Exercise 4.

3.

$$
\begin{array}{cccc}
X^\alpha,\ |\alpha| = d-2 & " & " & (i,j,k) = d-1,\ i,j,k \geq 0 \\
\left(\begin{array}{c|c|c|c}
X^\alpha f_1 & X^\alpha f_2 & X^\alpha f_3 & \Delta_{i,j,k}
\end{array}\right) & & & \\
\binom{d}{2} & \binom{d}{2} & \binom{d}{2} & \binom{d+1}{2}
\end{array}
\quad X^\alpha,\ |\alpha| = d + d - 2 = 2d-2
$$

$f_1, f_2, f_3\ (x_1, x_2, x_3)$ hom. of degree d.

$$\#\text{ columns} = 3\binom{d}{2} + \binom{d+1}{2} = d(2d-1) = \binom{2d}{2}\ \#\text{ rows}.$$

$\Rightarrow$ square matrix.

4. $\mathrm{Res}(f_1, f_2, f_3) = \pm \det(M)$. (compare degrees)

4. Fitting ideals

Goal: connect annihilators and matrices.

- F, G two free A-modules finite type

$F \simeq A^p \quad G \simeq A^q$

$\phi : F \to G$ A-linear.

$\phi : A^p \xrightarrow{(a_{ij})} A^q$

$p \geq q$

For all $\nu \leq \min(rk(F), rk(G))$ we set

det. ideals associated to a map.

$\boxed{\det_\nu(\phi)}$ = ideal generated by the ν-minors of a matrix of ϕ

↑ ideals, indep of the choice of matrix

$$\underset{\shortparallel}{A} \supseteq \det_1(\phi) \supseteq \det_2(\phi) \supseteq \det_3(\phi) \supseteq \dots \supseteq \det_{\min(rk(F), rk(G))}(\phi)$$

$\det_0(\phi)$ ideal generated by entries

$(\det(\phi) = 1)$

Convention.

Proposition: Let $\boxed{M}$ be a finitely generated A-module and suppose given two presentations:

$$q \geq p \qquad A^{q} \xrightarrow{(a_{ij})} A^{p} \xrightarrow{(x_1 - x_p)} \boxed{M} \to 0$$

$$s \geq r \qquad A^{s} \xrightarrow{(b_{ij})} A^{r} \xrightarrow{(y_1 - y_r)} \boxed{M} \to 0$$

ν fixed.

Then for all $\nu \geq 0$ we have $\det_{p-\nu}(a_{ij}) = \det_{r-\nu}(b_{ij})$ (exercise)!

$\Rightarrow$ **Definition:** The ν^{th} Fitting of M is defined as

$$\mathcal{F}_\nu(M) := \det_{p-\nu}(\phi) \qquad \nu \geq 0.$$

(It is indep. on the choice of presentation).

Example: $\mathcal{F}_0(M)$ (initial Fitting ideal)

$\rightsquigarrow$ generated by the "max minors" of a presentation matrix

We have the following properties.

i) We have an increasing sequence $\mathcal{F}_0(M) \subseteq \mathcal{F}_1(M) \subseteq \dots \subseteq A$

ii) If M is generated by q elts then $\mathcal{F}_q(M) = A = \mathcal{F}_t(M)$ $\forall t \geq q$

$\mathcal{F}_q = (\det_0(a_{ij})) = A \Leftarrow A^s \xrightarrow{(a_{ij})} A^{q(x_1 - x_q)} \to M \to 0$ $t \gg 0$

iii) $\mathcal{F}_0(M) \subseteq \mathrm{ann}_A(M)$

iv) For all $\nu \geq 0$, $\mathrm{ann}_A(M) . \mathcal{F}_\nu(M) \subseteq \mathcal{F}_{\nu-1}(M)$

v) If M is generated by q elements then

$$\mathrm{ann}_A(M)^q \subseteq \mathcal{F}_0(M) \subseteq \mathrm{ann}_A(M)$$

vi) If $A \to R$ is a ring morphism then

$$\mathcal{F}_j(M \otimes_A R) = \mathcal{F}_j(M) . R$$

(stability under change of basis). (forget)

Proof: i), ii) ok.

iii) We choose a presentation of M:

$$A^r \xrightarrow{(a_{ij})} A^q \xrightarrow{(x_1 - x_q)} M \to 0 \qquad \underline{r \geq q} \quad (\text{add columns})$$

N: be a $q \times q$ sub-matrix of (a_{ij})

$$ {}^tN \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} = 0 \quad (*)$$

(Recall: the columns of (a_{ij}) are syzygies of $(x_1 - x_q)$

$$(x_1\ x_2 \dots x_q) \cdot \begin{pmatrix} \vdots & a_{ij} \\ \vdots & \end{pmatrix} \Big\updownarrow q = 0$$

$\longleftrightarrow$ q columns
)

Multiply $(*)$ by the cofactor matrix:

$$\underbrace{\mathrm{cof}\left({}^{t}N\right)\cdot N}_{\shortparallel} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} = 0$$

$$\begin{pmatrix} \det(N) & & 0 \\ & \ddots & \\ 0 & & \det(N) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} = 0 \qquad \Rightarrow \quad x_i \cdot \det(N) = 0$$

$$\det(N) \in \mathrm{ann}_A(M).$$

$$\Rightarrow \quad \mathcal{F}_0(M) \subseteq \mathrm{ann}_A(M).$$

$\square$