C6 – Implicitization of plane rational curves

* Given a parameterization $\mathbb{A}^1 \overset{\phi}{\dashrightarrow} \mathbb{A}^2$

$$t \longmapsto \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)} \right)$$

compute an equation of the closure of the image of ϕ, and determine, if possible, pre-images of a point on this image.

We turn this question to projective setting:

We suppose given a map

$$\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^2 \qquad (\text{over } \mathbb{C})$$

$$(s:t) \longmapsto (f_0(s,t) : f_1(s,t), f_2(s,t))$$

where $f_i(s,t)$ are homogeneous polynomials of degree $d \geq 1$.

Without loss of generality, we assume that $\gcd(f_0, f_1, f_2) = 1$. (otherwise, clear denominators).

We are looking for an equation $F(x_0, x_1, x_2) = 0$ that defines the image of ϕ, assuming this is a curve (and not a point) that we denote by $\mathcal{C}$.

Fact: $d = \deg(F) \cdot \deg(\phi)$ where $\deg(\phi)$ is the number of pre-images of a general point on $\mathcal{C}$.

$d = \deg(f_i)$

This can be seen by pulling back a line in $\mathbb{P}^2$.

In particular, if ϕ is generically injective, then $\deg(\mathcal{C}) = \deg(F) = d$.

1) First computation by means of Sylvester resultant:

Prop: $$\mathrm{Res}(x_0 f_1 - x_1 f_0, x_0 f_2 - x_2 f_0) = x_0^d F(x_0, x_1, x_2)^{\deg(\phi)}.$$

(observe that $\mathrm{Res}(-,-)$ is of degree $2d$ by construction!).

Proof: We only show the vanishing property, i.e. $V(F.x_0) = V(Re)$.

Consider the projections

$$\begin{tikzcd}
\mathbb{P}^1\times\mathbb{P}^2 \arrow[rr, "\pi_2"] & & \mathbb{P}^2 \\
V(x_0f_1-x_1f_0,\, x_0f_2-x_2f_0) \arrow[u, phantom, "\cup"] \arrow[rr] \arrow[d, "\pi_1"'] & & V(Res(-,-)) \\
\mathbb{P}^1 & &
\end{tikzcd}$$

~~Let $(x_0:x_1:x_2) \in V(Res(-,-))$~~. It is clear that any point such that $x_0=0$ belongs to $V(Re(-,-))$ because $f_0(s,t)$ always has roots (actually $\underline{d}$ roots).

It is also clear that any point $(x_0:x_1:x_2)$ in the image of ϕ, assuming $x_0\neq 0$, is also in $V(Res(-,-))$ because then $f_0(s,t)\neq 0$ (because otherwise $f_0=f_1=f_2(s,t)=0$) and hence $\frac{x_1}{x_0}=\frac{f_1}{f_0}$ and $\frac{x_2}{x_0}=\frac{f_2}{f_0}$.

We conclude that $V(x_0F(x_0,x_1,x_2)) \subset V(Res(-,-))$

Conversely, if $Rs(-,-)$ vanishes, we have similar conclusion depending on the fact that $f_0(s,t)=0$ or not. $\square$

$\underline{Rk}$: It is important to notice that the fibers of π_1 are either a point on $\mathcal{C}$ (graph of $\mathcal{C}$) or the line $x_0=0$ (d times, at all roots of f_0).

In order to remove the extraneous factor "x_0^d" we would need to get exactly the graph of ϕ, i.e. the condition $(x_0:x_1:x_2)=(f_0:f_1:f_2)$.

This corresponds to the ideal $I_K = (x_0f_1-x_1f_0,\, x_0f_2-x_2f_0,\, x_1f_2-x_2f_1)$, i.e. $graph(\phi) = V(I_K)\subset \mathbb{P}^1\times\mathbb{P}^2$. When f_0 vanishes one still have two nonzero equations, hence the graph of ϕ.

How to eliminate (s,t) from these three equations?
Answer: use syzygies!

2) Graph & syzygies

- A syzygy of (f_0, f_1, f_2) is a triple (g_0, g_1, g_2) in $R = k[s,t]$ such that $\sum f_i g_i \equiv 0$. ($\deg g_i$ are the same as f_i's are homogeneous of degree d).

We identify a syzygy with a linear form
$$x_0 g_0(s,t) + x_1 g_1(s,t) + x_2 g_2(s,t).$$

<u>Rk</u>: This "moving line" in $\mathbb{P}^1 \times \mathbb{P}^2$ goes through the points $(s:t) \times \phi(s,t)$. $(*)$

- The generators of I_K are "obvious" syzygies. As already observed, they define the graph of ϕ.

$$\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^2 & \\
 & \mathrm{Graph}(\phi) = V(I_K) \arrow[u, phantom, "\cup"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, phantom, shift left=4ex, "\circlearrowleft"] & \\
 & \mathbb{P}^1 \arrow[r, "\phi"] & \mathcal{C} \subset \mathbb{P}^2
\end{tikzcd}$$

- Now, let I_S be the ideal generated by all syzygies in $R[x_0, x_1, x_2]$. Clearly, $I_K \subset I_S$, so $V(I_S) \subset V(I_K) = \mathrm{graph}(\phi)$. But actually any point in the graph is also in $V(I_S)$ by $(*)$, so this is an equality: $V(I_S) = \mathrm{Graph}(\phi)$.

<u>Rk</u>: One can be more precise: Since $\sum f_i g_i = 0$ we have
$$f_0(x_0 g_0 + x_1 g_1 + x_2 g_2) = g_1(x_1 f_0 - x_0 f_1) + g_2(x_2 f_0 - x_0 f_2)$$
(and so on).

Why is it better to use I_S ?

THM (Hilbert–Burch). Let $I = (f_0, f_1, f_2) \subset R = k[s,t]$. The ideal I has the following F.F.R.:

$$0 \to \bigoplus_{i=1}^{2} R(-d-\mu_i) \longrightarrow R(-d)^3 \longrightarrow R \longrightarrow R/I \to 0$$

where $\mu_1 + \mu_2 = d$, $\mu_i \geq 0$.

Proof: First by Hilbert Syz. THM the F.F.R is of the form

$$0 \to \bigoplus_{i=1}^{n} R(-a_i) \to R(-d)^3 \to R \to R/I \to 0. \qquad (*)$$

Now, $HP(R/I, t) = 0$ as the f_i's have no common roots ($\Leftrightarrow (x_0, x_1, x_2)^N \subset I$ for some N).

Applying the HP to the exact sequence $(*)$ we see immediately that $n = 2$ and moreover we have.

$$0 = HP(R, t) - HP(R(-d)^3, t) + HP(R(-a_1) \oplus R(-a_2), t)$$
$$= t+1 - 3(t-d+1) + (t-a_1+1) + (t-a_2+1)$$
$$= 1 + 3d - 3 - a_1 + 1 - a_2 + 1 = 3d - a_1 - a_2$$

i.e. $0 = 3d - (d+\mu_1) - (d+\mu_2)$ and hence $d = \mu_1 + \mu_2$. $\square$.

As a consequence, there exists two syzygies

$$\left.\begin{aligned} L_1 &= x_0 p_0 + x_1 p_1 + x_2 p_2 \quad \deg p_i = \mu_1 \\ L_2 &= x_0 q_0 + x_1 q_1 + x_2 q_2 \quad \deg q_i = \mu_2 \end{aligned}\right\} \bigoplus_{i=1}^{2} R(-d-\mu_i) \xrightarrow{\begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}} R(-d)^3$$

such that $I_S = (L_1, L_2)$. Only two generators!

Corollary: $\mathrm{Res}(L_1, L_2) = F(x_0, x_1, x_2)^{\deg(\phi)}$

Proof: $V(I_S)$ defines the graph so the vanishing of $\mathrm{Res}(L_1, L_2)$ is exactly the one of the curve. Then by comparing degrees we conclude.

Rks:
- $\mathrm{Sylv}(L_1, L_2)$ is a $d\times d$-matrix, hence $\mathrm{Res}(L_1, L_2)$ is of degree d by construction
- L_1 and L_2 are two moving lines that defines the curve $\mathcal{C}$ everywhere: compare with $x_0 f_1 - x_1 f_0$, $x_0 f_2 - x_2 f_0$.

Exemple: the circle $(s:t) \mapsto (s^2+t^2 : s^2-t^2 : 2st)$.

syzygies $\begin{pmatrix} -s & -t \\ s & -t \\ t & s \end{pmatrix}$ $\quad L_1 = (-s)x_0 + s x_1 + t x_2$

$L_2 = (-t)x_0 + (-t)x_1 + s x_2$

$\rightsquigarrow \mathrm{Sylv}(L_1, L_2) = \begin{array}{cc} L_1 & L_2 \\ \end{array}$ $\left(\begin{array}{cc} x_1 - x_0 & x_2 \\ x_2 & -x_0 - x_1 \end{array}\right)\begin{array}{c} s \\ t \end{array}$ $\quad \det = x_0^2 - x_1^2 - x_2^2$.

3) Pre-images and eigenvalues.

Question: given a point P on the curve $\mathcal{C}$, how can we find its corresponding parameters via ϕ?

Considering that the graph is defined by L_1 and L_2, it is clear that, given $P \in \mathbb{P}^2$

$$\gcd\left(L_1(s,t;P), L_2(s,t;P)\right) = \prod_{i=1}^{r_P} (\beta_i s - \alpha_i t)^{m_i}$$

where $\phi(\alpha_i : \beta_i) = P$

$\Sigma m_i = m_P(\mathcal{C}) \leftarrow$ (can be used as a definition).

It is possible to replace the above GCD computation by an eigenvalue problem. The gain is to allow "approximate computations".

The problem can be rephrased as follows: given two polynomials $f(x)$ and $g(x)$ of degree m and n in $\mathbb{C}[x]$, determine their common roots, including multiplicities.

<u>Fact</u>: $\operatorname{rank} \mathrm{Sylv}(f,g) = m+n - \deg \underbrace{\gcd(f,g)}_{\substack{\text{common roots with multiplicities.} \\ \prod_{i=1}^{r}(x-\alpha_i)^{m_i}}}$

(<u>exercise</u>)

(same proof as for the main properties of Sylvester resultant).

- Now, recall the construction of $\mathrm{Sylv}(f,g)$:

$$\begin{cases} f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0 \\ g(x) = b_n x^n + \cdots \qquad + b_0 \end{cases}$$

$$\mathrm{Sylv}(f,g)^T \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{m+n-1} \end{pmatrix} = \begin{pmatrix} f \\ xf \\ \vdots \\ x^{n-1} f \\ g \\ xg \\ \vdots \\ x^{m-1} g \end{pmatrix}$$

- Suppose that $\deg(\gcd(f,g)) = 1 \left(= \sum_{i=1}^{r} m_i\right)$. Let $\Delta = \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{m+n-1} \end{pmatrix}$ be a basis of the kernel of $\mathrm{Sylv}(f,g)^T$. Then the common root z of f and g satisfies $\delta_1 - z\delta_0 = 0$. Explain why!

- Suppose now that all $m_i = 1$ but we have r common roots $\alpha_1, \dots, \alpha_r$. As in the previous case, natural elements in the kernel of $Syl(f,g)^T$ are

$$V_{m+n-1}(\alpha_i) = \begin{pmatrix} 1 \\ \alpha_i \\ \vdots \\ \alpha_i^{m+n-1} \end{pmatrix}.$$

The matrix built from these vectors is actually ~~a~~ (close to a) Vandermonde matrix; ~~so~~ we have:

$$\det \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_r \\ | & & \vdots \\ \alpha_1^{r-1} & \cdots & \alpha_r^{r-1} \end{pmatrix} = \prod_{i<j} (\alpha_i - \alpha_j) \neq 0.$$

It follows that $V_{m+n-1}(\alpha_1), \dots, V_{m+n-1}(\alpha_r)$ span the kernel of $Syl(f,g)^T$ (they are linearly indep. and the dim of this kernel is r).

Define the square matrices Δ_0 and Δ_1 as follows:

$$\Delta_0 \quad \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ | & | & & | \\ \alpha_1^{r-1} & \alpha_2^{r-1} & \cdots & \alpha_r^{r-1} \\ \alpha_1^{r} & \alpha_2^{r} & \cdots & \alpha_r^{r} \\ | & | & & | \\ \alpha_1^{m+n-1} & \alpha_2^{m+n-1} & \cdots & \alpha_r^{m+n-1} \end{pmatrix} \quad \Delta_1$$

Proposition:

$$\det(\Delta_1 - t\Delta_0) = c \cdot \prod_{i=1}^{r} (t - \alpha_i)$$

where $c \neq 0 \in \mathbb{C}$.

Proof:

$$\prod(t-\alpha_i)\cdot\prod(\alpha_i-\alpha_j) = \det \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \alpha_1 & & \alpha_r & t \\ | & & | & \vdots \\ \alpha_1^{r} & & \alpha_r^{r} & t^{r} \\ \alpha_1^{r+1} & & \alpha_r^{r+1} & t^{r+1} \end{pmatrix} \underset{\substack{\uparrow \\ \text{from bottom} \\ \text{to top:} \\ \times(-t)\text{ up row.}}}{=} \det \begin{pmatrix} 1 & \text{—} & 1 & 1 \\ & & & 0 \\ & \Delta_1 - t\Delta_0 & & | \\ & & & 0 \end{pmatrix} = (-1)^{r+1} \det(\Delta_1 - t\Delta_0)$$

$\square$.

- The general case, i.e. common roots $\alpha_1, \dots, \alpha_r$ with multiplicity $m_1, \dots, m_r$ can be treated similarly as follows.

Define
$$V_d(\alpha; k) := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha & 1 & \ddots & \vdots \\ \alpha^2 & 2\alpha & & 0 \\ \vdots & \vdots & & \frac{(d-1)!}{(d-k)!}\alpha^{d-k} \\ \alpha^d & d\alpha^{d-1} & & \frac{d!}{(d-k+1)!}\alpha^{d-k+1} \end{pmatrix}$$
"derivative by column, k-1 times."

One can check that if α is a root of multiplicity m of f, g then the column of $V_{m+n-1}(\alpha; m)$ are all in the kernel of $\mathrm{Syl}(f,g)^T$:

$$\left(1\ \alpha\ \alpha^2 \ - \ \alpha^{m+n-1}\right) \cdot \mathrm{Syl}(f,g) = \left(f\ \ \alpha f \cdots \alpha^{n-1} f\ \ g\ \ \alpha g \cdots \alpha^{n-1} g\right)$$
$$\left(0\ 1\ 2\alpha \cdots (m+n-1)\alpha^{m+n-2}\right) \cdot \mathrm{Syl}(f,g) = \left(f'\ (\alpha f)' \cdots (\alpha^{n-1} f)'\ \ g' \cdots (\alpha^{n-1} g)'\right)$$
$$\vdots$$
etc

vanishes by assumption.

Finally, our candidate for the kernel of $\mathrm{Syl}(f,g)^T$ is the matrix

$$\left(V_{m+n-1}(\alpha_1; m_1) \,\middle|\, \cdots \,\middle|\, V_{m+n-1}(\alpha_r; m_r) \right)$$

Why is this matrix of rank $\sum_{i=1}^{r} m_i$? Because the determinant of its top ~~bloc~~ square bloc is a generalized Vandermonde matrix and its determinant is known to be equal to $\prod_{i<j} (\alpha_i - \alpha_j)^{m_i m_j}$

Prop: Define Δ_0 and Δ_1 as previously (to square bloc and top square bloc shifted by one row down), then
$$\det(\Delta_1 - t\Delta_0) = c \cdot \prod_{i=1}^{r} (t - \alpha_i)^{m_i}. \qquad c \neq 0,\ c \in \mathbb{C}.$$
$\square$

- To conclude, let us illustrate on an example

Ex: $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$

$$(s:t) \longmapsto (\underbrace{t^3}_{f_0} : \underbrace{st(s-t)}_{f_1} : \underbrace{s(s-t)(2s-5t)}_{f_2})$$

- The point $(1:0:0)$ has two pre-images: $(0:1)$ and $(1:0)$. We aim to recover them.

- Syzygies: computation give $\mu_1 = 1$ $\mu_2 = 2$ and

$$Sylv(L_1, L_2) = \begin{array}{cc} \begin{pmatrix} 2x_1 & 0 & x_0 \\ -5x_1 - x_2 & 2x_1 & -x_0 \\ 0 & -5x_1 - x_2 & -x_1 \end{pmatrix} & \begin{array}{c} s^2 \\ st \\ t^2 \end{array} \\ \begin{array}{ccc} sL_1 & tL_1 & L_2 \end{array} & \end{array}$$

(USE M2)

Evaluate at $P = (1:0:0)$ $\rightsquigarrow$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{c} s^2 \\ st \\ t^2 \end{array}$ rank 1, so 2 roots.

So a basis of the kernel of the transpose is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{array}{c} t^2 \\ st \\ s^2 \end{array}$

As $t = 1$, we choose the basis $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{array}{c} 1 \\ s \\ s^2 \end{array}$

So $\Delta_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Delta_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

Now $(\Delta_1 - s\Delta_0) = \begin{pmatrix} 1 & -s \\ 1-s & 0 \end{pmatrix}$ and $\det(\Delta_1 - s\Delta_0) = s(1-s)$ as expected.

□.