C8) <u>Properties of the resultant</u>

1) R a commutative ring and $l_1, .., l_n \in R[x_1, .., x_n]_1$ linear forms.

$$l_i = \sum_{j=1}^{n} u_{ij} x_j \qquad i = 1..n$$

Then $\mathrm{Res}(l_1, .., l_n) = \det(u_{ij})$.

<u>Proof</u>: Assume we are in the generic setting over $\mathbb{Z}$.

$x_1 \det(u_{ij}) \in (l_1, .., l_n) \Rightarrow \det(u_{ij}) \in \mathcal{Q}(I) = (\mathrm{Res}(l_1 .. l_n))$.

It follows that $\mathrm{Res}(l_1 .. l_n) \mid \det(u_{ij})$

Moreover $\deg_{l_i} \det(u_{ij}) = 1$ and $\underset{\substack{\uparrow \\ \text{depends on} \\ \text{all coeff.}}}{1 \leq} \deg_{l_i}(\mathrm{Res}) \underset{\substack{\uparrow \\ \text{Macaulay matrices.}}}{\leq}$

so $c.\mathrm{Res}(l_1, .., l_n) = \det(u_{ij})$ with $c \in \mathbb{Z}$.

By specialization $l_i \to x_i$ we get $1 = c \cdot 1$ □.

2) <u>Divisibility</u>. R a commutative ring and $(f_1, .., f_n), (g_1, .., g_n)$ two sequences of hom. polynomials in $R[x_1, .., x_n]$ such that $g_i \in (f_1, .., f_n) \subset R[x_1, .., x_n]$ for all i then

$$\mathrm{Res}(f_1, .., f_n) \mid \mathrm{Res}(g_1, .., g_n) \quad \text{in } R.$$

<u>Proof</u>: $g_i = \sum_{j=1}^{n} h_{ij} f_j$ by assumption. By specialization, one can assume that both f_j's and h_{ij}'s have indeterminate coefficients over $\mathbb{Z}$.

Then $x_n^N \mathrm{Res}(g_1, .., g_n) \underset{\substack{\uparrow \\ \text{inertia form} \\ \text{and specialization}}}{\in} (g_1, .., g_n) \underset{\substack{\uparrow \\ \text{assumption.}}}{\subset} (f_1, .., f_n)$

and hence $\mathrm{Res}(f_1 .. f_n) \mid \mathrm{Res}(g_1, .., g_n)$ (f_i's are generic). □.

3) Multi-degree of the resultant

$\mathrm{Res}(f_1,\dots,f_n) \in {}_kA[x_1,\dots,x_n] = k[u_{i,\alpha}][x_1,\dots,x_n]$ is homogeneous in the coeff $u_{i,\alpha}$ of f_i, for each i, of degree $\dfrac{d_1\cdots d_n}{d_i}$.

Proof: • The existence of the Macaulay determinants imply that $\deg_{f_i} \mathrm{Res}(f_1\dots f_n) \leq \dfrac{d_1\cdots d_n}{d_i}$ $\forall i$

• Now, specialize each f_i into a product of linear forms:

$$f_i \overset{\rho}{\longmapsto} g_i := \prod_{j=1}^{d_i} l_{i,j} \qquad l_{i,j} \text{ generic}$$

By divisibility we have $\mathrm{Res}(l_{1,j_1},\dots,l_{n,j_n}) \mid \mathrm{Res}(g_1,\dots,g_n)$

$\forall j_1,\dots,j_n$.

We deduce that

$$\prod_{\substack{1\leq j_1\leq d_1\\ \vdots \\ 1\leq j_n\leq d_n}} \mathrm{Res}(l_{1,j_1},\dots,l_{n,j_n}) \mid \mathrm{Res}(g_1\dots g_n) \qquad (*)$$

Moreover, $\deg_{l_{i,j_k}} \mathrm{Res}(l_{1,j_1},\dots,l_{n,j_n}) = 1$ $\forall i,j,k$.

We have $d_1\cdots d_n$ terms in the product $(*)$. In addition the coeffs of f_i become of degree d_i in the coeff of l_{i,j_k} via ρ.

Therefore we conclude that $\deg_{f_i} \mathrm{Res}(f_1\dots f_n) \geq \dfrac{d_1\cdots d_n}{d_i}$ $\square$

4) Multiplicativity

If $f_i = f_i' f_i''$ then

$$\mathrm{Res}(f_1,\dots,f_i,\dots,f_n) = \mathrm{Res}(f_1,\dots,f_i',\dots,f_n)\,\mathrm{Res}(f_1,\dots,f_i'',\dots,f_n).$$

(See Cox-Little-O'Shea).

5) Invariance under permutation.

Let $\sigma \in S_n$: $\mathrm{Res}(f_{\sigma(1)},\dots,f_{\sigma(n)}) = \varepsilon(\sigma)^{d_1\dots d_n}\,\mathrm{Res}(f_1,\dots,f_n)$

Proof: By the first property on determinants, we have

$$\mathrm{Res}(x_{\sigma(1)},\dots,x_{\sigma(n)}) = \varepsilon(\sigma)\cdot 1 = \varepsilon(\sigma).$$

Then, it is clear that $\mathrm{Res}(f_{\sigma(1)},\dots,f_{\sigma(n)}) \mid \mathrm{Res}(f_1,\dots,f_n)$ and conversely, so $\mathrm{Res}(f_{\sigma(1)},\dots,f_{\sigma(n)}) = c\cdot \mathrm{Res}(f_1,\dots,f_n)$ where $c=\pm 1$.

To determine c we specialize $f_i \to x_i^{d}$ for all i.

$$\mathrm{Res}(x_{\sigma(1)}^{d_{\sigma(1)}},\dots,x_{\sigma(n)}^{d_{\sigma(n)}}) = c\cdot \mathrm{Res}(x_1^{d_1},\dots,x_n^{d_n})$$

i.e $\quad \mathrm{Res}(x_{\sigma(1)},\dots,x_{\sigma(n)})^{d_1\dots d_n} = c\cdot \mathrm{Res}(x_1,\dots,x_n)^{d_1\dots d_n} = c$

i.e $\quad \varepsilon(\sigma)^{d_1\dots d_n} = c.$ $\square$.

6) Section by a hyperplane.

Let $f_1,\dots,f_{n-1} \in R[x_1,\dots,x_n]$, R a commutative ring, then

$$\mathrm{Res}(f_1,\dots,f_{n-1},x_n) = \mathrm{Res}(\bar f_1,\dots,\bar f_{n-1}) \text{ with } \bar f_i = f_i(x_1,\dots,x_{n-1},0).$$

Proof: On can assume that we are in generic setting over $\mathbb{Z}$.

$x_1^N\,\mathrm{Res}(f_1,\dots,f_{n-1},x_n) \in (f_1,\dots,f_{n-1},x_n)$ as $\mathrm{Res}(f_1,\dots,f_{n-1},x_n)$ is an inertia form

We have $x_1^N\,\mathrm{Res}(f_1,\dots,f_{n-1},x_n) = \sum_{i=1}^{n-1} h_i f_i + h_n\cdot x_n$.

Specializing x_n to 0 we get $x_1^N\,\mathrm{Res}(f_1,\dots,f_{n-1},x_n) \in (\bar f_1,\dots,\bar f_{n-1})$

As $\bar f_i$ are generic pol. we deduce that $\mathrm{Res}(f_1,\dots,f_{n-1},x_n)$ is an inertia form for $\bar f_1,\dots,\bar f_{n-1}$ and hence $\mathrm{Res}(\bar f_1,\dots,\bar f_{n-1}) \mid \mathrm{Res}(f_1,\dots,f_{n-1},x_n)$.

We conclude by comparing the degrees and specializing $\bar f_i \to x_i^{d_i}$. $\square$.

7) Invariance under elementary transformations

$$Res(f_1, \ldots, f_i + \sum_{j\neq i} h_{ij} f_j, \ldots, f_n) = Res(f_1, \ldots, f_i, \ldots, f_n).$$

(exercise).

8) Macaulay formula

Recall that for all $t \geq \delta+1$, $\delta = \sum_{i=1}^{n}(d_i-1)$, we have built the Macaulay determinants

$$D_i(t) \in A = k[U_{i,\alpha}] \; ; \; \deg_{f_i} D_i(t) = \frac{d_1 \ldots d_n}{d_i}.$$

They are inertia forms and are hence divisible by the resultant:

$$D_i(t) = Res(f_1, \ldots, f_n) \cdot H_i(t) \quad i = 1, \ldots, n.$$

By degree property, it is clear that $H_i(t)$ is indep of the coeff of f_i. This implies the following fact.

$$Res(f_1 \ldots f_n) = \gcd(D_1(t), \ldots, D_n(t)), \; k \text{ UFD}.$$

Nevertheless, $H_i(t)$ can be computed as follows:

For all $t \geq \delta+1$ set

$$Dod(n;t) = \{ x^\alpha \text{ such that } \exists\, i \neq j : \alpha_i \geq d_i \text{ and } \alpha_j \geq d_j \}$$

and let $\Delta(\underline{f};t)$ the submatrix of $\mathbb{M}(\underline{f};t)$ indexed by $Dod(n;t)$.

THM (Macaulay): For all $t \geq \delta+1$ we have

$$\det(\mathbb{M}(\underline{f};t)) = Res(f_1, \ldots, f_n) \det(\Delta(\underline{f};t)).$$

(see also e.g. Cox-Little-O'Shea). → It remains true after any specialization. □

Example: Let us see the formula in the case $f_1, f_2, f_3(x_1, x_2, x_3)$ and $d_1 = d_2 = 1$, $d_3 = 2$.

We expect $\deg_{f_1} \mathrm{Res} = 2$, $\deg_{f_2} \mathrm{Res} = 2$, $\deg_{f_3} = 1$.

$S = \sum_{i=1}^{3} (d_i - 1) + 1 = 2$.

$$\mathbb{M} := \begin{array}{c} \begin{array}{cccccc} x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2 \end{array} \\ \left(\begin{array}{ccc|cc|c} a_1 & 0 & 0 & 0 & 0 & c_1 \\ a_2 & a_1 & 0 & b_1 & 0 & c_2 \\ a_3 & 0 & a_1 & 0 & b_1 & c_3 \\ 0 & a_2 & 0 & b_2 & b_2 & c_4 \\ 0 & a_3 & a_2 & b_3 & 0 & c_5 \\ 0 & 0 & a_3 & 0 & b_3 & c_6 \end{array}\right) \\ \begin{array}{cccccc} x_1f_1 & x_2f_1 & x_3f_1 & x_2f_2 & x_3f_2 & f_3 \end{array} \end{array} \begin{array}{c} x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_2^2 \\ x_2x_3 \\ x_3^2 \end{array}$$

$f_1 = a_1x_1 + a_2x_2 + a_3x_3 \qquad f_2 = b_1x_1 + b_2x_2 + b_3x_3$

$f_3 = c_1x_1^2 + c_2x_1x_2 + \dots + c_6x_3^2$

Looking at the degrees, we know that one column of f_1 is superfluous.

The only "dodo" monomial is x_1x_2, so the Macaulay formula gives the equality

$$\det(\mathbb{M}) = a_1 \cdot \mathrm{Res}(f_1, f_2, f_3). \qquad \square.$$

Proof of the Macaulay formula:

The proof is by induction on $n \geq 1$. We also assume that we are in the generic setting over $\mathbb{Z}$.

- $n = 1$: $\mathrm{Res}(ux_1^{d_1}) = u$, $\mathbb{M} = (u)$, $\Delta = \emptyset$ so the formula is ok. ($n = 2$ can also be done, but more complicated).

- Assume that the formula holds for $n-1$.

We have $\det(\mathbb{M}) = \mathrm{Res}(f_1,\dots,f_n)\cdot H$ with H indep of coeffs f_n.

We specialize $f_n \mapsto x_n^{d_n}$:

$$\det(\mathbb{M}(f_1\dots f_{n-1}, x_n^{d_n}; t)) = \mathrm{Res}(f_1,\dots,f_{n-1}, x_n^{d_n})\cdot H$$
$$= \mathrm{Res}(\bar{f}_1,\dots,\bar{f}_{n-1})^{d_n}\cdot H \quad \text{(see properties).}$$

We write $\mathbb{M}(f_1,\dots,f_{n-1}, x_n^{d_n}; t)$ by choosing the following order for the basis:

$$\mathrm{Mon}(n;t) = \underbrace{\{x^\alpha \mid |\alpha|=t, \alpha_n=0\}}_{\mathrm{Mon}_0} \cup \underbrace{\{x^\alpha \mid |\alpha|=t, \alpha_n=1\}}_{\mathrm{Mon}_1} \cup \dots \cup \underbrace{\{x^\alpha \mid |\alpha|=t, \alpha_n=d_n-1\}}_{\mathrm{Mon}_{d_n-1}} \cup$$

$$\underbrace{\{x^\alpha \mid |\alpha|=t, \alpha_n\geq d_n, i(\alpha)<n\}}_{W_1} \cup \underbrace{\{x^\alpha \mid |\alpha|=t, \alpha_n\geq d_n, i(\alpha)=n\}}_{W_2}$$

$$\begin{array}{cc}
\begin{array}{cccccc}
\mathrm{Mon}_0 & \mathrm{Mon}_1 & \cdots & \mathrm{Mon}_{d_n-1} & W_1 & W_2
\end{array} & \\
\left(\begin{array}{cccccc}
K_0 & 0 & 0 & 0 & 0 & 0 \\
 & K_1 & 0 & 0 & 0 & 0 \\
 & & \ddots & 0 & 0 & 0 \\
 & * & & K_{d_n-1} & 0 & 0 \\
 & & & & L_1 & 0 \\
 & * & & & * & Id
\end{array}\right) &
\begin{array}{c}
\mathrm{Mon}_0 \\ \mathrm{Mon}_1 \\ \\ \mathrm{Mon}_{d_n-1} \\ W_1 \\ W_2
\end{array}
\end{array}$$

because $\dfrac{x^\alpha}{x_{i(\alpha)}^{d_{i(\alpha)}}} f_{i(\alpha)} = \dfrac{x^\alpha}{x_{i(\alpha)}^{d_{i(\alpha)}}}\left[\bar{f}_{i(\alpha)}(x_1,\dots,x_{n-1}) + \text{"terms of deg} \geq 1 \text{ in } x_n\text{"}\right]$

Thus, the block K_ℓ identifies to $\mathbb{M}(\overline{f}_1, \ldots, \overline{f}_{n-1}; t-\ell)$

Notice that $t-\ell \geq d_1 + \cdots + d_{n-1} - (n-1) + 1 = d_1 + \cdots + d_{n-1} - n + 2$

since $t \geq d_1 + \cdots + d_n - n + 1$ and $\ell \leq d_n - 1$.

We obtain: $\det\left(\mathbb{M}(f_1, \ldots, f_{n-1}, x_n^{d_n}; t)\right) = \det(L_1) \prod_{\ell=0}^{d_n-1} \det\left(\mathbb{M}(\overline{f}_1, \ldots, \overline{f}_{n-1}; t-\ell)\right)$

and hence, applying our inductive hypothesis,

$$\det\left(\mathbb{M}(f_1, \ldots, f_{n-1}, x_n^{d_n}; t)\right) = \det(L_1)\, \mathrm{Res}(\overline{f}_i)^{d_n} \prod_{\ell=0}^{d_n-1} \det\left(\Delta(\overline{f}_1, \ldots, \overline{f}_{n-1}; t-\ell)\right)$$

so that $\quad H = \det(L_1) \prod_{\ell=0}^{d_n-1} \det\left(\Delta(\overline{f}_1, \ldots, \overline{f}_{n-1}; t-\ell)\right) \quad (*)$

Now, to conclude we have to compare $(*)$ and $\det(\Delta(f_1, \ldots, f_n; t-\ell))$.

We have $\mathrm{Dod}(n;t) \cap \{x^\alpha \mid |\alpha| = t \text{ and } i(\alpha) = n\} = \emptyset$ so

$$\Delta(f_1, \ldots, f_n; t) = \Delta(f_1, \ldots, f_{n-1}, x_n^{d_n}; t).$$

Setting $\mathrm{Dod}_\ell = \mathrm{Dod}(n;t) \cap \mathrm{Mon}_\ell$ for $\ell = 0, \ldots, d_n - 1$ and observing that $\mathrm{Dod} \cap \mathrm{Mon}_{\geq d_n} = W_1$, we have the following decomposition:

$$\begin{array}{c|cccc} & \mathrm{Dod}_0 & \cdots & \mathrm{Dod}_{d_n-1} & W_1 \\ \hline \mathrm{Dod}_0 & K'_0 & 0 & 0 & 0 \\ \vdots & * & \ddots & 0 & 0 \\ \mathrm{Dod}_{d_n-1} & * & & K'_{d_n-1} & 0 \\ W_1 & * & * & & L_1 \end{array}$$

where $K'_\ell \leftrightarrow \Delta(\overline{f}_1, \ldots, \overline{f}_{n-1}; t-\ell)$

So we get

$$\det\left(\Delta(f_1, \ldots, f_{n-1}, x_n^{d_n}; t)\right) = \det(L_1) \prod_{\ell=0}^{d_n-1} \det\left(\Delta(\overline{f}_1, \ldots, \overline{f}_{n-1}; t-\ell)\right) \overset{\text{by } (*)}{=} H$$

$\square$.

9) Application to parameterized surfaces

Suppose given a parameterization of a surface $\mathcal{S}$ in $\mathbb{P}^3$:

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$

$$(s:t:u) \mapsto (f_1(s,t,u) : f_2 : f_3 : f_4).$$

f_i are homogeneous polynomials of degree $d \geq 1$

$\in \mathbb{K}[s,t,u]_d$ $\quad$ $\mathbb{K}$ alg. closed field.

We assume that $\overline{\mathrm{Im}(\phi)} = \mathcal{S} \subset \mathbb{P}^3$.

Rk: We notice that $\mathcal{S}$ is irreducible: it corresponds to principal ideal (H) which is the kernel of the map

$$\mathbb{K}[x,y,z,w] \to \mathbb{K}[s,t,u]$$
$$x \mapsto f_1$$
$$y \mapsto f_2$$
$$z \mapsto f_3$$
$$w \mapsto f_4$$

$\deg(\mathcal{S}) = \deg(H)$

intersection points with a line. $\quad$ degree of a polynomial

Lemma: If $\phi : \mathbb{P}^2 \to \mathbb{P}^3$ is regular (i.e. $V(f_1, \dots, f_4) = \emptyset$ in $\mathbb{P}^2$) and $\phi : \mathbb{P}^2 \to \mathcal{S}$ generically injective, then $\deg \mathcal{S} = d^2$.

Proof: A ~~point~~ line in $\mathbb{P}^2$ is determined by a system of equations

$$\begin{cases} a_1 x + b_1 y + c_1 z + d_1 w = 0 \\ a_2 x + b_2 y + c_2 z + d_2 w = 0 \end{cases}$$

Its intersection with $\mathcal{S}$ gives the system $\begin{cases} a_1 f_1 + b_1 f_2 + c_1 f_3 + d_1 f_4 = 0 \\ a_2 f_1 + b_2 f_2 + \dots + d_2 f_4 = 0 \end{cases}$

By Bézout et by our assumption this is d^2 points. $\square$

Rk: More generally: $\quad \deg(\phi) \cdot \deg(\mathcal{S}) = d^2 - \sum_{p \in V(f_1, \dots, f_4)} e_p \quad$ e_p: mult. of I.

Proposition: Under the assumptions of the previous lemma, we have

$$\mathrm{Res}(f_1 - x f_4, f_2 - y f_4, f_3 - z f_4) = H(x,y,z,1)$$

Rk: - If $V(f_i - f_4) \neq \phi$ then $\mathrm{Res}(-,-,-) \equiv 0$.
- If $\deg(\phi) > 1$ then $\mathrm{Res}(-,-,-) = H^{\deg(\phi)}$.

Proof (sketch of): As for curves, it is not hard to see that $H(x,y,z,1)$ vanishes if and only if $\mathrm{Res}(-,-,-)$ vanishes (since $V(f_i) = \phi$).
Since H is irreducible we have

$$\mathrm{Res}(f_1 - x f_4, f_2 - y f_4, f_3 - z f_4) = H^{\alpha} \quad \alpha \in \mathbb{N}.$$

We know that $\deg(H) = d^2$, so it suffices to show that $\deg \mathrm{Res}(-,-,-) \leq d^2$.

Now, by change of coord. one can assume that $H(x,y,z,w)$ does not go through the point $(1:0:0:0)$; this implies

$$H(x,y,z,1) = H_{d^2}(x,y,z) + H_{d^2-1}(x,y,z) + \cdots \quad (\text{hom. components}).$$

where $H_{d^2}(1,0,0) \neq 0$, i.e. $H_{d^2} = c.x^{d^2} + \cdots \quad c \neq 0$.

Consequently, $H(x,0,0,1) = c.x^{d^2} +$ terms of $\deg < d^2 - 1$.

But $\mathrm{Res}(f_1 - x f_4, f_2, f_3) = H(x,0,0,1)^{\alpha}$ and

$\deg_x \mathrm{Res}(f_1 - x f_4, f_2, f_3) \leq d^2$. Hence $\alpha = 1$ $\square$

(Rk: if one admits that $\deg(\phi) . \deg(H) = d^2$, then this proof shows that $\alpha = \deg(\phi)$.)

Remark: Computing the implicit equation of a param. surface eases the resolution of intersection problems:

- given $P \in \mathbb{P}^3$: does P belongs to $\mathcal{S}$?
- given a rational curve $t \mapsto \alpha(t)$, what are the intersection points with $\mathcal{S}$?

Exercice: A more compact formula than the Macaulay formula. (see exercise sheet).

10) Applications to geometry.

- **Poisson's formula** $\quad k$ an alg. closed field.

Suppose given
$$\begin{cases} f_i \in k[x_1, \dots, x_n]_{d_i \geq 1} & i = 1, \dots, n-1 \\ f, g \in k[x_1, \dots, x_n]_{d \geq 1} & \end{cases}$$

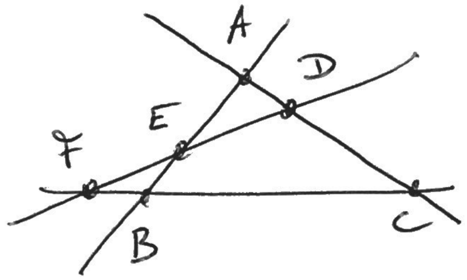
such that $V(f_1, \dots, f_{n-1}, g) = \emptyset$ $\left(\Rightarrow \operatorname{Res}(f_1, \dots, f_{n-1}, g) \neq 0\right)$,

then
$$\frac{\operatorname{Res}(f_1, \dots, f_{n-1}, f)}{\operatorname{Res}(f_1, \dots, f_{n-1}, g)} = \prod_{\xi \in V(f_1, \dots, f_{n-1})} \left(\frac{f}{g}(\xi)\right)^{\mu_\xi}$$

Historically, one of the first definition of the resultant.
($g = x_n$ provides a definition by iteration).

- An illustrative example in plane geometry

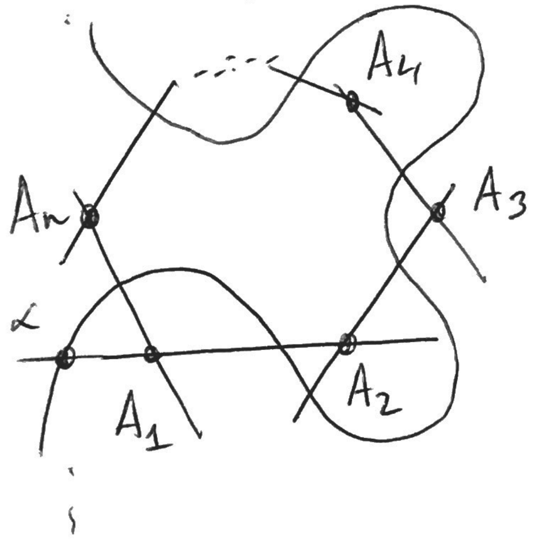
- Ménélaüs theorem



THM: $\frac{\overline{DA}}{\overline{DC}} \times \frac{\overline{FC}}{\overline{FB}} \times \frac{\overline{EB}}{\overline{EA}} = 1.$

- An algebraic generalization:

THM: Suppose given a closed polygone $(A_i)_{i=1,\dots,n}$ and a plane alg. curve $\mathcal{C}$ of degree d (of equation $f=0$).



THM:

$$\prod_{i=1}^{n} \left(\prod_{\alpha \in (A_i, A_{i+1}) \cap \mathcal{C}} \frac{\overline{\alpha A_i}}{\overline{\alpha A_{i+1}}} \right) = 1.$$

$\dashv$.