M2 MPA - Computational Algebraic Geometry

Mid-term exam - duration: one hour

October 22nd, 2020

Exercise 1

1. Let Q be a primary ideal and let P be the radical of Q. Show that P is a prime ideal and that P is the smallest prime ideal containing Q.

2. Let $Q = (x^2, xy) \subset \mathbb{C}[x, y, z]$. Show that the radical of Q is prime. Is Q a primary ideal?

Exercise 2 Let $f(x, y)$ be a homogeneous polynomial of degree $d \geq 1$ in the graded ring $R = \mathbb{C}[x, y]$ and denote by I the ideal generated by f, i.e. $I = (f) \subset R$. Describe the Hilbert function of R/I. What is its Hilbert polynomial?

Exercise 3 Let $f_1 = xy - x$, $f_2 = x^2 - y$ in $\mathbb{C}[x, y]$ with the grlex ordering and $x > y$. Build a Gröbner basis of the ideal $I = (f_1, f_2)$.

Exercise 4 Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let f and g be two homogeneous polynomials of positive degree d and e respectively. We assume that f and g have no common factor in R and we denote by I the ideal generated by f and g, i.e. $I = (f, g) \subset R$.

1. Show that R/I has a finite free resolution of the form

$$0 \to F_2 \to F_1 \to F_0 \to R/I \to 0.$$

   Describe explicitly the graded free R-modules F_i and the maps in this finite free resolution.

2. What is the Hilbert series of R? What are the Hilbert series of F_0, F_1 and F_2?

3. Deduce that the Hilbert series of R/I is of the form $P(t)/(1-t)^{n-2}$ where $P(t) \in \mathbb{Z}[t]$ is such that $P(1) \neq 0$.

4. Finally, deduce that $V(I)$ is of dimension $n - 2$ and degree $P(1)$. What is the value of $P(1)$ in terms of d and e? (hint: $1/(1-t)^{n-2}$ is the Hilbert series of a polynomial ring in $n - 2$ variables).

Exercise 5 Show that the rank of the Sylvester matrix of two polynomials $f(x), g(x) \in \mathbb{C}[x]$ of degree m and n respectively is equal to $m + n - \deg(\gcd(f, g))$, where $\deg(\gcd(f, g))$ is the degree of the greatest common divisor of $f(x)$ and $g(x)$.