M2 MPA - Computational Algebraic Geometry

Mid-term exam - duration: one hour

October 22nd, 2020

Exercise 1

1. Let Q be a primary ideal and let P be the radical of Q. Show that P is a prime ideal and that P is the smallest prime ideal containing Q.

2. Let $Q = (x^2, xy) \subset \mathbb{C}[x, y, z]$. Show that the radical of Q is prime. Is Q a primary ideal?

Solution 1

1. (See notes of the first lecture). Let f, g be such that $fg \in P = \sqrt{Q}$. Then there exits an integer m such that $f^m g^m \in Q$. As Q is primary this implies that either $f^m \in Q$, hence $f \in \sqrt{Q}$, or either there exits an integer m' such that $g^{mm'} \in Q$, hence $g \in \sqrt{Q}$. We deduce that $P = \sqrt{Q}$ is a prime ideal.
   Now, Let P' be a prime ideal such that $Q \subset P'$. Then we have $\sqrt{Q} = P \subseteq \sqrt{P'} = P'$.

2. We have $(x^2) \subset Q \subset (x)$ so the radical of Q is equal to (x), which is a prime ideal. Q is not primary because $xy \in Q$ but $x \notin Q$ and $y^n \notin Q$ for any integer n (because any element in Q is divisible by x).

Exercise 2 Let $f(x, y)$ be a homogeneous polynomial of degree $d \geq 1$ in the graded ring $R = \mathbb{C}[x, y]$ and denote by I the ideal generated by f, i.e. $I = (f) \subset R$. Describe the Hilbert function of R/I. What is its Hilbert polynomial?

Solution 2 For any integer $i = 0, \ldots, d-1$ we have $\mathrm{HF}(R/I, i) = \mathrm{HF}(R, i) = i+1$, in particular $\mathrm{HF}(R/I, d-1) = d$. Then, for all $i \geq d$ we have

$$\mathrm{HF}(R/I, i) = \mathrm{HF}(R, i) - \mathrm{HF}(I, i) = i + 1 - (i - d + 1) = d.$$

The Hilbert polynomial is hence the constant polynomial that is equal to d.

Exercise 3 Let $f_1 = xy - x$, $f_2 = x^2 - y$ in $\mathbb{C}[x, y]$ with the grlex ordering and $x > y$. Build a Gröbner basis of the ideal $I = (f_1, f_2)$.

Solution 3 We start with $G = \{f_1, f_2\}$. We first compute the S-polynomial

$$S(f_1, f_2) = xf_1 - yf_2 = -x^2 + y^2.$$

Then we reduce it by the division algorithm

$$S(f_1, f_2) = -f_2 + y^2 - y.$$

We define $f_3 = y^2 - y$ and add it to G which is now $G = \{f_1, f_2, f_3\}$. Now, we get

$$S(f_1, f_3) = yf_1 - xf_3 = 0$$

and

$$S(f_2, f_3) = y^2 f_2 - x^2 f_3 = x^2 y - y^3.$$

The reduction of $S(f_2, f_3)$ by G gives

$$S(f_1, f_3) = yf_2 - yf_3 + 0$$

and hence G is a Grobner basis.

Exercise 4 Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let f and g be two homogeneous polynomials of positive degree d and e respectively. We assume that f and g have no common factor in R and we denote by I the ideal generated by f and g, i.e. $I = (f, g) \subset R$.

1. Show that R/I has a finite free resolution of the form

$$0 \to F_2 \to F_1 \to F_0 \to R/I \to 0.$$

   Describe explicitly the graded free R-modules F_i and the maps in this finite free resolution.

2. What is the Hilbert series of R? What are the Hilbert series of F_0, F_1 and F_2?

3. Deduce that the Hilbert series of R/I is of the form $P(t)/(1-t)^{n-2}$ where $P(t) \in \mathbb{Z}[t]$ is such that $P(1) \neq 0$.

4. Finally, deduce that $V(I)$ is of dimension $n-2$ and degree $P(1)$. What is the value of $P(1)$ in terms of d and e? (hint: $1/(1-t)^{n-2}$ is the Hilbert series of a polynomial ring in $n-2$ variables).

Solution 4

1. $F_0 = R$ and $F_1 = R(-d) \oplus R(-e)$ as the first map is given by the generators, that is to say

$$\partial_1 : F_1 \to F_0 : (p, q) \mapsto pf + qg.$$

   The kernel of ∂_1 corresponds to couples p, q such that $pf + qg = 0$. Since f and g have no common factors and that R is a UFD we deduce that f divides q, i.e. $q = fq'$, and g divides p, i.e. $p = gp'$. In addition we have $gp'f + fg'g = 0$ from we deduce that $p' + q' = 0$. Therefore, the kernel of ∂_1 corresponds to elements of the form $h(-g, f)$ where h is any homogeneous polynomials; it is hence isomorphic to R. Taking into account the grading we get $F_2 = R(-d-e)$ and the map

$$F_2 \to F_1 : h \mapsto h(-g, f)$$

   is injective, so that the resolution stops at the second step.

2. From the definition: $\mathrm{HS}(R, t) = \mathrm{HS}(F_0, t) = 1/(1-t)^n$, $\mathrm{HS}(F_1, t) = (t^d + t^e)/(1-t)^n$ and $\mathrm{HS}(F_2, t) = t^{d+e}/(1-t)^n$.

3. Applying Hilbert series to the exact sequence obtained in the first question we get

$$\begin{aligned}\mathrm{HS}(R/I,t) &= \mathrm{HS}(F_0,t) - \mathrm{HS}(F_1,t) + \mathrm{HS}(F_2,t) \\ &= \frac{1-t^d-t^e+t^{d+e}}{(1-t)^n} = \frac{(1-t^d)(1-t^e)}{(1-t)^n} \\ &= \frac{(1+t+\cdots+t^{d-1})(1+t+\cdots+t^{e-1})}{(1-t)^{n-2}} =: \frac{P(t)}{(1-t)^{n-2}}.\end{aligned}$$

We have $P(1) = de \neq 0$.

4. We know that

$$\mathrm{HS}(\mathbb{C}[x_1,\ldots,x_{n-2}],t) = \frac{1}{(1-t)^{n-2}} = (1 + (n-2)t + \cdots + \binom{n-3+i}{n-3} t^i + \cdots)$$

where $\binom{n-3+i}{n-3} = \frac{(i+n-3)(i+n-4)\cdots(i+1)}{(n-3)!}$ is a polynomial in i of degree $n-3$ and leading coefficient equal to $1/(n-3)!$. Now, if $P(t) := \sum_{j=0}^{l} c_j t^j$ then the coefficient of t^i in $\mathrm{HS}(R/I,t)$ is equal to

$$\sum_{j=0}^{l} c_j \binom{n-3+i-j}{n-3},$$

assuming that i is sufficiently high. This is a polynomial of degree $n-3$ in i and its leading coefficient is equal to

$$\sum_{j=0}^{l} c_j \times \frac{1}{(n-3)!} = \frac{P(1)}{(n-3)!} = \frac{de}{(n-3)!}.$$

The Hilbert polynomial $\mathrm{HP}(R/I,i)$ of R/I is hence a polynomial of degree $n-3$ and leading coefficient $de/(n-3)!$. It follows that $V(I) \subset \mathbb{P}^{n-1}$ is of dimension $n-3$ (codimension 2) and of degree de.

Exercise 5 Show that the rank of the Sylvester matrix of two polynomials $f(x), g(x) \in \mathbb{C}[x]$ of degree m and n respectively is equal to $m+n-\deg(\gcd(f,g))$, where $\deg(\gcd(f,g))$ is the degree of the greatest common divisor of $f(x)$ and $g(x)$.

Solution 5 The Sylvester is the matrix of the map of $\mathbb{C}$-vector spaces

$$\mathbb{C}[x]_{<n} \oplus \mathbb{C}[x]_{<m} \to \mathbb{C}[x]_{<m+n} : (u,v) \mapsto uf + vg. \tag{1}$$

Let h be the gcd of f and g and denote by δ its degree. We have $f = hf'$ and $g = hg'$ with f' and g' coprime polynomials.
Now, let (u,v) be an element in the kernel of (1). We have $uf + vg = 0 = h(uf' + vg')$. We deduce that f' divides v, i.e. $v = f'v'$, and g' divides u, i.e. $u = g'u'$. Moreover we get $uf' + vg' = f'g'(u' + v') = 0$ and hence $v' = -u'$. Finally, we proved that $(u,v) = u'(g', -f')$ where u' is a polynomial in $\mathbb{C}[x]_{<\delta}$. As any element of this form is obviously in the kernel of (1) we deduce that the kernel of the Sylvester matrix is of dimension δ, and hence its rank of dimension $m+n-\delta$.