Exercises - Regular Sequences

M2 MPA - Computational Algebraic Geometry

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A sequence $\{a_1, \dots, a_s\}$ of elements in a commutative ring R is called a *regular sequence* if

i) $(a_1, \dots, a_s) \neq R$

ii) a_i is a nonzerodivisor in $R/(a_1, \dots, a_{i-1})$ for all $i = 1, \dots, s$.

An ideal generated by a regular sequence in called a *complete intersection ideal*.

Exercise 1 Show that the property of being a regular sequence depends on the order of the elements. For that purpose, one can consider the ring $R = k[x, y, z]$ and the two sequences $(x, y(1-x), z(1-x))$ and $(y(1-x), z(1-x), x)$.

Exercise 2 Let $\{a_1, \dots, a_s\}$ be a regular sequence in R. Show that the sequence obtained by permutation of a_i and a_{i+1} is regular if and only if a_{i+1} is not a zerodivisor in $R/(a_1, \dots, a_{i-1})$.

Exercise 3 Graded Nakayama Lemma
Let $R = \oplus R_i$ a graded ring and M a graded R-module such that $M_i = 0$ for i sufficiently negative. Set $R_+ := \oplus_{i>0} R_i$. Show that if $R_+ M = M$ then $M = 0$.

Exercise 4 Let $\{a_1, \dots, a_s\}$ be a regular sequence of homogeneous elements in a graded ring R. Then, show that this sequence remains regular after any permutation of its elements (hint: use previous exercises).

Exercise 5 Let $\{a_1, \dots, a_s$ be a regular sequence in a commutative ring R. Show that for all

$$g \in Syz(a_1, \dots, a_s) := \{(h_1, \dots, h_s) \in R^s \text{ such that } \sum h_i a_i = 0\}$$

there exists a skew-symmetric matrix M with coefficients in R such that

$$g = M \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix}$$