Course 9 – Homological aspects of elimination.

1. Resultant ideal and annihilators.

- Given a commutative ring A and polynomials $f_1, \dots, f_r \in A[x_1, \dots, x_n]$ homogeneous of degree $d_i \geq 1$, one considered the elimination ideal

$$\mathfrak{a} := (I : \mathfrak{m}^\infty)_0 = (I : \mathfrak{m}^\infty) \cap A$$
$$= \{ a \in A \text{ such that } \exists m \in \mathbb{N} \ a x_i^m \in I \ \forall i \}$$

with the notation $I = (f_1, \dots, f_r)$ and $\mathfrak{m} = (x_1, \dots, x_n)$.
This ideal provides necessary and sufficient condition(s) for the existence of a common root of the f_i's under specialization.

- Set $B := A[x_1, \dots, x_n]/I$. It is a graded ring. We define

$$H^0_{\mathfrak{m}}(B) := \{ b \in B \text{ such that } \exists k \in \mathbb{N} \ \mathfrak{m}^k . b \underset{B}{=} 0 \}$$

This is a homogeneous ideal of B; we clearly have the isomorphism

$$(I : \mathfrak{m}^\infty)/I \simeq H^0_{\mathfrak{m}}(B).$$

In particular, $(I : \mathfrak{m}^\infty)_0 = H^0_{\mathfrak{m}}(B)_0 = \mathfrak{a}$ (we assume $A \cap I = 0$).

- Link with annihilators.

For any couple $(\nu, t) \in \mathbb{N}^2$ we define the A-linear map

$$\theta_{\nu, t} : B_\nu \longrightarrow \mathrm{Hom}_A(B_t, B_{t+\nu})$$
$$b \longrightarrow \left| \begin{array}{l} B_t \to B_{t+\nu} \\ c \mapsto b.c \end{array} \right.$$

By definition of $H^0_{\mathfrak{m}}(B)$, for all $\nu \in \mathbb{N}$ we have

$$H^0_{\mathfrak{m}}(B)_\nu = \bigcup_{t \in \mathbb{N}} \operatorname{Ker}(\theta_{\nu,t}).$$

Moreover, $\forall (\nu,t) \in \mathbb{N}^2$, we have

$$\operatorname{Ker}(\theta_{\nu,t}) \subset \operatorname{Ker}(\theta_{\nu,t+1})$$

because the multiplication $B_1 \otimes B_t \to B_{t+1}$ is surjective, so if $b \in \operatorname{Ker}(\theta_{\nu,t})$ then $b.c = 0$ for all $c \in B_{t+1}$.

$\rightsquigarrow$ $H^0_{\mathfrak{m}}(B)_\nu$ is a union of kernels that form an increasing sequence.

Now, recall that $\operatorname{ann}_A(B_t) = \operatorname{Ker}(\theta_{0,t})$ $\forall t$. Thus, setting $\nu = 0$ we get

$$\mathfrak{a} = H^0_{\mathfrak{m}}(B)_0 = \bigcup_{t \geq 0} \operatorname{ann}_A(B_t)$$

with $(0) = \operatorname{ann}_A(B_0) \subset \operatorname{ann}_A(B_1) \subset \cdots \subset \operatorname{ann}_A(B_t) \subset \operatorname{ann}_A(B_{t+1}) \subset \cdots$

<u>Proposition</u>: Let $\eta \in \mathbb{N}$ such that $H^0_{\mathfrak{m}}(B)_\eta = 0$, then

$$\operatorname{ann}_A(B_\eta) = \operatorname{ann}_A(B_{\eta+t}) = H^0_{\mathfrak{m}}(B)_0 = \mathfrak{a} \quad \forall t \geq 0.$$

<u>Proof</u>: We know that $\operatorname{ann}_A(B_\nu) \subset \operatorname{ann}_A(B_{\nu+t})$ $\forall t \geq 0$.
Let $a \in \operatorname{ann}_A(B_{\nu+t})$, i.e. $a.B_{\nu+t} = 0$. By def. of $\theta_{\nu,t}$ we see that $a.B_\nu \subset \operatorname{Ker}(\theta_{\nu,t})$. Therefore, if $\operatorname{Ker}(\theta_{\nu,t}) = 0$ then $a \in \operatorname{ann}_A(B_\nu)$ and hence $\operatorname{ann}_A(B_\nu) = \operatorname{ann}_A(B_{\nu+t})$.
But the assumption $H^0_{\mathfrak{m}}(B)_\eta = 0$ implies that $\operatorname{Ker}(\theta_{\eta,t}) = 0$ $\forall t \geq 0$ and the result is proved. $\square$

$\rightsquigarrow$ $\mathfrak{a}$ is the annihilator of a graded component of B; we need to control B and $H^0_{\mathfrak{m}}(B)$.

2. Tools from homological algebra

- Exact sequences.

- A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of commutative groups is exact if $\ker g = \mathrm{Im}\, f$
- A sequence $X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} \cdots$ is exact if $\ker f_n = \mathrm{Im}\, f_{n+1}$ $\forall n$
- A short exact sequence is an exact sequence of the form

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

- A morphism of long exact sequences is a family of morphisms $(h_n : X_n \to Y_n)_n$ such that $h_n \circ f_{n+1} = g_{n+1} \circ h_{n+1}$ $\forall n$

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & X_n \arrow[r, "f_n"] \arrow[d, "h_n"] \arrow[dr, phantom, "\circlearrowleft"] & X_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Five lemma:

$$\begin{tikzcd}
X_{n+2} \arrow[r, "f_{n+2}"] \arrow[d, "h_{n+2}"] & X_{n+1} \arrow[r] \arrow[d, "h_{n+1}"] & X_n \arrow[r, "f_n"] \arrow[d, "h_n"] & X_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & X_{n-2} \arrow[d, "h_{n-2}"] & (\text{exact}) \\
Y_{n+2} \arrow[r, "g_{n+2}"'] & Y_{n+1} \arrow[r] & Y_n \arrow[r, "g_n"'] & Y_{n-1} \arrow[r] & Y_{n-2} & (\text{exact})
\end{tikzcd}$$

$\leftarrow$ morphism of exact seq.

If $h_{n+2}, h_{n+1}, h_{n-1}$ and h_{n-2} are isomorphisms, then h_n is an isomorphism.

Proof: (chasing diagram) we prove injectivity and leave surjectivity as an exercise.

- Let $x \in \ker(h_n)$, then $g_n \circ h_n(x) = h_{n-1} \circ f_n(x) = 0$.
  Since h_{n-1} is an isomorphism then $f_n(x) = 0$. By exactness at X_n, this implies that $\exists y \in X_{n+1}$ such that $x = f_{n+1}(y)$
- Now $g_{n+1} \circ h_{n+1}(y) = 0$ so by exactness at Y_{n+1}

$h_{n+1}(y) \in \mathrm{Im}\, g_{n+2}$: $\exists\ z \in Y_{n+2}$ such that $g_{n+2}(z) = h_{n+1}(y)$.
It follows that $y = h_{n+1}^{-1} \circ g_{n+2}(z) = f_{n+2} \circ h_{n+2}^{-1}(z)$
and hence $y \in \mathrm{Im}(f_{n+2}) = \mathrm{Ker}(f_{n+1})$. Consequently, $x = 0$ $\square$.

- Chain complexes.

- A chain complex is a sequence of morphisms

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

such that $d_n \circ d_{n+1} = 0\ \forall n$ $\left(\mathrm{Im}(d_{n+1}) \subset \mathrm{Ker}(d_n) \right)$.
The maps "d_n" are called the differentials.

Rk: a long exact sequence is a particular chain complex.

- Let $(X_\bullet, d_\bullet)$ a chain complex. We set

$$\left|\begin{array}{ll} Z_n = Z_n(X_\bullet) = \mathrm{Ker}(d_n) & \text{cycles} \\ B_n = B_n(X_\bullet) = \mathrm{Im}(d_{n+1}) & \text{boundaries} \\ H_n = H_n(X_\bullet) = Z_n/B_n & \text{homology groups} \end{array}\right.$$

Rk: $H_n(X_\bullet) = 0\ \forall n \iff (X_\bullet, d_\bullet)$ is a long exact sequence

Ex: $0 \to M \xrightarrow{f} N \to 0$ then $\left| \begin{array}{l} H_0 = N/\mathrm{Im}(f) = \mathrm{coker}\, f \\ H_1 = \mathrm{Ker}(f). \end{array}\right.$

- Morphism of complexes.

Let $(X_\bullet, d_\bullet)$ and $(Y_\bullet, d'_\bullet)$ be two complexes.

A morphism between them is given as a family $(f_n : X_n \to Y_n)_n$ such that we have commutative diagrams

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowright"] & X_n \arrow[d, "f_n"] \\ Y_{n+1} \arrow[r, "d'_{n+1}"'] & Y_n \end{tikzcd} \quad \forall n.$$

The maps f_n send $Z_n(X_\bullet)$ to $Z_n(Y_\bullet)$ and $B_n(X_\bullet)$ to $B_n(Y_\bullet)$.
Thus $(f_\bullet)$ induce morphisms

$$H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet) \quad \forall n.$$

Similarly, concepts of sub-groups, kernel, image, etc... from group theory extends to similar concepts for chain complexes.
For instance $\mathrm{Ker}(f_\bullet : X_\bullet \to Y_\bullet)$ is a chain complex.

$$\begin{tikzcd} \mathrm{Ker}(f_{n+1}) \arrow[r, dashed] \arrow[d, hook] & \mathrm{Ker}(f_n) \arrow[d, hook] & \\ X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r] \arrow[d, "f_n"] & X_{n-1} \\ Y_{n+1} \arrow[r, "d'_{n+1}"'] & Y_n \arrow[r] & Y_{n-1} \end{tikzcd}$$

$d_{n+1}(\mathrm{Ker}(f_{n+1})) \subset \mathrm{Ker}(f_n)$.

- <u>Long exact sequence of homology</u>.

Suppose given a short exact sequence of complexes

$$X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{g_\bullet} Z_\bullet$$

($f_\bullet$ injective, $g_\bullet$ surjective, $\mathrm{Ker}\, g_\bullet = \mathrm{Im}\, f_\bullet$), then we have a long exact sequence of the form

$$\cdots H_{n+1}(Z_\bullet) \xrightarrow{\partial_{n+1}} H_n(X_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(Y_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(Z_\bullet) \xrightarrow{\partial_n} H_{n-1}(X_\bullet) \to \cdots$$

where the morphisms $(\partial_n)_{n\in\mathbb{N}}$ are called "connecting morphisms".

Proof: It is technical and can be found in many book; we skip it $\square$.

3. The Koszul complex.

- Let A be a commutative ring. For all $x \in A$ we define the complex

$$K_\bullet(x;A): \quad 0 \to K_1 = A \xrightarrow{x} K_0 = A \to 0$$
$$a \mapsto a.x$$

More generally given a sequence of elements in A $(x_1, \dots, x_n) = \underline{x}$ we define the Koszul complex as follows:

$$K_i(\underline{x}; A) = \Lambda^i(A^n) \quad \text{(exterior algebra)}$$

If $A^n = \bigoplus_{i=1}^n Ae_i$ (canonical basis), we have $K_0 = A$ and for all $p \geq 1$

$$K_p = \bigoplus_{1 \leq i_1 < \dots < i_p \leq n} A\, e_{i_1} \wedge \dots \wedge e_{i_p}$$

The differentials are given by

$$d_p : K_p \to K_{p-1}$$
$$e_{i_1} \wedge \dots \wedge e_{i_p} \mapsto \sum_{k=1}^{p} (-1)^{k+1} x_{i_k}\, e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_p}$$

Lemma: $d_{p-1} \circ d_p = 0$

Proof: (exercise).

Example: $K_1(\underline{x}; A) \to K_0(\underline{x}; A)$ is the map

$$\bigoplus_{i=1}^{n} A \to A$$
$$(y_1, \dots, y_n) \to \sum x_i y_i$$

(this should remind you something ...)

Remark: If A is a graded ring then $K_\bullet(\underline{x}; A)$ is also graded providing $x_1, \dots, x_n$ are homogeneous elets.

Let $\deg(x_i) = d_i$ then $K_0 = A$, $K_1 = \bigoplus_{i=1}^{n} A(-d_i)$ and

$$K_p = \bigoplus_{1 \leq i_1 < \dots < i_p \leq n} A(-d_{i_1} - \dots - d_{i_p})$$

Finally, given a A-module M, we define $K_\bullet(\underline{x}; M)$ as $K_\bullet(\underline{x}; A) \otimes_A M$.

- Homology of the Koszul complex

Two easy remarks:

i) $H_p(\underline{x}; M)$ is a sub-quotient of a direct sum of copies of M, hence $\mathrm{ann}_A(M)$ annihilates $H_p(\underline{x}; M)$ $\forall p$.

ii) $H_0(\underline{x}; M) = M/(\underline{x}).M$; it is annihilated by $(\underline{x})$.

Proposition: (i) The ideals $\mathrm{ann}_A(M)$ and $(\underline{x})$ of A annihilate the homology of $K_\bullet(\underline{x}; M)$.

(ii) If $\underline{x}$ is a M-regular sequence then $H_p(\underline{x}; M) = 0$ $\forall p \geq 1$.

Proof: i) we define $\sigma_p^j : K_p \to K_{p+1}$ (of degree d_j).

$$e_{i_1} \wedge \dots \wedge e_{i_p} \to e_j \wedge e_{i_1} \wedge \dots \wedge e_{i_p}$$

and one can check that for any $x \in K_p(\underline{x}; M)$ we have (exercise)

$$d_{p+1}(\sigma_p^j(\underline{x})) + \sigma_{p-1}^j(d_p(\underline{x})) = x_j \cdot x .$$

We deduce that $(\underline{x})$ annihilate $H_p(\underline{x}; M)$ $\forall p$.

ii) We proceed by induction on n.

$\underline{n=1}$: $H_1(x_1; M) = \mathrm{Ker}(M \xrightarrow{x_1} M) = 0$

Suppose the claimed property true for $n-1$.

Let $x' = (x_1, \dots, x_{n-1})$ and $x = (x_1, \dots, x_n)$.

We have an exact sequence of complexes

$$0 \to K_\bullet(x'; M) \xrightarrow{i_\bullet} K_\bullet(\underline{x}; M) \xrightarrow{\pi_\bullet} \overleftarrow{K_\bullet}(x'; M) \to 0 \quad (*)$$

where $\overleftarrow{K_\bullet}(x'; M)$ is the left shift: $\overleftarrow{K_p} = K_{p-1}$ and $\overleftarrow{d_p} = d_{p-1}$.

The map i_p are canonical injection and the map π_p is defined as

$$\pi_p(e_{i_1} \wedge \dots \wedge e_{i_p}) = \begin{cases} e_{i_1} \wedge \dots \wedge e_{i_{p-1}} & \text{if } i_p = n \\ 0 & \text{otherwise.} \end{cases}$$

Exactness by rows $0 \to K_p(X') \xrightarrow{i_p} K_p(X) \xrightarrow{\pi_p} K_{p-1}(x') \to 0$ is essentially by definition. The fact that the differentials give commutative diagram is easy to check.

Now, from $(*)$ we obtain a long exact sequence of homology:

$$\cdots \to H_p(x';M) \xrightarrow{\partial_{p+1}} H_p(x';M) \xrightarrow{i_p} H_p(x;M) \xrightarrow{\pi_p} H_{p-1}(x';M) \to \cdots$$

(The connecting morphism $\delta_\bullet$ is actually a multiplication by $(-1)^p \cdot x_n$; $\delta_{p+1}(y) = y \cdot (-1)^p x_n$. (to be checked).)

Now, from our assumption we get that

$$H_p(x;M) = 0 \quad \forall p > 1.$$

Moreover, the end of this exact sequence is

only this map is useful here.

$$\cdots \to \underbrace{H_1(x';M)}_{\substack{=0 \\ \text{by assumption}}} \xrightarrow{i_1} H_1(x;M) \xrightarrow{\pi_1} \underbrace{H_0(x';M)}_{\substack{\| \\ M/(x').M}} \xrightarrow{\times x_n} \underbrace{H_0(x';M)}_{\substack{\| \\ M/(x').M}} \to \cdots$$

As x_n is not a zero divisor, the map on the right is injective and hence $H_1(x;M) = 0$. □

• <u>Application to the resultant.</u>

$A = \mathbb{Z}[U_{i,\alpha}]$ $\quad C = A[X_1, \dots, X_n]$ $\quad f_i = \sum_{|\alpha| = d_i \geq 1} U_{i,\alpha} X^\alpha \quad i = 1, \dots, n$

We have seen that $(f_1, \dots, f_n)$ is a C-regular sequence so we deduce that $H_p(K_\bullet(f_1, \dots, f_n; C)) = 0 \quad \forall p \geq 1$.

Moreover, $H_0(K_\bullet(\underline{f}; C)) = C/(f_1, \dots, f_n) = B$ (previous notation).

Therefore, the Koszul complex is a free graded resolution of B:

$$0 \to \underbrace{C(-\Sigma d_i)}_{\substack{\| \\ K_n}} \to \cdots \to \underbrace{\bigoplus_{i=1}^{n} C(-d_i)}_{\substack{\| \\ K_1}} \to \underbrace{C}_{\substack{\| \\ K_0}} \to B \to 0$$

Taking a graded slice of this complex we obtain a free resolution of ~~the~~ free A-modules of B_ν, $\nu \in \mathbb{N}$.

Example: $n=2$ $\nu = d_1+d_2-1$

$$0 \to C(-d_1-d_2) \to C(-d_1)\oplus C(-d_2) \to C' \to B \to 0$$

graded slice

$$0 \to C_{d_2-1}\oplus C_{d_1-1} \to C_{d_1+d_2-1} \to B_{d_1+d_2-1} \to 0$$

sylvester matrix.

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. Now, two questions:

1) why did we choose d_1+d_2-1 in the above example?
How to control $H^0_m(B)_2$? ⇝ study of local cohomology

2) what is the link between $\mathrm{ann}_A(B_\nu)$ and the determinant of the matrix in the above example?
⇝ Fitting ideals.

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• THM: With the above assumptions we have

$$H^0_m(B)_\nu = 0 \quad \text{for all } \nu \geq \sum_{i=1}^n (d_i-1)+1$$

Proof. requires Cech complex and spectral sequences ... skip it for the moment. □

Corollary: For any $\nu \geq \sum_{i=1}^n (d_i-1)+1$ we have

$$\mathrm{ann}_A(B_\nu) = q = (\mathrm{Res}(f_1,\ldots,f_n)) \subset A.$$

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4 - Fitting ideals

- Let F, G be two free A-modules of finite type and $\phi : F \to G$ a A-morphism. For all $\nu \leq \max(rk(F), rk(G))$ we set

$\det_\nu(\phi) :=$ ideal generated by the ν-minors of a matrix of ϕ.

Rks: - These are ideals of A
- They are indep. of the choice of the matrix of ϕ
- $\det(\phi) = 1$
- $$\det_{\nu<0}(\phi) = A = \dots = \underset{\det_0(\phi)}{A} \supseteq \det_1(\phi) \supseteq \dots \supseteq \det_{\max(rk(F), rk(G))}(\phi) \supseteq 0 \supseteq \dots \supseteq 0.$$

The important property of Fitting ideals is the following:

Proposition: Let M be a finitely generated A-modules and suppose given two presentations

$$A^q \xrightarrow{(a_{ij})} A^p \xrightarrow{(x_1 - x_p)} M \to 0$$
$$A^s \xrightarrow{(b_{ij})} A^r \xrightarrow{y_1 - y_r} M \to 0$$

Then for all $\nu \geq 0$ we have $\det_{p-\nu}(a_{ij}) = \det_{r-\nu}(b_{ij})$.

Def: The ν^{th} Fitting invariant of M is defined as

$$F_\nu(M) := \det_{p-\nu}(\phi) \qquad \nu \geq 0.$$

(It is indep. of the choice of a presentation of M).

Proof: • First, one can assume that $p=r$ because:

write $y_j = \sum_{i=1}^{p} \alpha_i x_i$, $j=1..r$, we have a presentation

$$A^q \oplus A^r \xrightarrow{\left(\begin{array}{c|c} a_{ij} & * \\ \hline 0 & \begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\end{array}\right)} A^p \oplus A^r \xrightarrow{(x_1 - x_p, y_1 - y_r)} M \to 0$$

and we see that $\det_{p-\nu}(a_{ij}) = \det_{p+r-\nu}\left(\begin{array}{c|c} a_{ij} & * \\ \hline 0 & \begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\end{array}\right)$ $\forall \nu \geq 0$

With a similar argument,

$$\det_{r-\nu}(b_{ij}) = \det_{p+r-\nu}\left(\begin{array}{c|c} b_{ij} & *' \\ \hline 0 & \begin{smallmatrix}1 & \\ & \ddots & \\ & & 1\end{smallmatrix}\end{array}\right).$$

• So, assume $p=r$.

If $q=s$ then (a_{ij}) and (b_{ij}) are matrices of the same map and hence the result follows.

If $q \neq s$, say $s \geq q$: $A^s = A^{s-q} \oplus A^q$ and acting on columns we can obtain

$$A^s \to A^{r=p} = A^{s-q} \oplus A^q \xrightarrow{0 \oplus \phi} A^p$$

where $\phi : A^q \to A^p$ $\square$

We have the following properties:

Let M be a finitely generated A-mod.

(i) We have an increasing sequence $\mathcal{F}_0(M) \subseteq \mathcal{F}_1(M) \subseteq \cdots \subseteq A$

(ii) If M is generated by q elts then $\mathcal{F}_q(M) = A$ $(= \mathcal{F}_t(M)\ \forall t \geq q)$

(iii) $\mathcal{F}_0(M) \subseteq \operatorname{ann}_A(M)$

(iv) For all $\nu > 0$, $\operatorname{ann}_A(M) \cdot \mathcal{F}_\nu(M) \subseteq \mathcal{F}_{\nu-1}(M)$

(v) If M is generated by q elements then

$$\operatorname{ann}_A(M)^q \subseteq \mathcal{F}_0(M) \subseteq \operatorname{ann}_A(M)$$

(vi) If $A \to R$ is a ring morphism, then

$$\mathcal{F}_\nu(M \otimes_A R) = \mathcal{F}_\nu(M) \cdot R$$

(stability under change of basis).

Proof: exercises — classical topic. □.

Application to the resultant.

We have seen that $\operatorname{ann}_A(B_\nu) = (\operatorname{Res}(f_1 - f_n))$ $\forall \nu \geq \sum_{i=1}^{n}(d_i - 1) + 1$

Now, we have shown that $V(\operatorname{ann}_A(B_\nu)) = V(\mathcal{F}_0(B_\nu))$ $\forall \nu \geq 0$

Actually, one can show that

$$(\gcd \text{ minors of } \mathcal{F}_0(B_\nu)) = (\operatorname{Res}(f_1 - f_n)) \quad \forall \nu \geq \sum_{i=1}^{n}(d_i - 1) + 1$$

This property can be seen indirectly by means of Macaulay matrices that are elements in $\mathcal{F}_0(B_\nu)$.

In conclusion, the first map of the Koszul complex in degree $\nu \geq \sum(d_i - 1) + 1$ give a matrix-based representation of the resultant:

$\operatorname{Res}(f_1 - f_n) = 0 \iff$ the rank of this matrix drops.