Course 10 - Elimination matrices, Implicitization

1) Recap: resultant and the Koszul complex

Consider n generic homogeneous polynomials

$$f_i(x_1,\dots,x_n) = \sum_{|\alpha|=d_i} u_{i,\alpha} x^\alpha \qquad i=1\dots n$$

$$\mathfrak{m} = (x_1,\dots,x_n) \subset C$$

$A = \mathbb{Z}[u_{i,\alpha}]$ $\quad C = A[x_1,\dots,x_n]$ $\quad f_i \in C_{d_i}$, $\quad I = (f_1,\dots,f_n) \subset C$

- We have defined the resultant $\mathrm{Res}(f_1,\dots,f_n)$ as the generator of $\mathfrak{a} = (I : \mathfrak{m}^\infty) \cap A$ such that $\mathrm{Res}(x_1^{d_1},\dots,x_n^{d_n}) = 1$.

- Let $B := C/I$; it is a graded ring. We have seen that

$$\mathrm{ann}_A(B_\nu) = \mathfrak{a} = (\mathrm{Res}(f_1,\dots,f_n)) \quad \forall \nu \geq \left(\sum_{i=1}^n (d_i-1)\right)+1$$

- We have seen that $V_{\mathbb{A}^N}(\mathrm{ann}_A(B_\nu)) = V_{\mathbb{A}^N}(\mathcal{F}_0(B_\nu))$ $\forall \nu \geq 0$ where $\mathbb{A}^N$ is the affine space associated to A ($\mathbb{A}^N = \mathrm{Spec}(A)$).

Actually, the codimension one component of $\mathcal{F}_0(B_\nu)$, without embedded components is exactly $\mathfrak{a}$ for all $\nu \geq (\sum(d_i-1))+1$, which means, in term of equations,

$$\left(\text{gcd of generators of } \mathcal{F}_0(B_\nu)\right)$$

$$\shortparallel$$

$$[\mathcal{F}_0(B_\nu)] = \mathfrak{a} \quad \forall \nu \geq \left(\sum(d_i-1)\right)+1.$$

notation: "divisor associated to".

- The ideal $\mathcal{F}_0(B_\nu)$ is interesting because it can be "computed" from a presentation of B_ν:

we have
$$\bigoplus_{i=1}^{n} C(-d_i) \xrightarrow{(f_1 - f_n)} C \to B \to 0$$

so
$$\bigoplus_{i=1}^{n} C_{\nu - d_i} \xrightarrow{\text{"}(f_1,\dots,f_n)\text{"}} C_\nu \to B_\nu \to 0$$
$$\downarrow$$

$$\mathbb{M}_\nu := \left(\underbrace{x^\alpha \cdot f_1}_{\substack{x^\alpha \\ |\alpha| = \nu - d_1}} \quad \cdots \quad \underbrace{x^\alpha \cdot f_n}_{\substack{x^\alpha \\ |\alpha| = \nu - d_n}} \right) \Big\} \text{ monomials } x^\alpha \ |\alpha| = \nu$$

→ The gcd of maximal minors of $\mathbb{M}_\nu$ (of size $\#\{x^\alpha : |\alpha| = \nu\}$) is equal to $\mathrm{Res}(f_1, \dots, f_n)$

→ Even better: the specialization of $\mathbb{M}_\nu$ for a specific choice of polynomial is not full rank ⟺ the resultant vanishes. We say that $\mathbb{M}_\nu$ is an elimination matrix

"Very useful in applications" | polynomial evaluation ⟷ rank estimation
(Res is hard to compute) ($\mathbb{M}_\nu$ is easy to build).

<u>Remark:</u> The Macaulay determinants $D_i(\nu)$ are max minors of $\mathbb{M}_\nu$.

Example: $n=2$ $\quad \nu \geq d_1+d_2-1$

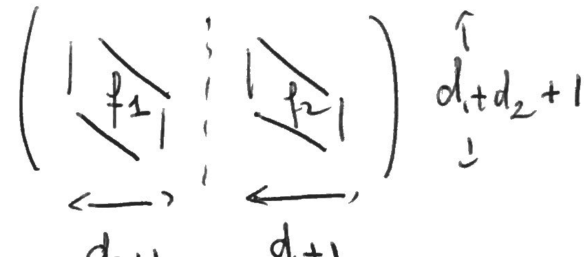
$$\mathbb{M}_{d_1+d_2-1} = \left(C_{d_2-1} \oplus C_{d_1-1} \longrightarrow C_{d_1+d_2-1} \right)$$

this is the Sylvester matrix (square!)

$$\mathbb{M}_{d_1+d_2} = \left(C_{d_2} \oplus C_{d_1} \longrightarrow C_{d_1+d_2} \right)$$

not square!

but the rank property still holds!



When n is arbitrary, in general $\mathbb{M}_\nu$ is not square, even for $\nu = \left(\sum (d_i-1)\right)+1$. The Macaulay formula can be used to compute it, but there is a more general tool that uses the Koszul complex.

2- Determinant of complexes.

- A be a domain, let $\phi : A^n \to A^n$ be a morphism of A-module. It is not hard to see that

$$\phi \text{ injective} \iff \det(\phi) \neq 0$$

So the determinant measure the failure of exactness of the sequence $0 \to A^n \xrightarrow{\phi} A^n$.

- Suppose given a long exact sequence of free A-modules (A a domain)

$$0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \xrightarrow{\phi_1} F_0 \qquad (*)$$

Set $r_k := \operatorname{rank}(F_k)$.

Assume that $\overset{H_0(F_\bullet)}{\overset{\shortparallel}{\operatorname{coker}(\phi_1)}}$ is such that $\operatorname{ann}_A(\operatorname{coker}(\phi_1)) \neq 0$

then $(*)$ is exact after $-\otimes_A \overset{K}{\overset{\shortparallel}{\operatorname{Frac}(A)}}$ and hence

$\sum (-1)^i r_i = 0$ $\quad$ / $0 \to F_n \otimes K \to \cdots \to F_0 \otimes K \to 0$ is exact).

Now, we decompose $(*)$ as follows:

- Set $F_n := F_n^{(0)}$; it is of rank r_n.
- ϕ_n is injective, so $\phi_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix}$ with $\det(c_n) \neq 0$ (choice of basis over K).

$$\Rightarrow \quad 0 \to F_n^{(0)} \xrightarrow{\begin{pmatrix} c_n \\ d_n \end{pmatrix}} F_{n-1} = F_{n-1}^{(0)} \oplus F_{n-1}^{(1)} \quad \text{with} \left| \begin{array}{l} \operatorname{rank} F_{n-1}^{(0)} = r_n \\ \operatorname{rank} F_{n-1}^{(1)} = r_{n-1} - r_n \end{array} \right.$$

- Now, any element of F_{n-1} can be viewed as an element of $F_{n-1}^{(1)}$ modulo $\operatorname{Im}(\phi_n) = \operatorname{Ker}(\phi_{n-1})$ (at least over K).

We deduce that there exists a decomposition

$$F_{n-2} = F_{n-2}^{(0)} \oplus F_{n-2}^{(1)} \quad \text{such that} \left| \begin{array}{l} \operatorname{rank} F_{n-2}^{(0)} = \operatorname{rank} F_{n-1}^{(1)} \\ \qquad = r_{n-1} - r_n \\ \operatorname{rank} F_{n-2}^{(1)} = r_{n-2} - r_{n-1} + r_n \end{array} \right.$$

and $\quad \phi_{n-1} = \left(\begin{array}{c|c} a_{n-1} & c_{n-1} \\ \hline b_{n-1} & d_{n-1} \end{array} \right)$ with $\det(c_{n-1}) \neq 0$

we have so far

$$0 \to F_n^{(0)} \xrightarrow{\begin{pmatrix} c_n \\ d_n \end{pmatrix}} F_{n-1}^{(0)} \oplus F_{n-1}^{(1)} \xrightarrow{\left(\begin{array}{c|c} a_{n-1} & c_{n-1} \\ \hline b_{n-1} & d_{n-1} \end{array}\right)} F_{n-2}^{(0)} \oplus F_{n-2}^{(1)}$$

$\det(c_n) \neq 0$ $\qquad\qquad$ $\det(c_{n-1}) \neq 0$

- Continuing this way, we end with the map

$$F_2^{(0)} \oplus F_2^{(1)} \xrightarrow{\left(\begin{array}{c|c} a_2 & c_2 \\ \hline b_2 & d_2 \end{array}\right)} F_1^{(0)} \oplus F_1^{(1)} \xrightarrow{(a_1 \ c_1)} F_0 \qquad \text{with } \det(c_1) \neq 0$$

because $\sum (-1)^i r_i = 0$.

<u>THM</u> The product $\displaystyle\prod_{i=1}^{n} \det(c_i)^{(-1)^{i+1}} = \frac{\det(c_1) \cdot \det(c_3) \cdots}{\det(c_2) \cdot \det(c_4) \cdots}$

is independent of the choice of decomposition and is an element in A (up to multiplication by an invertible elmt in A). It is denoted $\det(F_0)$.

Moreover $(\det(F_0)) =$ "codimension one part of $H_0(F_0)$"

$=$ gcd of generators of $\mathrm{ann}_A(H_0(F_0))$ $\quad$ (A UFD).

This construction generalizes the concept of the determinant of a matrix.

- Application to the resultant.

The Koszul complex $K_\bullet(f_1,\dots,f_n;C)$ is acyclic, i.e. $H_p(K_\bullet)=0$ for all $p\geq 1$ and $H_0(K_\bullet)=B$.

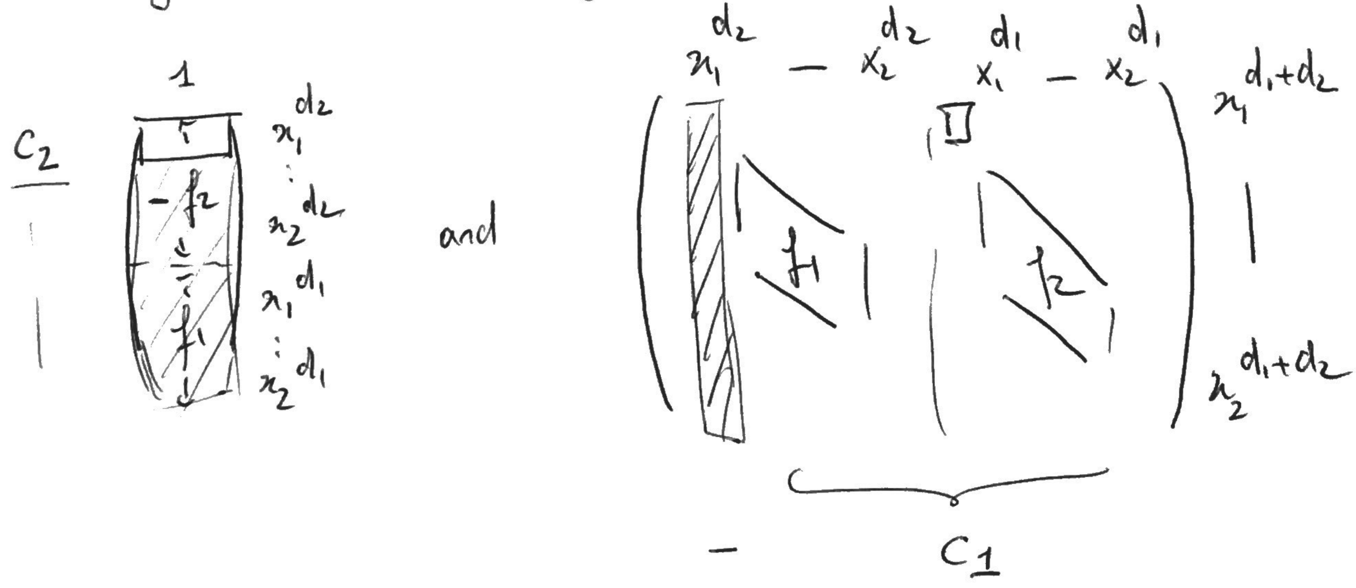
Therefore $\left(\det\left(K_\bullet(f_1,\dots,f_n;C)_\nu\right)\right) \underset{\substack{\uparrow \\ \text{ideals in } A}}{=} \left(\mathrm{Res}(f_1,\dots,f_n)\right)$

for all $\nu \geq \left(\sum_{i=1}^{n}(d_i-1)\right)+1$.

Example: (come back) $n=2$ $\nu=d_1+d_2$

$$C_0=A \xrightarrow{\text{“}\binom{-f_2}{f_1}\text{”}} C_{d_2}\oplus C_{d_1} \xrightarrow{\text{“}(f_1\ f_2)\text{”}} C_{d_1+d_2}$$

writting the matrices we get



$$\frac{\det(C_1)}{\det(C_2)} = \pm\,\mathrm{Rs}(f_1,f_2) \quad \text{in } A=\mathbb{Z}[\text{coeff-}f_1,f_2].$$

3) Implicitization and syzygies

- We considered the implicitization of parameterized plane curves and saw that syzygies play a key role. Our goal is to extend this to the case of surfaces. We will sketch the theory that is much more involved than in the case of curves.

- Suppose given a parameterized surface $\mathcal{S}$:

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^3 \\ (x_0:x_1:x_2) & \longrightarrow & (f_0:f_1:f_2:f_3) \end{array}$$

Let $(T_0:T_1:T_2:T_3)$ the coordinates of $\mathbb{P}^3$. We have already discuss the fact that if the base locus is empty then

$$\mathrm{Res}(T_0f_1-T_1f_0,\, T_0f_2-T_2f_0,\, T_0f_3-T_3f_0) = T_0^{2d^2}\, \underbrace{H(T_0,T_1,T_2,T_3)^{dg(\phi)}}_{\text{impl. eq.}}$$

(can be computed as $\det(K_\bullet)$).

- It turns out that in many cases the base locus is not empty; recall that $dg(\phi)\cdot dg(\mathcal{S}) = d^2 - \sum_{p\in\mathcal{B}} e_p$ $\quad\mathcal{B}$: base locus

So we have to develop an alternative method.

- Equations $T_0f_i - T_if_0$ are such that they are equal to 0 when $T_i \to f_i$ i.e., they are syzygies! We will now try to use other, non trivial, syzygies.

- Framework: $\mathbb{P}^{n-1} \overset{\phi}{\dashrightarrow} \mathbb{P}^n$

$$(x_0 : \dots : x_{n-1}) \longmapsto (f_0 : \dots : f_n)$$

such that:
- f_i are homogeneous pol. of degree $d \geq 1$
- $\overline{\mathrm{Im}(\phi)}$ is a hypersurface $\mathcal{H}$
- $\mathcal{B} = V(I) \subset \mathbb{P}^{n-1}$ is finite, $I = (f_0, \dots, f_n)$.

(we still have $\deg(\phi) \cdot \deg(\mathcal{H}) = d^{n-1} - \sum_{p \in \mathcal{B}} e_p$).

Algebraically, we have the k-algebra map

$$k[T_0, \dots, T_n] \xrightarrow{h} k[x_0, \dots, x_{n-1}]$$
$$T_i \longmapsto f_i$$

and $\ker(h)$ is the defining ideal of our hypersurface

$$\ker(h) = \{ P(T_0, \dots, T_n) \text{ such that } P(f_0, \dots, f_n) \equiv 0 \}.$$

It is a principal ideal generated by an irreducible polynomial (A a domain) called the implicit equation ($H(T_0, \dots, T_n)$).

- The Rees algebra. Set $A = k[x_0, \dots, x_{n-1}]$

Consider the map $\beta : A[T_0, \dots, T_n] \longrightarrow A[z]$
$$T_i \longmapsto f_i z$$

$\mathrm{Rees}_A(I)$ is defined as $\dfrac{A[T_0, \dots, T_n]}{\ker(\beta)}$.

It is a domain. It is sometimes denoted as

$$\mathrm{Rees}_A(I) = A \oplus I \oplus I^2 \oplus \cdots$$

"grading in z". It is bi-graded.

We notice that $\ker(\beta) \cap k[T_0, \dots, T_n] = \ker(h)$

So the equations of the Rees algebra contain the equation of our hypersurface!

However, it is very hard to compute these equations in general: it can be shown that

$$\ker(\beta) = (T_0 - f_0 Z, \dots, T_n - f_n Z) \cap A[T_0, \dots, T_n]$$

so a G.B. computation with lex order $Z > x_0 > \dots > x_{n-1} > T_0 > \dots > T_n$ will provide, in theory, an equation of $\mathcal{H}$.

- <u>The Symmetric algebra</u>

The elements in $\ker(\beta)$ that are linear in $T_0, \dots, T_n$ are of the form $\sum_i a_i T_i$, $a_i \in A$ and satisfy $\sum a_i f_i = 0$.
They are in correspondance with the syzygies of $I = (f_0, \dots, f_n)$.
Si $K := (\sum a_i T_i,\ a_i \in A : \sum a_i f_i) \cdot A[T_0, \dots, T_n]$
The symmetric algebra of I is defined as

$$Sym_A(I) \simeq A[T_0, \dots, T_n] / K$$

We notice that $Sym_A(I) \to Rees_A(I) \to 0$ by construction. They are both bi-graded.

Notice also that now $K \cap k[T_0, \dots, T_n] = 0$, unless $\mathcal{H}$ is a hyperplane.

The strategy is to eliminate equations of $Sym_A(I)$ $x_0, \dots, x_{n-1}$ from
(One may think of equations in K as $f_0, f_1, \dots, f_r$).

THM: Assume that the base points are finite and locally complete intersection (locally defined by 2 equations), then

$$\ker(h) = \ker(\beta) \cap k[T_0, \dots, T_n] = (K : m^\infty) \cap k[T_0, \dots, T_n],$$
$$m = (x_0, \dots, x_{n-1})$$

and for all $\nu \geq (n-1)(d-1) - \underbrace{\operatorname{indeg}(I : m^\infty)}_{\text{smallest degree of a hypersurface going through } \mathcal{B}}$

then $\ker(h) = \operatorname{ann}_{k[T_0, \dots, T_n]} \left(Sym_A(I)_{\nu} \right)$.
ν ← degree w.r.t. $(x_0, \dots, x_{n-1})$.

$= [\mathcal{F}_0 (Sym_A(I)_\nu)]$ ← max minors of a presentation matrix.

- Suppose

$$\bigoplus_{i=1}^{r} A(-d-\mu_i) \xrightarrow{S = \begin{pmatrix} \uparrow & \dots & \uparrow \\ \downarrow & & \downarrow \end{pmatrix}} A(-d) \xrightarrow{f_0, \dots, f_n} A \longrightarrow A/I \longrightarrow 0$$

basis of syzygies of I.

$(T_0, \dots, T_n) \cdot S = (L_1, \dots, L_r)$ generators of K. So :

$$\bigoplus_{i=1}^{r} A[T_0, \dots, T_n](-\mu_i, -1) \xrightarrow{(L_1, \dots, L_r)} A[T_0, \dots, T_n] \longrightarrow Sym_A(I) \longrightarrow 0$$

is a presentation.

From here, we get a presentation of $Sym_A(I)_\nu$.

Example: The sphere: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$

$$(x_0 : x_1 : x_2) \to (x_0^2 + x_1^2 + x_2^2 : x_0^2 - x_1^2 - x_2^2 : 2x_0x_1 : 2x_0x_2)$$

$\mathcal{B} = (x_0, x_1^2 + x_2^2)$ two cyclic points (aligned)

$\mathcal{Z}_0(\mathrm{Sym}_A(I)_\nu)$ is OK for all $\nu \geq 2 \times (2-1) - 1 = 1$

$$(\mathbb{M}2) \qquad \bigoplus_{i=1}^{4} A[T_0,..,T_3](-1,-1) \longrightarrow A[T_0,..,T_3] \to \mathrm{Sym}_A(I) \to 0$$

We get the matrix $\mathbb{M}_1 = \begin{pmatrix} T_2 & -T_0 + T_1 & 0 & T_3 \\ -T_0 - T_1 & T_2 & T_3 & 0 \\ 0 & T_3 & -T_2 & -T_0 - T_1 \end{pmatrix}$

$$\downarrow$$

$$\bigoplus_{i=1}^{\nu} A_0[\underline{T}] \longrightarrow A_1[\underline{T}] \to \mathrm{Sym}_A(I)_1 \to 0$$

The rank of $\mathbb{M}_1$ drops exactly for those points in $\mathbb{P}^3$ that belong to the sphere!.

$$P \in \text{Sphere} \iff \operatorname{rank} \mathbb{M}_1(P) < 3 .$$

Next question: do we have a complex that plays the same role as the Koszul complex for the resultant?

Yes, this is the approximation complex of cycles ...