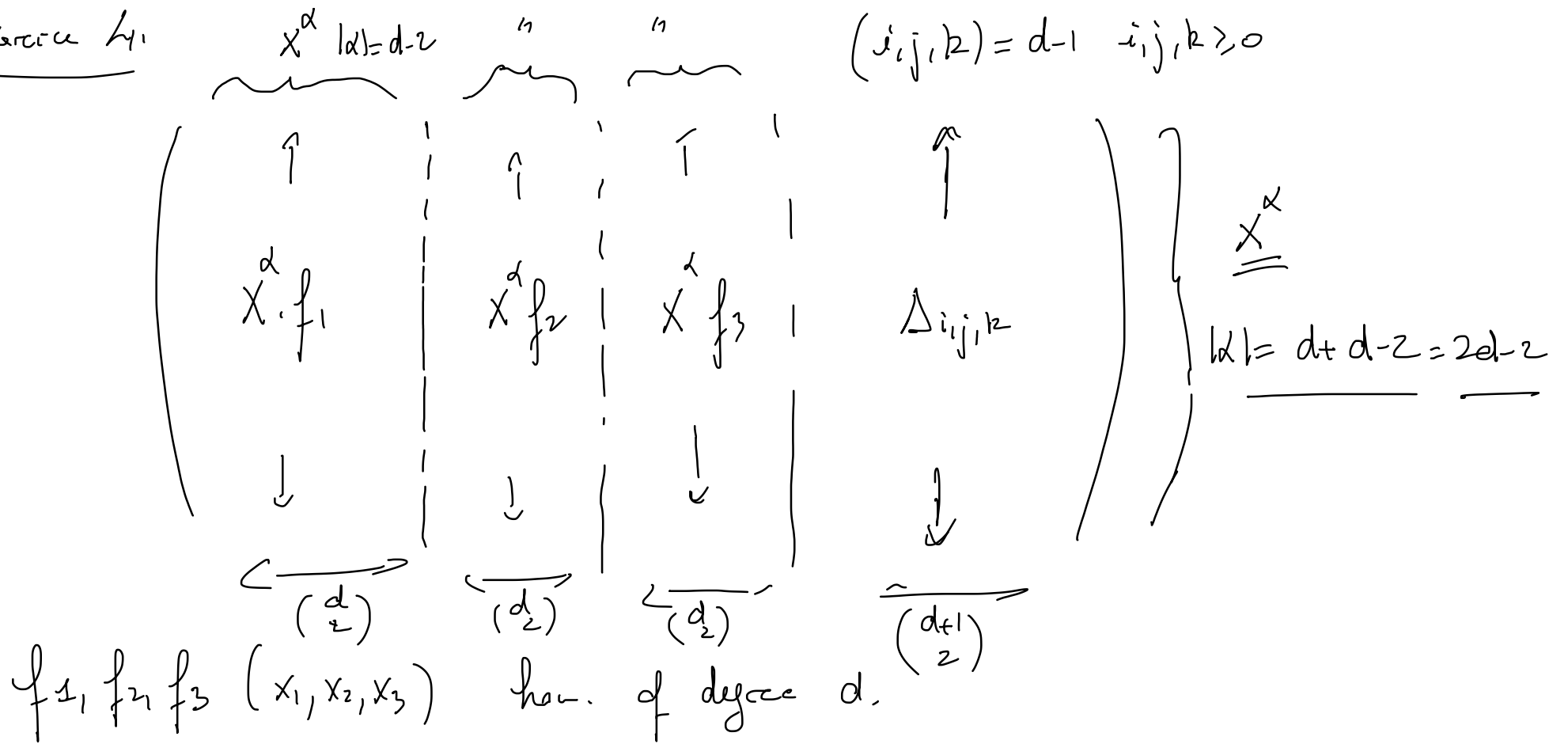


Esercizio 4.

3.



$$\# \text{ columns} = 3 \binom{d}{2} + \binom{d+1}{2} = d(2d-1) = \binom{2d}{2} \quad \# \text{ rows.}$$

\Rightarrow square matrix.

$$4. \quad \text{Rs}(f_1, f_2, f_3) = \pm \det(\underline{M}). \quad (\text{compare degrees})$$

4. Fitting ideals

Goal: correct annihilators and matrices.

- F, G two free A -modules finite type

$$\begin{aligned} F &\cong A^p & G &\cong A^q \\ \phi &: A^p \xrightarrow{(a_{ij})} A^q \\ p &\geq q \end{aligned}$$

$\phi: F \rightarrow G$ A -linear.

For all $\nu \leq \min(\text{rk}(F), \text{rk}(G))$ we set

det. ideals
associated
to a map.

$\boxed{\det_\nu(\phi)}$ = ideal generated by the ν -minors of a matrix of ϕ
 \uparrow ideals, indep of the choice of matrix

$$\boxed{A} \supseteq \underbrace{\det_1(\phi)}_{\substack{\uparrow \\ \text{ideal generated by entries}}} \supseteq \underbrace{\det_2(\phi)} \supseteq \underbrace{\det_3(\phi)} \supseteq \dots \supseteq \underbrace{\det(\phi)}_{\min(\text{rk}(F), \text{rk}(G))}$$

$\det(\phi) = 1$
convention.

Proposition: Let M be a finitely generated A -module
and suppose given two presentations:

$$\begin{array}{c} \text{--- } q \geq p \\ A^{\oplus q} \xrightarrow{(a_{ij})} A^{\oplus p} \longrightarrow M \longrightarrow 0 \end{array}$$

$$\begin{array}{c} \text{--- } r \geq v \\ A^{\oplus r} \xrightarrow{(b_{ij})} A^{\oplus v} \longrightarrow M \longrightarrow 0 \end{array}$$

v fixed.

Then for all $v \geq 0$ we have $\det_{p-v} (a_{ij}) = \det_{r-v} (b_{ij})$
(exercise).

\Rightarrow Definition: The v^{th} Fitting of M is defined as

$$\underline{F}_v(M) := \det_{p-v} (\phi) \quad v \geq 0.$$

(\underline{F}_v is indep. on the choice of presentation).

Example: $\underline{F}_0(M)$ (initial Fitting ideal)

\mathbb{Z} generated by the "max minors" of a presentation matrix

We have the following properties.

i) We have an increasing sequence $\overline{F}_0(M) \subseteq \overline{F}_1(M) \subseteq \dots \subseteq A$

ii) If M is generated by g elts then $\overline{F}_g(M) = A = \overline{F}_+ (M) \quad \forall + \geq g$

→ iii) $\overline{F}_0(M) \subseteq \text{ann}_A(M)$ $\overline{F}_g = \frac{(\det_0(a_{ij}))}{= A} \leq A \xrightarrow{s(a_{ij})} A \xrightarrow{g(x_1, \dots, x_n)} M \rightarrow 0 \quad \vdash \gg 0$

→ iv) For all $j \geq 0$, $\text{ann}_A(M) \cdot \overline{F}_j(M) \subseteq \overline{F}_{j-1}(M)$

→ v) If N is generated by g elements then

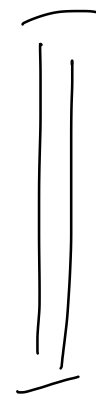
$$\text{ann}_A(M)^g \subseteq \overline{F}_0(M) \subseteq \text{ann}_A(N)$$

→ vi) If $A \rightarrow R$ is a ring morphism then

$$\overline{F}_j(M \otimes_A R) = \overline{F}_j(M) \cdot R$$

(Stability under change of basis).

(forget)



Proof: i), ii) ok.

iii) We choose a presentation of Π :

$$A \xrightarrow{r} A^{q \times (n-q)} \Pi \rightarrow \circ \quad \xrightarrow{r \geq q} \text{(add columns)}$$

N : be a $q \times q$ sub-matrix of (a_{ij})

$$z_N \begin{pmatrix} x_1 \\ \vdots \\ 1 \\ \vdots \\ x_q \end{pmatrix} = 0 \quad (\neq)$$

(Recall: the columns of (a_{ij}) are syzygies of (x_1, \dots, x_q))

$$\left(\begin{array}{c} \underline{x_1} \quad \underline{x_2} \quad \dots \quad \underline{x_q} \end{array} \right) \cdot \left(\begin{array}{c} \vdots \\ \vdots \\ a_{ij} \\ \vdots \end{array} \right) \Bigg|_{\substack{\uparrow q \\ \leftarrow q \\ \text{column}}} = 0$$

Multiply (*) by the cofactor matrix:

$$\underbrace{\text{cof}(N)}_I \cdot N \cdot \begin{pmatrix} x_1 \\ 1 \\ x_n \end{pmatrix} = 0$$

$$\begin{pmatrix} \det(N) & & 0 \\ & \ddots & \\ 0 & & \det(N) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 1 \\ x_n \end{pmatrix} = 0$$

$$\Rightarrow x_i \cdot \det(N) = 0$$

$$\det(N) \in \text{ann}_A(V).$$

$$\Rightarrow \overline{\mathcal{F}_0(N)} \subseteq \text{ann}_A(V).$$

□