R(-d) \to R.$$
Now, as $V(I) = \emptyset$ the Hilbert polynomial of R/I is the null polynomial. It follows that
$$\begin{aligned} 0 = \mathrm{HP}(R/I,t) &= \mathrm{HP}(R,t) - \sum_{i=0}^n \mathrm{HP}(R(-d),t) + \sum_{i=1}^N \mathrm{HP}(R(-d-\mu_i)) \\ &= (t+1) - (n+1)(t-d+1) + \sum_{i=1}^N (t-d-\mu_i+1) \\ &= (N-n)t + 1 + (n+1-N)(d-1) - \sum_{i=1}^N \mu_i. \end{aligned}$$
This implies that $N = n$ and then that $d = \sum_{i=1}^n \mu_i$. Notice that if the f_i's are linearly independent then $\mu_i > 0$ for all i, otherwise we have a syzygy with constant coordinates.

2. For this particular example, we have $\sum_{i=1}^{3} \mu_i = 3$ and $\mu_i \geq 1$ for all i because $f_0, \ldots, f_3$ are clearly linearly independent. So we need to find three syzygies of degree 1 which are linearly independent. Looking at the definition of the f_i's, it appears that $tf_i - sf_{i+1} = 0$ for all $i = 0, 1, 2$. These three syzygies are clearly linearly independent and hence they provide a basis of the syzygy module of I.

3. Following the lectures, the equations of the symmetric algebra are obtained by writing syzygies and replacing the f_i's by the x_i's. Here we get:

$$L_i = tx_i - sx_{i+1}, \quad i = 0, 1, 2.$$

It follows that $\mathrm{Sym}_R(I) = R[x_0, x_1, x_2, x_3]/(L_0, L_1, L_2)$.

4. By taking the component of degree 1 with respect to s, t of $\mathrm{Sym}_R(I)$, we get

$$\oplus_{i=0}^{2} R_{1-\mu_i}[x_0, \ldots, x_3](-1) \xrightarrow{(L_0, L_1, L_2)_1} R_1[x_0, \ldots, x_3] \to \mathrm{Sym}_R(I)_1 \to 0$$

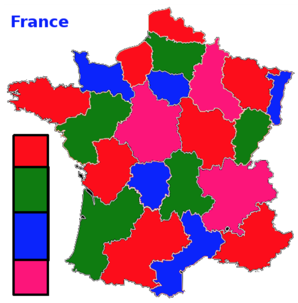
where the matrix is given by

$$\begin{pmatrix} -x_1 & -x_2 & -x_3 \\ x_0 & x_1 & x_2 \end{pmatrix}.$$

The variety cut out by the 2-minors of this matrix is exactly the curve $\mathcal{C}$ (which is known as the twisted cubic).

Exercise 4

The famous "Four Color Theorem" shows that only four colors are needed to color planar map so that no bordering regions have the same color. Typical examples are a colored world map, a colored map of the states of the USA, or a colored map of the French regions (see the side picture). In this exercise, we will provide a method to determine if three colors are sufficient for a particular map.



1. Could you provide a simple planar map to illustrate that three colors are not always enough to color it so that no bordering regions have the same color?

2. The three colors are represented by a complex cubic root of the unit and each region is represented by a variable x_i. Justify that for each region we have the polynomial equation

$$x_i^3 - 1 = 0.$$

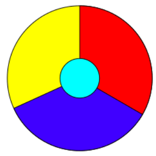
3. Let x_j and x_k be two neighboring regions. As neighboring regions cannot have the same color, show that x_j and x_k must satisfy a polynomial equation of degree 2.
(Hint: use that $x_j^3 - x_k^3 = 0$).

4. Deduce from the previous questions that there exists a polynomial system such that a map with n regions can be colored with three colors if and only if there exists at least one solution to this polynomial system.

5. Given a particular map, explain how you would use a computer algebra system to determine if it can be colored with three colors.

Solution 4

1. It is easy tp design small maps that cannot be colored with three colors. An example (from wikipedia):



2. The equation $x_i^3 = 1$ has three distinct complex roots $\{1, j, j^2\}$ that can be bijectively associated to three colors.

3. If x_j and x_k are two neighboring regions then the variables x_j and x_k are not allowed to take the same value. Since

$$0 = x_j^3 - x_k^3 = (x_j - x_k)(x_j^2 + x_j x_k + x_k^2)$$

we deduce that the polynomial $x_j^2 + x_j x_k + x_k^2$ vanishes if and only if $x_j \neq x_k$, always under the assumption that $x_j^3 = 1$ and $x_k^3 = 1$.

4. For all i we set $f_i = x_i^3 - 1$ and for all couple (j, k) we set $g_{j,k} = x_j^2 + x_j x_k + x_k^2$. These are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Consider the algebraic affine variety V defined by all the f_i's and the $g_{j,k}$'s such that x_j and x_k are neighboring regions. We deduce that the map can be colored with three colors, so that no bordering regions have the same color, if and only if $V \neq \emptyset$.

5. To conclude, by Hilbert Nullstellensatz we have to decide if $1 \in I$, where I is the ideal generated by the equations defining V. This can be done by computing a Grobner basis of I, a task that can/must be done with a computer algebra system.

Exercise 5 Let M be a finitely generated A-module, show that if M is generated by q elements then

$$\mathrm{ann}_A(M)^q \subset \mathcal{F}_0(M) \subset \mathrm{ann}_A(M).$$

Solution 5
Choose a presentation of M:

$$A^r \xrightarrow{(a_{i,j})} A^q \xrightarrow{(x_1,\ldots,x_q)} M \to 0. \tag{2}$$

Notice that one can always assume $r \geq q$ since one can easily add trivial columns to the matrix $(a_{i,j})$. Let N be a $q \times q$ submatrix of $(a_{i,j})$. Then

$$N^T \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} = 0.$$

Multiplying this equality on the left by the matrix of cofactors of N^T we get that $\det(N) x_i = 0$ for all i, from we deduce that $\mathcal{F}_0(M) \subset \mathrm{ann}_A(M)$.

Now, let $a \in \mathrm{ann}_A(M)$. Starting from (2), we have the following presentation of M:

$$A^q \oplus A^r \xrightarrow{(a \cdot \mathrm{Id}_q | a_{i,j})} A^q \xrightarrow{(x_1, \ldots, x_q)} M \to 0$$

where $a \cdot \mathrm{Id}_q$ is the diagonal matrix where all the diagonal elements are equal to a. Thus, for all integer $\nu \geq 0$ we have (Fitting ideals are independent on the choice of the presentation matrix)

$$\det_{q-\nu+1}(a_{i,j}) = \det_{q-\nu+1}(a \cdot \mathrm{Id}_q | a_{i,j}) \supset a \cdot \det_{q-\nu}(a_{i,j}).$$

It follows that for all $\nu \geq 1$

$$\mathrm{ann}_A(M)\mathcal{F}_\nu(M) \subset \mathcal{F}_{\nu-1}(M). \tag{3}$$

To conclude, we use (3) iteratively to get that

$$\mathrm{ann}_A(M)^q \mathcal{F}_q(M) \subset \mathrm{ann}_A(M)^{q-1}\mathcal{F}_{q-1}(M) \subset \cdots \subset \mathcal{F}_0(M)$$

and we observe that $\mathcal{F}_q(M) = A$ since M is generated by q elements and that $\mathcal{F}_0(M) \subset \mathrm{ann}_A(M)$, as already proved.