C3 – Free resolutions and regular sequences

Goal: "use free modules to represent any finitely generated mod"

- free modules are the nicest possible models
- We have a nice formula for the Hilbert series.

1) Projective modules.

Def: P a R-mod.
P is projective if for any surjective R-mod map $A \xrightarrow{f} B \to 0$ and any $P \xrightarrow{g} B$, $\exists\, h : P \to A$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ A \arrow[r, "f"'] & B \arrow[r] & 0 \end{tikzcd}$$

commutes. (i.e. $g = f \circ h$)

Lemma: TFAE

i) P is projective

ii) Every exact sequence
$$0 \to N \to M \xrightarrow{f} P \to 0$$
splits (i.e. $\exists\, h : P \to M$ such that $f \circ h = Id$)

iii) $\exists\, K$ such that $P \oplus K \simeq F$ for some free module F.

Proof: i) $\Rightarrow$ ii) by definition: $\exists h$

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "Id"] \\ M \arrow[r, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

ii) $\Rightarrow$ iii) Classical property (exercice).

$0 \to N \to M \to P \to 0$ splits $\Leftrightarrow$ $M \simeq P \oplus N$

Now, consider the exact sequence

$0 \to Ker(\pi) \to \bigoplus_{p \in P} R \xrightarrow{\pi} P \to 0$ then $\bigoplus_{p\in P} R \simeq P \oplus Ker(\pi)$.

iii) $\Rightarrow$ i) Let F such that $P \oplus K \simeq F$.

Suppose given

$$\begin{tikzcd} & P \arrow[d, "g"] \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

One has

$$\begin{tikzcd} & b_i \quad F = P \oplus K \simeq \bigoplus R b_i \arrow[d] \arrow[ddl, "h"'] \\ & P \arrow[d, "g"] \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

$m_i \longrightarrow g(b_i)$

So $\exists\, h_{|P} : P \to A$ commutes.

$\square$

* Projective modules are very interesting modules because they allow to get commutative diagrams.

* Even better: * Over a local ring, projective $\Rightarrow$ free

exercice 4.11 | * finitely generated graded proj. module over $k[x_1, \dots, x_n]$ is a graded free module.

Eisenbud's book [Commutative algebra]

2) Free resolutions

- R denotes a polynomial ring over a field.

__Example:__ $R = k[x,y,z]$ and $I = (x^3+y^3+z^3)$
We have seen that

$$HF(R/I, i) = HF(R,i) - HF(R(-3), i)$$

We have actually a graded exact sequence:

$$0 \to R(-3) \xrightarrow{\cdot(x^3+y^3+z^3)} R \to R/I \to 0 .$$

What happens when we sliced with the ideal (x).

$J = I + (x)$:

$$0 \to R(-4) \xrightarrow{\begin{pmatrix} x^3+y^3+z^3 \\ -x \end{pmatrix}} R(-1) \oplus R(-3) \xrightarrow{(x,\ x^3+y^3+z^3)} R \to R/J \to 0$$

check exactness here.

As a consequence:

$$HP(R/J, i) = HP(R,i) - HP(R(-1), i) - HP(R(-3), i) + HP(R(-4), i)$$

$$= \binom{i+2}{2} - \binom{i+1}{2} - \binom{i-1}{2} + \binom{i-2}{2}$$

$$= \frac{(i+2)(i+1)}{2} - \frac{(i+1)i}{2} - \frac{(i-1)(i-2)}{2} + \frac{(i-2)(i-3)}{2} = 3 .$$

→ __Exercise:__ Bézout THM. f, g of degree d, e, no common factors.

$$0 \to R(-d-e) \to R(-d) \oplus R(-e) \to R \to R/(f,g) \to 0$$

$$HP(R/(f,g), i) = d.e .$$

$\square$.

THM (Hilbert Syz. THM): M a finitely generated graded module over $R = k[x_1, \dots, x_n]$, then there exists a graded exact sequence of modules

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

where the F_i's are finitely generated and free.

Proof: postponed to next lectures (homol. algebra).

$\rightsquigarrow$ Macaulay 2

Betti tables.

	F_0	F_1	F_2	
total:	1	2	1	← ranks
0 :	1	1	.	
1 :	.	.	.	
2 :	.	1	1	

$$R \leftarrow \begin{matrix} R(-1) \\ \oplus \\ R(-3) \end{matrix} \leftarrow R(-4)$$

Remark: The existence of free resolutions is easy:

- M finitely generated $\Rightarrow$ $R^{k_0} \xrightarrow{\phi_0} M \to 0$ exists.
- $\ker \phi_0$ is finitely generated because R is noetherian
- So $R^{k_1} \xrightarrow{\phi_1} R^{k_0} \xrightarrow{\phi_0} M \to 0$ exists, and so on.

Key point of Hilbert Syz THM is that M has a finite free resolution over a polynomial ring. (of length $\leq n$).

Let $T = \mathbb{R}[x]/(x^2)$, then a free resolution of $(x) \subset T$ is

$$\cdots \to T(-2) \xrightarrow{\cdot(x)} T(-1) \xrightarrow{\cdot(x)} (x) \to 0$$

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• <u>Minimal</u> free <u>resolutions</u>

A free resolution is called minimal if there are no constant terms in any of the maps (all entries belongs to $\bigoplus_{i \geq 1} R_i$).

This is because if a constant term appear, then the maps can be simplified:

$$0 \to R(-3) \xrightarrow{\begin{bmatrix} y \\ -1 \end{bmatrix}} R(-2) \oplus R(-3) \xrightarrow{(x^2, yx^2)} I \to 0$$

$$\rightsquigarrow \quad 0 \to R(-2) \xrightarrow{(x^2)} I \to 0.$$

Other example: $0 \to R \xrightarrow{1} R \to 0$ can be added to any other exact sequence:

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots$$

$$F_{i+1} \to F_i \oplus R \xrightarrow{\begin{pmatrix} d_i & 0 \\ 0 & 1 \end{pmatrix}} F_{i-1} \oplus R \to F_{i-2} \to \cdots$$

• Matrices in a free resolutions are not unique (as choice of generators for ideals) but the free modules that appear in a minimal free resolution <u>are</u> (Eisenbud, THM 20.2).

- Hilbert Series.

Let $R = k[x_1, \ldots, x_n]$ and M with a F.F.R.

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F_k \simeq \bigoplus_{i=1}^{r_k} R(-a_{k,i})$ $\quad r_k = \operatorname{rank} F_k$.

- The fact that the Hilbert function of M become a polynomial in large degrees is because this is obviously true for free modules.

Actually: $HP(M,i) = \sum_{j=0}^{n} (-1)^j HP(F_j, i)$

$$= \sum_{j=0}^{n} (-1)^j \sum_{i=1}^{r_j} \binom{n-1+i-a_{ji}}{n-1}.$$

- $HS(R,t) = 1/(1-t)^n$ so $HS(R(-a),t) = t^a/(1-t)^n$

and hence $HS(F_k, t) = \dfrac{\sum_{i=1}^{r_k} t^{a_{ki}}}{(1-t)^n}$

We recover that $HS(M,t) = P(M,t)/(1-t)^n$ with $P(M,t) \in \mathbb{Z}[t,t^{-1}]$.

3) Regular sequences

Goal: "understand what happens when slicing with a hyperplane or hypersurface"

Lemma: $I \subseteq R$ be a homogeneous ideal and $f \in R_d$. Then, we have a graded exact sequence

$$0 \to R(-d)/(I:f) \to R/I \to R/I+(f) \to 0.$$

Proof:

$$0 \to (I,f)/I \to R/I \to R/(I,f) \to 0 \text{ is}$$

clearly exact. Now, the multiplication by f

$$R(-d) \xrightarrow{\times f} (I,f)/I \to 0$$

has kernel equals to $(I:f)$, so $\dfrac{R(-d)}{(I:f)} \simeq \left(I,f/I\right)$.

$\llcorner$

- Let $f \in R$.

f is a nonzerodivisor on M if $f.m \neq 0 \ \forall m \neq 0 \in M$.

$\Rightarrow$ f is a nonzerodivisor on $R/I \iff (I:f) = I$.

Therefore, suppose that

$$HP(R/I, i) = \frac{a_m}{m!} i^m + \dots$$

If f is a homogeneous linear form which is not a zero divisor on R/I, then by the above lemma we deduce:

$$HP\left(R/(I,f), i\right) = HP(R/I, i) - HP(R/I, i-1)$$
$$= \frac{a_m}{(m-1)!} i^{m-1} + \dots$$

$\Rightarrow$ dimension drops by one and degree unchanged, as claimed in the previous lecture.

[$\Rightarrow$ By repeating this process, we arrive at a constant HP].

- The previous argument shows what we expected but the problem is about the existence of nonzerodivisor.

Indeed, f is a nonzerodivisor $\Leftrightarrow$ $(I:f) = I$.

Let $I = \bigcap_{i=1}^{r} Q_i$ a primary decomposition.

We have seen that if $f \notin P_i$ then $(Q_i : f) = Q_i$

So since $(I:f) = \bigcap (Q_i : f)$ we might have an issue if I has a m-primary component, $m = (x_1, \dots, x_n)$.

(e.g. $(m^2 : \ell) = m$ with ℓ a linear form)

To overcome this situation, one can proceed as follows.

- If I has a m-primary component, define I' to be the same ideal but remove this component:

$$I = \bigcap_{i=1}^{r} Q_i \text{ and } Q_r \text{ } m\text{-primary then } I' = \bigcap_{i=1}^{r-1} Q_i .$$

Now, $HP(R/_I, i) = HP(R/_{I'}, i)$ because we have an exact sequence $0 \to R/_{I \cap J} \to R/_I \oplus R/_J \to R/_{I+J} \to 0$ for any ideals I, J, and because $HP(R/_{Q_r}, i) = 0$.

(the component Q_r, which is such that $\sqrt{Q_r} = m$, is geometrically irrelevant).

- Now, we claim that there exists ~~...~~ a linear form $f \in R_1$ such that $f \notin \bigcup_{i=1}^{r-1} P_i$, where $P_i = \sqrt{Q_i}$.

This implies that f is a nonzero divisor of $R/_{I'}$ and we are done.

The existence of this linear form is ~~a~~ consequence of the following lemma.

<u>Lemma</u> (prime avoidance): If $I \subseteq \bigcup_{i=1}^{n} P_i$, with P_i prime, then $I \subseteq P_i$ for some I.
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Indeed, $m \not\subseteq \bigcup_{i=1}^{r-1} P_i$ (union of associated primes of I')

So there must be a linear form $f \in m_1$: $f \notin \bigcup_{i=1}^{r-1} P_i$.

Proof of lemma: We prove that $I \not\subseteq P_i \ \forall i \Rightarrow I \not\subseteq \bigcup_{i=1}^{n} P_i$.

We proceed by induction on n; the case $n=1$ is trivial.

Now, suppose that $I \not\subseteq P_i \ \forall i$ and $I \subseteq \bigcup_{i=1}^{n} P_i$.

By inductive assumption, $I \not\subseteq \bigcup_{j \neq i} P_j$, this for all i.

This mean that for any i $\exists\, x_i$ such that

$$x_i \in I \quad \text{but} \quad x_i \notin \bigcup_{j \neq i} P_j$$

In addition, one may assume $x_i \in P_i$ otherwise we have a contradiction.

So for any i $\exists\, x_i \in I \cap P_i$ such that $x_i \notin \bigcup_{j \neq i} P_j$.

Now, let $$x = \sum_{i=1}^{n} x_1 \dots \hat{x}_i \dots x_n$$

$x \in I$ by construction.

Pick k: $x_1 \dots \hat{x}_k \dots x_n \notin P_k$ because $x_j \notin \bigcup_{i \neq j} P_i \supset P_k$ $\quad j \neq k$

and P_k is prime

It follows that $x \notin P_k$ since all the other monomials in x belong to P_k.

In conclusion, for any k, $x \notin P_k$ so $x \notin \bigcup_{k=1}^{n} P_k$, a contradiction.

$\square$

- Reguler sequences.

In the previous paragraph we sliced I by a hypersurface defined by $f=0$. In this section, we start from scratch, i.e from a hypersurface and then we iterate.

- Let M be a graded R-module. A regular sequence on M is a sequence of homogeneous polynomials

$$\{f_1, \dots, f_m\}$$

such that
- f_1 is a nonzerodivisor on M
- f_i is a nonzerodivisor on $M / (f_1, \dots, f_{i-1}) \cdot M$ $\quad i \geq 1$

Example:
- $\{x, y\}$ is a reg. seq. in $R = k[x, y]$
- $\{x_1, \dots, x_n\}$ —— $R = k[x_1, \dots, x_n]$

- We one computes a free resolution for an ideal generated by a regular sequence, one should get only the trivial relations:

$I = (f_1, f_2, f_3)$

$$0 \to R \xrightarrow{\begin{pmatrix} f_3 \\ -f_2 \\ f_1 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -f_2 & -f_3 & 0 \\ f_1 & 0 & -f_3 \\ 0 & f_1 & f_2 \end{pmatrix}} R^3 \xrightarrow{(f_1\ f_2\ f_3)} R \to R/I \to 0$$

(guess of a resolution).

- Bézout THM can be stated by saying that the ideal generated by 2 polynomials without common factors is a complete intersection

(An ideal generated by a reg. seq is called a complete intersection).

(Exercise: write a F.F.R and compute the H.P. (already done before))

Exercice: see what happens with three polynomials in $\mathbb{P}^3$: compute the Hilbert polynomial of a reg. seq of 3 polynomials (ms M2.) using our guessed resolution.

This called the Koszul complex (we will come back to it later.)