SUFFICIENT STABILITY CONDITIONS FOR TIME-VARYING NETWORKS OF TELEGRAPHER'S EQUATIONS OR DIFFERENCE DELAY EQUATIONS

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Abstract. We give a sufficient condition for exponential stability of a network of lossless telegrapher's equations, coupled by linear time-varying boundary conditions. The sufficient conditions is in terms of dissipativity of the couplings, which is natural for instance in the context of microwave circuits. Exponential stability is with respect to any L^p-norm, $1 \leq p \leq \infty$. This also yields a sufficient condition for exponential stability to a special class of linear time-varying difference delay systems which is quite explicit and tractable. One ingredient of the proof is that L^p exponential stability for such difference delay systems is independent of p, thereby reproving in a simpler way some results from [4].

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1. Introduction

The stability of electrical circuits operating at high frequency, that is, when delays induced by wires cannot be neglected, has received a lot of attention in the last decades, see for example references [2, 16]. At such an operating regime, wires should be considered as transmission lines, and it is customary to model each of them by a lossless telegrapher's equation (a 1-D hyperbolic partial differential equation, in short: PDE) where voltage and current are functions of abscissa and time. The other elements in the circuit, some of which may be active and nonlinear (transistors, diodes), induce couplings between the boundary conditions of these PDE consisting of a system of both differential and non-differential equations with finite-dimensional state, obtained by applying the classical laws of electricity, at each node, to the boundaries that "touch" this node.

Periodic solutions for such infinite dimensional dynamical systems occur naturally in several contexts; for instance, they arise spontaneously in the case of oscillators, or through periodic forcing in the case of amplifiers (the forcing is the signal to be amplified, represented for instance by a periodic voltage source). Assuming a periodic solution, one may linearize the equations around the latter to investigate its local exponential stability, based on the exponential stability of the first order approximation. The linearized system consists of the original collection of telegrapher's equations (which are linear already), coupled at their nodes (*i.e.* the endpoints of a line) by a set of linear differential and non-differential equations with periodic coefficients, obtained by linearizing the initial couplings, see [18]. To this linear system, one associates a *high frequency limit system* (in short: HFLS), where the linear differential equation at each node degenerates into a linear, time-varying but

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non-differential relation (*i.e.* there is no dynamics in the couplings at infinite frequencies), so that the state of the HFLS reduces to currents and voltages in the lines. The behavior of the HFLS is crucial to the stability of the linearized system, because the solution operator of the latter is, in natural functional spaces, a compact perturbation of the solution operator to the HFLS, see [13, ch. 3, thm. 7.3] and [12]. In particular, the stability of the HFLS is essentially necessary to the stability of the linearized system.

The HFLS is a system of lossless 1-D telegrapher's equations, with linear couplings that depend on time in a periodic manner. With this application in mind, the present paper is devoted, more generally, to the stability of lossless 1-D telegrapher's equations with linear time varying couplings whose coefficients are measurable and uniformly essentially bounded with respect to time, but not necessarily periodic. As is well known, integrating the telegrapher's equation yields an expression of the general solution in terms of two (essentially arbitrary) functions of one variable, and this allows one to recast the original system as a time-varying linear *difference delay system*; the two frameworks are equivalent to study issues of stability.

Stability of networks of hyperbolic PDEs has been addressed extensively, including more general systems of conservation laws than telegrapher's equations (possibly nonlinear), but almost[1] only when the boundary conditions (*i.e.* the couplings) consist of *time-independent* relations, see [1, 5] and the bibliography therein. Another possible, different application of these criteria is to stabilization of such equations with control, like in [7,14] for instance. As far as methods are concerned, Lyapunov functions are a classical tool to obtain sufficient stability conditions, see [1] where they are applied to certain systems of hyperbolic PDEs with conservation laws, or for instance [11], where Lyapunov functions are constructed through linear matrix inequalities, to retarded delay systems. We are not aware of attempts in this direction for difference delay systems.

In another connection, a typical way of obtaining necessary and sufficient stability conditions for a time-invariant network of telegrapher's equations is to apply the Henry-Hale theorem [13, 15] or variants thereof (*cf.* Section 3) to the equivalent difference delay system with constant coefficients. However, no analog for the time-dependent case seems to be known.

The main contribution of this paper is to establish sufficient conditions for exponential stability of networks of telegrapher's equations, in the form of a dissipativity assumption on the couplings at each node of the network, which is fairly natural in a circuit-theoretic context. We also derive sufficient conditions for exponential stability of time-varying difference delay systems, that are a consequence of the former and of independent interest. To our knowledge, this is the first result of this kind in the time-varying case. The proof, which involves going back and forth between the PDE formulation and the difference delay system formulation, has interesting features that should be useful in other contexts as well. Roughly speaking, we rely on classical energy estimates to first obtain a Lyapunov function in the L^2 sense for each telegrapher's equation, using the dissipativity condition at each node; this allows us to show L^2 exponential stability of the system of PDE, therefore also of the associated delay system. In a second step, we deduce from the L^2 exponential stability of the difference delay system its exponential stability in the L^∞ sense (and in fact in the L^p sense for all $p \in [1, \infty]$). This second step is actually subsumed under the work in [4], but we feel our derivation is simpler and worthy in its own. Note that applications to the local stability of a periodic trajectory in an electrical network indeed require L^∞ stability (or C^0 stability) and not just L^2 stability, for the state along a perturbed trajectory of the linearized system must remain uniformly close to the state along the periodic trajectory of the original system, in order that linearization remains meaningful. This paper makes no attempt at handling more general PDEs or coupling conditions. We rather tried to remain as elementary as possible in treating the problem at hand. In particular, our arguments for well posedness may fail for general hyperbolic 1-D equations, for which notions like broad solutions were introduced in [3] and used *e.g.* in [6], see also [1] for other approaches.

The paper is organised as follows. Section 2 introduces networks of telegrapher's equations coupled by time-varying boundary conditions, gives well-posedness results that we could not find in the literature, discusses

[1] As an anonymous reviewer pointed out to us, although the paper [5] deals with local stability of an equilibrium point for nonlinear *time-invariant* hyperbolic systems, it contains a statement (Lemma 3.2) about stability of smoothly *time-varying* linear systems of hyperbolic PDEs for some Sobolev norm. We discuss this further in Section 3.2 and sketch in Section 4.4 how the proof of that lemma may be adapted here.

the construction of equivalent difference delay systems and defines the notions of stability under examination here. Section 3 contains our main result, both in terms of networks of telegrapher's equations and in terms of difference delay equations, while Section 4 is devoted to the proofs.

2. Problem statement

2.1. A time-varying network of hyperbolic equations

Consider a directed graph with N edges and N' nodes, where N and N' are two positive integers. Nodes are numbered by integers $p \in \{1, \cdots, N'\}$, and edges by integers $k \in \{1, \cdots, N\}$.

Figure 2.1 represents a graph with 3 nodes and 4 edges whose only purpose is to illustrate the definitions.

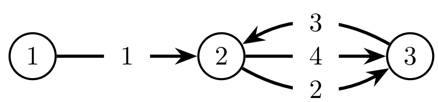

Figure 1. A graph that induces coupling boundary conditions for (1) with $N = 4$.

Each edge figures a telegrapher equation. More precisely, we see edge number k as a copy of the real segment $[0, 1]$ (*i.e.* a transmission line of unit length) on which two real function $v_k(t, .)$ and $i_k(t, .)$ are defined for each time $t>0$ (the voltage and the current) in such a way that the lossless telegrapher's equation is satisfied:

$$\left\{ \begin{array}{l} C_k \dfrac{\partial v_k(t,x)}{\partial t} = -\dfrac{\partial i_k(t,x)}{\partial x}, \\ L_k \dfrac{\partial i_k(t,x)}{\partial t} = -\dfrac{\partial v_k(t,x)}{\partial x}, \end{array} \right. \qquad (t,x) \in \Omega\,, \tag{1}$$

where

$$\Omega \;=\; \{(t,x) \in \mathbb{R}^2,\ 0 < x < 1 \text{ and } 0 < t < +\infty\}, \tag{2}$$

and, for each k, L_k and C_k are two strictly positive numbers (the inductance and the capacity of the line k). In (3) we define the numbers τ_k and K_k, often called the *delay* and *characteristic impedance* of the line k, and we assume without loss of generality, possibly after a re-ordering of the edges, that the delays τ_k are increasing:

$$\tau_k = \sqrt{L_k C_k}\,, \quad K_k = \sqrt{C_k / L_k}\,, \quad 0 < \tau_1 \le \tau_2 \cdots \le \tau_N\,. \tag{3}$$

Each node couples the edges adjacent to it through boundary conditions involving, for each such edge, the endpoint of $[0, 1]$ that touches the node. More precisely, if

- $j(p)$ is the out-degree of the graph at node p (i.e. the number of edges outgoing from node p) and $\tilde{\jmath}(p)$ the in-degree of the graph at node p (i.e. the number of edges incoming at node p), and
- $k_1 < \cdots < k_{j(p)}$ are the labels of the edges *outgoing* from node p and $k_{j(p)+1} < \cdots < k_{j(p)+\tilde{\jmath}(p)}$ the labels of the edges *incoming at* node p,

then node p couples the equations (1) together by imposing $j(p)+\tilde{\jmath}(p)$ relations between the $2(j(p)+\tilde{\jmath}(p))$ entries of the following two vectors:

$$V_p(t)=\begin{pmatrix} v_{k_1}(t,0)\\ \vdots\\ v_{k_{j(p)}}(t,0)\\ v_{k_{j(p)+1}}(t,1)\\ \vdots\\ v_{k_{j(p)+\tilde{\jmath}(p)}}(t,1)\end{pmatrix},\qquad I_p(t)=\begin{pmatrix} -i_{k_1}(t,0)\\ \vdots\\ -i_{k_{j(p)}}(t,0)\\ i_{k_{j(p)+1}}(t,1)\\ \vdots\\ i_{k_{j(p)+\tilde{\jmath}(p)}}(t,1)\end{pmatrix}. \tag{4}$$

These relations read

$$V_p(t)=A_p(t)I_p(t), \tag{5}$$

where $t\mapsto A_p(t)$ is a map from $\mathbb{R}^+$ (the nonnegative real numbers) to the set of square $(j(p)+\tilde{\jmath}(p))\times(j(p)+\tilde{\jmath}(p))$ matrices which is assumed to be measurable and *bounded*. In many cases of interest (*e.g.* when modeling an electrical circuit), it is moreover *continuous*. Our results rest on the following condition characterizing *dissipativity* at (each) node p:

$$A_p(t)+A_p^*(t)\geq \alpha_p\, Id,\quad \alpha_p>0\,,\quad p\in\{1,\cdots,N'\}, \tag{6}$$

where superscript $*$ denotes the transpose of a matrix and α_p is independent of t. Inequality (6) is meant to hold between symmetric matrices, for a.e. t. We shall have an occasion to deal also with complex matrices, in which case superscript $*$ denotes the transpose conjugate. Here and below, the symbol Id stands for the identity operator or the identity matrix of appropriate size, while the context will keep the meaning clear.

Example 2.1. For the graph in Figure 2.1 it holds that $N=4$, i.e. we have four telegrapher's equations of the form (1), numbered with $k\in\{1,2,3,4\}$, and we have that $N'=3$, hence we get three sets of boundary conditions. Let us detail the latter.
- For $p=1$, we have $j(1)=1$, $\tilde{\jmath}(1)=0$, and we see from the graph that $k_1=1$,
- for $p=2$, we have $j(2)=2$, $\tilde{\jmath}(2)=1$, and we see from the graph that $k_1=2$, $k_2=4$, $k_3=3$,
- for $p=3$, we have $j(3)=1$, $\tilde{\jmath}(3)=2$, and we see from the graph that $k_1=3$, $k_2=2$ and $k_3=4$.

This yields three equations of the form (5) as follows, with $A_1(t)$ a scalar, $A_2(t)$ is a 4×4 matrix and $A_3(t)$ a 3×3 matrix:

$$v_1(t,0)=-A_1(t)\,i_1(t,0),\quad \begin{pmatrix} v_2(t,0)\\ v_4(t,0)\\ v_1(t,1)\\ v_3(t,1)\end{pmatrix}=A_2(t)\begin{pmatrix} -i_2(t,0)\\ -i_4(t,0)\\ i_1(t,1)\\ i_3(t,1)\end{pmatrix},\quad \begin{pmatrix} v_3(t,0)\\ v_2(t,1)\\ v_4(t,1)\end{pmatrix}=A_3(t)\begin{pmatrix} -i_3(t,0)\\ i_2(t,1)\\ i_4(t,1)\end{pmatrix}. \tag{7}$$

Remark 2.2 (On the minus signs in the vector I_p in (4))**.** We shall see later why (6) amounts to energy dissipation in some sense. This is one justification for the minus signs in the first entries of the vector $I_p(t)$ defined by (4): removing the minus signs, one may still express the (same) boundary conditions by an equation like (5) with a different $A_p(t)$, but then condition (6) will assume an intricate form. Alternatively, from a circuit-theoretic viewpoint, the minus signs are justified by Kirchoff's law of currents.

Remark 2.3 (On the normalization of line lengths)**.** We have assumed that the space variable x belongs to the interval $[0,1]$ for every k in equation (1) rather than $[0,\ell_k]$ for some positive ℓ_k. This is no loss of generality, for such a normalization can always be achieved by a linear change of variable on x. With this normalization, τ_k given by equation (3) has the meaning of a time delay.

Remark 2.4 (On the possibility of loops)**.** In the above framework, nothing prevents an edge from being both outgoing from, and incoming to a given node p. In this case, the index k of this edge appears twice in equation (4), once as a k_j with $j \leq j(p)$ and once as a $k_{j(p)+l}$ with $l \leq \tilde{j}$.

So far, we endowed a system consisting of N PDE, indexed by the edges of our graph (namely: (1)), with boundary conditions given by a collection of N' linear time-dependent relations, indexed by the nodes of the graph (namely: (5)). As a result, the boundary conditions at $x = 0$ and at $x = 1$ for a given telegrapher's equation of the form (1) are generally obtained from two different relations of the form(5). To compactify the notation, we shall rewrite the boundary conditions in lumped form, as a single linear relation between concatenated vectors $\mathbf{V}(t)$ and $\mathbf{I}(t)$ defined by:

$$v(x,t) = \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_N(x,t) \end{pmatrix}, \quad i(x,t) = \begin{pmatrix} i_1(x,t) \\ \vdots \\ i_N(x,t) \end{pmatrix}, \quad \mathbf{V}(t) = \begin{pmatrix} v(t,0) \\ \\ v(t,1) \end{pmatrix}, \quad \mathbf{I}(t) = \begin{pmatrix} -i(t,0) \\ \\ i(t,1) \end{pmatrix}, \tag{8}$$

that aggregate *all* boundary values of voltages and currents in the lines. Since the concatenation of all vectors $V_p(t)$ (resp. $I_p(t)$) defined in (4) contains each component of $\mathbf{V}(t)$ (resp. $\mathbf{I}(t)$) exactly once, as p ranges from 1 to N', there is a $2N \times 2N$ permutation matrix P_1 such that

$$\begin{pmatrix} V_1(t) \\ \vdots \\ V_{N'}(t) \end{pmatrix} = P_1\, \mathbf{V}(t)\,, \qquad \begin{pmatrix} I_1(t) \\ \vdots \\ I_{N'}(t) \end{pmatrix} = P_1\, \mathbf{I}(t)\,. \tag{9}$$

The set of equations (5), $1 \leq p \leq N'$, can now be written as

$$\mathbf{V}(t) = \mathbf{A}(t)\, \mathbf{I}(t) \tag{10}$$

with

$$\mathbf{A}(t) = P_1^{-1}\ \mathrm{diag}(A_1(t), \ldots, A_{N'})\, P_1 \tag{11}$$

where $\mathrm{diag}(A_1(t), \ldots, A_{N'}(t))$ is a block-diagonal $2N \times 2N$ matrix. Note that this "aggregated" notation may be understood as collapsing all the nodes into a single one; all edges are then "loops" as described in Remark 2.4. Clearly, the hypotheses on A_p made in (6) translate into the following assumption on the matrix $\mathbf{A}(t)$ that will be used throughout the paper :

Assumption 2.5. *The map $t \mapsto \mathbf{A}(t)$ is measurable and essentially bounded $[0, +\infty) \to \mathbb{R}^{2N\times 2N}$, moreover there exists a positive number α, independent of t, such that*

$$\mathbf{A}(t) + \mathbf{A}^*(t) \geq \alpha\, Id, \qquad \alpha > 0 \qquad t \in \mathbb{R}. \tag{12}$$

2.2. Well posedness of evolution problem in the L^p and C^0 cases

Equations (1) ($1 \leq k \leq N$) and (10)-(8) define a linear time-varying dynamical system, whose state at time t consists of a collection of $2N$ real functions on $[0,1]$, namely $x \mapsto v_k(t,x)$ and $x \mapsto i_k(t,x)$ for $1 \leq k \leq N$. Before we can study the stability of this dynamical system, we need to address the issue of well posedness, *i.e.* of existence and uniqueness of solutions given initial conditions $v_k(0,.)$ and $i_k(0,.)$ (the Cauchy problem). When the matrices $A_p(t)$ (or equivalently the matrix $\mathbf{A}(t)$) do not actually depend on t, well-posedness results are classical, see for instance the textbooks [1,8].

In the time-varying case, which is our concern here, a very definition of well-posedness seems hard to find in the literature, perhaps because the introduction of time dependent boundary conditions leads to a failure of classical semigroup theory. We shall consider two cases according to whether the state at time t consists of continuous functions or merely L^p-summable functions on $[0,1]$, $1 \le p \le \infty$.

To fix notation, we denote respectively by $\mathbb{N}$ and $\mathbb{R}$ the sets of nonnegative integers and real numbers. We also recall the notation $\mathbb{R}^+$ for nonnegative real numbers. We write the Euclidean norm of $x \in \mathbb{R}^l$ as $\|x\|$, and the Euclidean scalar product of $x, y \in \mathbb{R}^l$ as $\langle x, y\rangle$, irrespectively of l. We put $C^0(E)$ for the space of real continuous functions on any (topological) space E. When E is compact we endow $C^0(E)$ with the *sup* norm. Also, whenever $E \subset \mathbb{R}^l$ is measurable and $1 \le p < \infty$, we put $L^p(E)$ for the familiar Lebesgue space of (equivalent classes of a.e. coinciding) real-valued measurable functions on E whose absolute value to the p^{th} power is integrable, endowed with the norm $\|f\|_{L^p(E)} = (\int_E |f(x)|^p dx)^{1/p}$ where dx indicates the differential of Lebesgue measure (restricted to E). The space $L^\infty(E)$ corresponds to real, essentially bounded Lebesgue measurable functions, normed with the essential supremum of their absolute value on E. More generally, for F a Banach space with norm $\|.\|_F$, we let $C^0(E, F)$ be the space of F-valued continuous functions on E, and if E is compact we set $\|f\|_{C^0(E,F)} = \sup_E \|f\|_F$. In a similar way, $L^p(E, F)$ is the space of F-valued measurable functions f on E such that $\|f\|_F \in L^p(E)$. We also define locally integrable functions: $L^p_{loc}(E)$ designates the space of functions whose restriction $f_{|K}$ to any compact set $K \subset E$ belongs to $L^p(K)$. Likewise, we let $L^p_{loc}(E, F)$ be the space of F-valued measurable functions f on E such that $\|f\|_F \in L^p_{loc}(E)$. Since $\mathbb{R}^l$ is σ-compact, the topology of L^p-convergence on every compact set is metrizable on $L^p_{loc}(\mathbb{R}^l, F)$. The spectral norm of a linear operator $B : F_1 \to F_2$ between two Banach spaces is $|||B||| = \sup_{x \in F_1} \|Bx\|_{F_2}/\|x\|_{F_1}$, keeping the notation independent of F_1, F_2 for simplicity.

Next, let us make precise the meaning of (1) and (10) when v_k and i_k lie in $L^1_{loc}(\overline{\Omega})$, where Ω is defined by (2) and $\overline{\Omega}$ indicates the closure of Ω in $\mathbb{R}^2$. Later, we shall see this space is big enough to accomodate cases we have in mind. Note that $\overline{\Omega} = [0, \infty) \times [0, 1]$, and that $L^1_{loc}(\overline{\Omega})$ identifies with a subspace of $L^1_{loc}(\Omega)$, since $[0, \infty) \times [0, 1] \setminus \Omega$ has 2-D Lebesgue measure zero. Indeed, the latter set is just the boundary $\partial\Omega$ of Ω in $\mathbb{R}^2$:

$$\partial\Omega = (\{0\}\times(0,1)) \cup ([0,+\infty)\times\{0\}) \cup ([0,+\infty)\times\{1\}) \,. \tag{13}$$

Equation (1) is understood in the distributional sense as soon as $(v_k, i_k) \in L^1_{loc}(\Omega) \times L^1_{loc}(\Omega)$. That is, (v_k, i_k) is a solution to (1) if, for all C^∞-smooth functions $\varphi : \Omega \to \mathbb{R}$ with compact support, it holds that

$$\begin{aligned} &\iint_\Omega \Big(L_k\, i_k(t,x) \frac{\partial \varphi}{\partial t}(t,x) + v_k(t,x) \frac{\partial \varphi}{\partial x}(t,x) \Big) \mathrm{d}t\mathrm{d}x = 0 \,, \\ &\iint_\Omega \Big(C_k\, v_k(t,x) \frac{\partial \varphi}{\partial t}(t,x) + i_k(t,x) \frac{\partial \varphi}{\partial x}(t,x) \Big) \mathrm{d}t\mathrm{d}x = 0 \,. \end{aligned} \tag{14}$$

As to (10), the definition (8) of $\mathbf{V}$ and $\mathbf{I}$, as well as the choice of initial conditions $v_k(0,.)$ and $i_k(0,.)$, require that v_k and i_k extend in some way to $\partial\Omega$ described in (13), and this is where their membership to $L^1_{loc}(\overline{\Omega})$ (not just to $L^1_{loc}(\Omega)$) is useful. In fact, when $h \in L^1_{loc}(\overline{\Omega}) = L^1_{loc}([0,\infty) \times [0,1])$, we get from Fubini's theorem that $\tau \mapsto h(\tau, x)$ belongs to $L^1_{loc}([0,\infty))$ for a.e. $x \in [0,1]$ and that $s \mapsto h(t,s)$ lies in $L^1([0,1])$ for a.e. $t \in [0,\infty)$. For such x and t, we set

$$\widehat{h}(0,x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon h(s,x)\mathrm{d}s \,, \quad \widehat{h}(t,0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon h(t,s)\mathrm{d}s \,, \quad \widehat{h}(t,1) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{1-\varepsilon}^1 h(t,s)\mathrm{d}s \,, \tag{15}$$

whenever the limits exist.

Definition 2.6. We say that $h \in L^1_{loc}(\overline{\Omega})$ *has a strict extension to* $\partial\Omega$ if and only if the limits in (15) exist for almost all $x \in (0,1)$ and almost all $t \in (0,\infty)$, and then the functions $x \mapsto \widehat{h}(0,x)$, $t \mapsto \widehat{h}(t,0)$ and $t \mapsto \widehat{h}(t,1)$

define the strict extension of h to $\partial\Omega$, almost everywhere with respect to $\mathcal{H}^1$-Hausdorff measure[2], see *e.g.* [10, ch. 2] for the definition of Hausdorff measures.

Remark 2.7. Definition 2.6 may look strange at first glance, since when $h \in L^1_{loc}(\overline{\Omega})$ it seems to be defined already on $\partial\Omega \subset \overline{\Omega}$; but of course it is not so, because $\partial\Omega$ has 2-D Lebesgue measure zero, hence the values assumed by h there are immaterial. When the limits in (15) exist for a.e. x and t, they produce a specific definition of h on $\partial\Omega$, a.e. with respect to $\mathcal{H}^1$, that we call the strict extension. If $h : \Omega \to \mathbb{R}$ is continuous and extends continuously $\overline{\Omega} \to \mathbb{R}$, clearly the strict extension exists and it is the natural one. Even then, we sometimes use the notation $\widehat{h}(0,x)$, $\widehat{h}(t,0)$ and $\widehat{h}(t,1)$ for reasons of consistency, although writing $h(0,x)$, $h(t,0)$ and $h(t,1)$ is more appropriate in this case.

If all v_k and i_k have a strict extension to $\partial\Omega$, then we interpret the boundary conditions (10) to mean the following set of equalities between (a.e. defined) measurable functions of a single variable t:

$$\begin{pmatrix} \widehat{v}_1(t,0) \\ \vdots \\ \widehat{v}_N(t,0) \\ \widehat{v}_1(t,1) \\ \vdots \\ \widehat{v}_N(t,1) \end{pmatrix} = \mathbf{A}(t) \begin{pmatrix} -\widehat{\imath}_1(t,0) \\ \vdots \\ -\widehat{\imath}_N(t,0) \\ \widehat{\imath}_1(t,1) \\ \vdots \\ \widehat{\imath}_N(t,1) \end{pmatrix}, \qquad \text{a.e. } t \in (0,\infty). \tag{16}$$

We can now state a well-posedness result for the system (1)-(10). Part I deals with solutions belonging to $L^1_{loc}([0,\infty), L^p([0,1]))$, and part II is about continuous solutions. They do not run completely parallel to each other, because continuity requires a compatibility relation on the initial conditions, see (18). The theorem is standard in nature but, as mentioned already, we could not find a reference in the literature for the case of *time-varying* boundary conditions (16). To connect the statement with the previous discussion, we observe that $L^1_{loc}([0,\infty), L^p([0,1])) \subset L^1_{loc}([0,\infty) \times [0,1]) = L^1_{loc}(\overline{\Omega})$ for $1 \le p \le \infty$, by Hölder's inequality and the Fubini theorem.

Theorem 2.8 (Well-posedness). *Let $\mathbf{A} : [0,\infty) \to \mathbb{R}^{2N\times 2N}$ meet Assumption 2.5 and $1 \le p \le \infty$.*
I) *If $i_k^0, v_k^0 \in L^p([0,1])$, $1 \le k \le N$, there is a unique map $(t,x) \mapsto (v_1(t,x), \dots, v_N(t,x), i_1(t,x), \dots, i_N(t,x))$ from Ω into $\mathbb{R}^{2N\times 2N}$ such that:*

- *$t \mapsto (v_1(t,.), \dots, v_N(t,.), i_1(t,.), \dots, i_N(t,.))$ belongs to $L^1_{loc}([0,\infty), (L^p([0,1]))^{2N})$ and v_k, i_k have a strict extension to $\partial\Omega$ satisfying the initial conditions*

$$\widehat{v}_k(0,x) = v_k^0(x), \;\; \widehat{\imath}_k(0,x) = i_k^0(x) \quad k = 0, \dots, N\,, \tag{17}$$

- *$(t,x) \mapsto (v_1(t,x), \dots, v_N(t,x), i_1(t,x), \dots, i_N(t,x))$ is a solution of (1)-(10), $1 \le k \le N$, in the sense of (14) and (16).*

II) *If, in addition, $t \mapsto \mathbf{A}(t)$ is continuous and $v_1^0, \dots, v_N^0, i_1^0, \dots, i_N^0$ are elements of $C^0([0,1])$ satisfying*

$$\begin{pmatrix} v_1^0(0) \\ \vdots \\ v_N^0(0) \\ v_1^0(1) \\ \vdots \\ v_N^0(1) \end{pmatrix} = \mathbf{A}(0) \begin{pmatrix} -i_1^0(0) \\ \vdots \\ -i_N^0(0) \\ i_1^0(1) \\ \vdots \\ i_N^0(1) \end{pmatrix}, \tag{18}$$

[2] $\mathcal{H}^1$ restricted to $\partial\Omega \subset \mathbb{R}^2$ is simply the measure whose restriction to each curve $\{0\}\times(0,1)$, $[0,+\infty)\times\{0\}$ and $[0,+\infty)\times\{1\}$ coincides with arc length.

then the map $(t,x)\mapsto(v_1(t,x),\dots,v_N(t,x),i_1(t,x),\dots,i_N(t,x))$ *from part* **I** *is continuous* $\overline{\Omega}\to\mathbb{R}^{2N}$ *(equivalently:* $t\mapsto(v_1(t,.),\dots,v_N(t,.),i_1(t,.),\dots,i_N(t,.))$ *is continuous* $[0,\infty)\to C^0([0,1])^{2N\times 2N}$*) and satisfies the initial conditions in the strong sense:*

$$v_k(0,x)=v_k^0(x),\;\; i_k(0,x)=i_k^0(x)\,,\qquad x\in[0,1],\quad k=0,\dots,N\,. \tag{19}$$

Remark 2.9. Assumption 2.5 is stronger than needed for the previous result to hold. In fact, it is enough for part I that the maps $t\mapsto\mathbf{A}(t)$ and $t\mapsto\left(I+\mathbf{A}(t)\,\mathbf{K}\right)^{-1}$ be well defined, measurable and bounded ($\mathbf{K}$ is defined in (28)), and for part II that they be continuous and bounded. We do not dwell on such generalizations.

The proof of Theorem 2.8 is given at the end of Section 2.3, after establishing the equivalence of (1)-(10) with a suitable difference delay system. As a first step in this direction, we stress below the special form of solutions to (1) in $L^p_{loc}(\Omega)$, and show they have a strict extension to $\partial\Omega$ if, moreover, they lie in $L^p_{loc}(\overline{\Omega})$.

Proposition 2.10. *Let* i_k *and* v_k *belong to* $L^p_{loc}(\Omega)$ *(resp.* $C^0(\Omega)$*) for some* $p\in[1,\infty]$*, and satisfy* (1) *on* Ω *in the sense of* (14)*. Then, the following properties hold.*

- (i) *There exists two functions* f_k *and* g_k *in* $L^p_{loc}((-\infty,1))$ *and* $L^p_{loc}((0,\infty))$ *(resp. in* $C^0((-\infty,1))$ *and* $C^0((0,\infty))$*) such that*

$$v_k(t,x)=f_k(x-\frac{t}{\tau_k})+g_k(x+\frac{t}{\tau_k})\,,\quad i_k(t,x)=K_k\left(f_k(x-\frac{t}{\tau_k})-g_k(x+\frac{t}{\tau_k})\right)\,, \tag{20}$$

for almost every (resp. every) (x,t) *in* Ω*, where* τ_k,K_k *are defined by* (3)*.*

- (ii) *If, in addition,* v_k *and* i_k *lie in* $L^p_{loc}(\overline{\Omega})$ *(resp. extend continuously* $\overline{\Omega}\to\mathbb{R}$*), then* f_k *and* g_k *lie in* $L^p_{loc}((-\infty,1])$ *and* $L^p_{loc}([0,\infty))$ *(resp. in* $C^0((-\infty,1])$ *and* $C^0([0,\infty))$*), moreover* v_k*,* i_k *have a strict extension to* $\partial\,\Omega$ *according to Definition 2.6. More precisely, we have that*

$$\begin{aligned}
\widehat{v}_k(t,0)&=f_k(-\frac{t}{\tau_k})+g_k(\frac{t}{\tau_k})\,, & \widehat{\imath}_k(t,0)&=K_k\left(f_k(-\frac{t}{\tau_k})-g_k(\frac{t}{\tau_k})\right)\,,\\
\widehat{v}_k(t,1)&=f_k(1-\frac{t}{\tau_k})+g_k(1+\frac{t}{\tau_k})\,, & \widehat{\imath}_k(t,1)&=K_k\left(f_k(1-\frac{t}{\tau_k})-g_k(1+\frac{t}{\tau_k})\right)\,,\\
\widehat{\imath}_k(0,x)&=K_k\left(f_k(x)-g_k(x)\right)\,, & \widehat{v}_k(0,x)&=f_k(x)+g_k(x)\,,
\end{aligned} \tag{21}$$

where the first four relations hold for almost all (resp. all) t *in* $(0,+\infty)$ *and the last two for almost all (resp. all)* x *in* $(0,1)$*.*

- (iii) *Conversely, if* f_k *and* g_k *lie in* $L^p_{loc}((-\infty,1))$ *and* $L^p_{loc}((0,\infty))$ *(resp. in* $C^0((-\infty,1))$ *and* $C^0((0,\infty))$*), then* v_k *and* i_k *given by* (20) *belong to* $L^p_{loc}(\Omega)$ *(resp.* $C^0(\Omega)$*) and satisfy* (1)*. If, moreover,* f_k *and* g_k *lie in* $L^p_{loc}((-\infty,1])$ *and* $L^p_{loc}([0,\infty))$ *(resp. in* $C^0((-\infty,1])$ *and* $C^0([0,\infty))$*), then* v_k *and* i_k *belong to* $L^p_{loc}(\overline{\Omega})$ *(resp.* $C^0(\overline{\Omega})$*) and* (21) *holds.*

Proof. The proof of point (i) rests on a simple, linear change of variables, valid even in the distributional sense: if we introduce new variables $r=x-t/\tau_k$, $s=x+t/\tau_k$ and define two functions f_k, g_k on $\Omega_1:=\{(r,s)\in\mathbb{R}^2,\ 0<r+s<2\text{ and }-\infty<r-s<0\}$, *via*

$$\begin{pmatrix}f_k(r,s)\\ g_k(r,s)\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2K_k}\\ \frac{1}{2} & -\frac{1}{2K_k}\end{pmatrix}\begin{pmatrix}v_k\Big(\dfrac{-r+s}{2}\tau_k\,,\dfrac{r+s}{2}\Big)\\ i_k\Big(\dfrac{-r+s}{2}\tau_k\,,\dfrac{r+s}{2}\Big)\end{pmatrix}\,,\quad\begin{pmatrix}v_k(t,x)\\ i_k(t,x)\end{pmatrix}=\begin{pmatrix}1 & 1\\ K_k & -K_k\end{pmatrix}\begin{pmatrix}f_k\big(x-t/\tau_k,x+t/\tau_k\big)\\ g_k\big(x-t/\tau_k,x+t/\tau_k\big)\end{pmatrix}\,,$$

f_k, g_k are in one-to-one correspondence with v_k, i_k, they are in $L^p_{loc}(\Omega_1)$ (resp. $C^0(\Omega_1,\mathbb{R})$) if and only if the latter are in $L^p_{loc}(\Omega)$ (resp. $C^0(\Omega,\mathbb{R})$), and System (1) gets transformed into the distributional identity

$$\frac{\partial f_k}{\partial s} = \frac{\partial g_k}{\partial r} = 0\,. \tag{22}$$

Equation (22) means that f_k does not depend on the second argument nor g_k on the first one, hence the form (20) for v_k and i_k. We turn to Point (*ii*). First, we observe that if v_k and i_k lie in $L^p_{loc}(\overline{\Omega})$ (resp. extend continuously $\overline{\Omega}\to\mathbb{R}$), then f_k and g_k lie in $L^p_{loc}((-\infty,1])$ and $L^p_{loc}([0,\infty))$ (resp. in $C^0((-\infty,1])$ and $C^0([0,\infty))$), by the change of variable formula (resp. by inspection). The case where v_k and i_k extend continuously $\overline{\Omega}\to\mathbb{R}$ is now obvious. To handle the case where $v_k, i_k \in L^p_{loc}(\overline{\Omega})$, recall that a (non-centered) Lebesgue point of a function $\ell \in L^1_{loc}(\mathbb{R})$ is a point $x\in\mathbb{R}$ such that $\lim_{|I|\to 0, I\ni x}\frac{1}{|I|}\int_I |\ell(y)-\ell(x)|\mathrm{d}y = 0$, where the limit is taken over all closed intervals I containing x and $|I|$ indicates the length of I. Let $\widetilde{f}_k$ and $\widetilde{g}_k$ be the extensions by 0 of f_k and g_k to the whole real line. Using (20) in (15), we see that the first two relations in (21) certainly hold for $t\in(0,\infty)$ such that $-t/\tau_k$ is a Lebesgue point of $\widetilde{f}_k$ and t/τ_k is a Lebesgue point of $\widetilde{g}_k$, the third and fourth relations if $1-t/\tau_k$ is a Lebesgue point of $\widetilde{f}_k$ and $1+t/\tau_k$ is a Lebesgue point of $\widetilde{g}_k$, and the last two if $x\in(0,1)$ is a Lebesgue point of both $\widetilde{f}_k$ and $\widetilde{g}_k$. Since almost all points are Lebesgue points of a given function in $L^1_{loc}(\mathbb{R})$ [10, thm. 1.34], while $L^p_{loc}(\mathbb{R})\subset L^1_{loc}(\mathbb{R})$ by Hölder's inequality, this proves Point (*ii*). Point (*iii*) is obvious, reverting computations. □

Remark 2.11. The weak formulation (14) defines solutions v_k, i_k to (1) as locally integrable functions $\Omega\to\mathbb{R}$, while Theorem 2.8 stresses their representation as functions $\mathbb{R}^+\to L^p([0,1])$. The two points of view are essentially equivalent by Fubini's theorem, but suggestive of different moods. In this connection, it is worth mentioning that if $p<\infty$, then the solution set forth in Part I of Theorem 2.8 not only belongs to $L^1_{loc}([0,\infty),(L^p([0,1]))^{2N})$, but in fact is continuous $[0,\infty)\to(L^p([0,1]))^{2N\times 2N}$. Indeed, granted that f_k and g_k lie in $L^p_{loc}((-\infty,1])$ and $L^p_{loc}([0,\infty))$ by Proposition 2.10, this follows from the very proof of the theorem (*cf.* (31) and (26) below) and the fact that $\tau\mapsto f(.-\tau)$ is continuous $\mathbb{R}\to L^p(\mathbb{R})$, whenever $f\in L^p(\mathbb{R})$, $p<\infty$.

2.3. Difference delay equations and their relation with networks of telegrapher's equations

A general linear time-varying difference delay equation in the variable z is of the form

$$z(t) = \sum_{i=1}^{M} D_i(t)\, z(t-\eta_i) \quad \text{for all (or almost all) } t\geq 0\,, \tag{23}$$

where the delays $0<\eta_1\leq\cdots\leq\eta_M$ are arranged in nondecreasing order, each $t\mapsto D_i(t)$ is a $d\times d$ matrix-valued function, and solutions $t\mapsto z(t)$ are $\mathbb{R}^d$-valued functions. Hereafter, we make the following assumption.

Assumption 2.12. *The maps $t\mapsto D_i(t)$ belong to $L^\infty(\mathbb{R}^+,\mathbb{R}^{d\times d})$ for $1\leq i\leq M$.*

Given initial conditions on $[-\eta_M,0]$, we recap existence and uniqueness of solutions to (23) in the following theorem. The existence of continuous solutions requires an additional continuity assumption on the D_i, as well as compatibility relations on the initial conditions; this is why we introduce the following space:

$$\mathcal{C} := \{\phi\in C^0([-\eta_M,0],\mathbb{R}^d)\mid \phi(0)=\sum_{i=1}^{M} D_i(0)\phi(-\eta_i)\}\,. \tag{24}$$

Theorem 2.13. *Let Assumption 2.12 hold and ϕ be an element of $L^p([-\eta_M,0],\mathbb{R}^d)$ with $1\leq p\leq\infty$.*

(i) *There is a unique solution z to (23) in $L^p_{loc}([-\eta_M,+\infty),\mathbb{R}^d)$ meeting the initial condition $z_{|[-\eta_M,0]}=\phi$.*

(ii) *If, moreover, the maps $D_i:\mathbb{R}^+\to\mathbb{R}^{d\times d}$ are continuous and $\phi\in\mathcal{C}$, then $z\in C^0([-\eta_M,+\infty),\mathbb{R}^d))$.*

Proof. This is a classical, elementary inductive argument, see *e.g.* [13]: for any $T \geq 0$, if a solution has been found on $[-\eta_M, T]$, it clearly can be extended to $[-\eta_M, T+\eta_1]$ in a unique manner using (23). When the $D_i(.)$ are continuous, $\phi \in \mathcal{C}$ is clearly necessary and sufficient for the unique solution to be continuous. $\square$

Remark 2.14 (merging repeated delays)**.** In (23), we allow for repeated delays, *i.e.* it may be that $\eta_i = \eta_{i+1}$ for some i. This to comply with (1)-(3), where it would be too restrictive to require that the numbers τ_k are distinct, and because we are about to convert (1)-(3) into (23) in such a way that $\eta_i = \tau_i$. However, when dealing with (23), it is better to avoid repetition by merging terms with the same delay. Since it will be needed in the statement of Theorem 3.6, let us formalize this: first, define an enumeration without repetition of the original list of delays, say, $0 < \widehat{\eta}_1 < \widehat{\eta}_2 < \cdots < \widehat{\eta}_{\widehat{M}}$ with $\widehat{M} \leq M$, then define for each j

$$\widehat{D}_j(t) = \sum_{\{i,\ \eta_i = \widehat{\eta}_j\}} D_i(t)\,. \tag{25}$$

It is clear that (23) can be re-written as $z(t) = \sum_{j=1}^{\widehat{M}} \widehat{D}_j(t)\, z(t-\widehat{\eta}_j)$, and if the η_i were distinct already, then the system is left unchanged.

We now construe the system of coupled telegrapher's equations from Sections 2.1 and 2.2 as a difference delay system of the form (23). For this, let $(v_k, i_k) \in L^1_{loc}(\overline{\Omega}) \times L^1_{loc}(\overline{\Omega})$ (resp. $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$) be, for $1 \leq k \leq N$, solutions of (1)-(10), observing from Proposition 2.10 (*ii*) that the boundary conditions (10) indeed make sense. Let f_k, g_k be as in Proposition 2.10, and define:

$$x_k(t) = f_k(-\frac{t}{\tau_k}) \quad\text{and}\quad y_k(t) = g_k(1+\frac{t}{\tau_k}). \tag{26}$$

The functions f_k and g_k lie in $L^1_{loc}((-\infty,1])$ and $L^1_{loc}([0,\infty))$ (resp. in $C^0((-\infty,1])$ and $C^0([0,\infty))$) by Proposition 2.10, therefore x_k and y_k lie in $L^1_{loc}([-\tau_k,+\infty))$ (resp. $C^0([-\tau_k,+\infty))$). Moreover, the boundary values of v_k and i_k are related to x_k and y_k as follows (substitute (26) in (21)):

$$\begin{cases} \widehat{v}_k(t,0) = x_k(t) + y_k(t-\tau_k)\,, \\ \widehat{\imath}_k(t,0) = K_k[x_k(t) - y_k(t-\tau_k)]\,, \\ \widehat{v}_k(t,1) = x_k(t-\tau_k) + y_k(t)\,, \\ \widehat{\imath}_k(t,1) = K_k[x_k(t-\tau_k) - y_k(t)]\,. \end{cases} \tag{27}$$

Plugging (27) in (16) gives us

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} + \begin{pmatrix} y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} = \mathbf{A}(t) \left[\begin{pmatrix} -K_1\, x_1(t) \\ \vdots \\ -K_N\, x_N(t) \\ -K_1\, y_1(t) \\ \vdots \\ -K_N\, y_N(t) \end{pmatrix} + \begin{pmatrix} K_1\, y_1(t-\tau_1) \\ \vdots \\ K_N\, y_N(t-\tau_N) \\ K_1\, x_1(t-\tau_1) \\ \vdots \\ K_N\, x_N(t-\tau_N) \end{pmatrix} \right].$$

Thus, if we define

$$\mathbf{K} = \operatorname{diag}(K_1, \ldots, K_N, K_1, \ldots, K_N), \qquad P_2 = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} \tag{28}$$

where Id has size $N \times N$, and observe that $P_2\mathbf{K} = \mathbf{K}P_2$ while noting that relation $\mathbf{K} = \mathbf{K}^* > 0$ together with the dissipativity condition (12) entail that $I + \mathbf{A}(t)\,\mathbf{K}$ is invertible, we obtain:

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}. \tag{29}$$

Setting $d = 2N$ and letting $z(t)$ be the vector $[x_1(t), \cdots, x_N(t), y_1(t), \cdots, y_N(t)]^*$ and, for each $i \in \{1, \cdots, , N\}$, the $d \times d$ matrix $D_i(t)$ have the same i^{th} and $(i+N)^{\text{th}}$ columns as the matrix $-\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2$, the other columns being zero, it is obvious that system (29) can be rewritten in the form (23) with $M = N$ and $\eta_i = \tau_i$, $1 \le i \le N$. As for initial conditions, we obtain from (26) and the last line of (21) that

$$x_k(t) = \frac{1}{2K_k}\left(K_k\, v_k^0(-\frac{t}{\tau_k}) + i_k^0(-\frac{t}{\tau_k})\right),\ y_k(t) = \frac{1}{2K_k}\left(K_k\, v_k^0(1+\frac{t}{\tau_k}) - i_k^0(1+\frac{t}{\tau_k})\right),\quad t \in [-\tau_k, 0], \tag{30}$$

where we note that both $-t/\tau_k$ and $1 + t/\tau_k$ range over $[0, 1]$ when t ranges over $[-\tau_k, 0]$. The only difference with the situation in Theorem 2.13 is that initial values for x_k, y_k are only provided over $[-\tau_k, 0]$ through (26) and (21), not over $[-\tau_N, 0]$. However, with the previous definitions of $z(t)$ and $D_i(t)$, $1 \le i \le N$, the values of x_k and y_k on $[-\tau_N, -\tau_k)$ when $\tau_k < \tau_N$ are unimportant to the dynamics of (23) for $t \ge 0$, because the columns of $D_i(t)$ other than i^{th} and $(i+N)^{\text{th}}$ are identically zero. Thus, we may pick initial conditions for x_k and y_k on $[-\tau_N, -\tau_k)$ arbitrarily, provided that we comply with summability or continuity requirements. For instance, we can extend x_k and y_k to $[-\tau_N, 0]$ using the operators $J_{[-\tau_k,0]}^{[-\tau_N,0]}$ defined as follows. For $a < b < c$ three real numbers, $J_{[a,b]}^{[a,c]}$ be an extension operator mapping functions on $[a, b]$ to functions on $[a, c]$ so that $L^p([a, b])$ gets mapped into $L^p([a, c])$ and $C^0([a, b])$ into $C^0([a, c])$, in a continuous manner. Such an operator is easily constructed by choosing a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ which is 1 on $(-\infty, b]$ and 0 on $[\min\{2b - a, c\}, +\infty)$; then, for $f : [a, b] \to \mathbb{R}$, define $J_{[a,b]}^{[a,c]} f$ to be f on $[a, b]$ and $\varphi(t)f(2b - t)$ for $t \in (b, c]$, where the product is interpreted as zero if $2b - t < a$. Similarly we define $J_{[b,c]}^{[a,c]}$ mapping functions on $[b, c]$ to functions on $[a, c]$.

We have now reduced the Cauchy problem for (1)-(10), $1 \le k \le N$, to the Cauchy problem for a particular equation of the form (23). Moreover, it is obvious from what precedes that initial conditions in $L^p([0, 1], \mathbb{R})$ (resp. $C^0([0, 1])$ meeting (18)) for v_k, i_k correspond to initial conditions in $L^p([-\tau_N, 0], \mathbb{R}^{2N})$ (resp. $\mathcal{C}$) for z, and that solutions v_k, i_k in $L^p_{loc}([0, \infty), L^p([0, 1]))$ (resp. $C^0([0, \infty), C^0([0, 1])))$ correspond to solutions $z \in L^p_{loc}([0, \infty), \mathbb{R}^{2N})$ (resp. $C^0([0, \infty), \mathbb{R}^{2N})$).

Proof of Theorem 2.8. The above discussion (starting after Theorem 2.13) shows that the function $(t, x) \mapsto (v_1(t, x), \ldots, v_N(t, x), i_1(t, x), \ldots, i_N(t, x))$ is a solution of (1)-(10)-(17) for Part I or (1)-(10)-(19) for part II if and only if

$$v_k(t, x) = x_k(t - x\tau_k) + y_k((x - 1)\tau_k + t)\,, \quad i_k(t, x) = K_k\left(x_k(t - x\tau_k) - y_k((x - 1)\tau_k + t)\right), \tag{31}$$

where $t \mapsto (x_1(t), \ldots, x_N(t), y_i(t), \ldots, y_N(t))$ is a solution of the difference delay system (29) in $L^p_{loc}([0, \infty), \mathbb{R}^{2N})$ or in $C^0([0, \infty), \mathbb{R}^{2N})$, with initial conditions given by (30) and extended if necessary to $[-\tau_N, 0]$ using the operator $J_{[-\tau_k,0]}^{[-\tau_N,0]}$ constructed just before this proof. The result now follows from Theorem 2.13. $\square$

2.4. Exponential stability: definitions

Definition 2.15. Let $\mathbf{A} : [0,\infty) \to \mathbb{R}^{2N\times 2N}$ meet Assumption 2.5 (resp. meet Assumption 2.5 and be continuous). For $1 \le p \le \infty$, System (1)-(10) is said to be L^p (resp. C^0) exponentially stable if and only if there exist $\gamma, K > 0$ such that, for all solutions given by Theorem 2.8 part I (resp. part II), one has:

$$\begin{aligned} &\left\| \left(\widehat{\imath}(t,\cdot), \widehat{v}(t,\cdot)\right)\right\|_{L^p([0,1],\mathbb{R}^2)} \le K e^{-\gamma t} \left\| \left(\widehat{\imath}(0,\cdot), \widehat{v}(0,\cdot)\right)\right\|_{L^p([0,1],\mathbb{R}^2)}, \text{ for all } t \ge 0 \\ &(\text{ resp. } \left\| \left(i(t,\cdot), v(t,\cdot)\right)\right\|_{C^0([0,1],\mathbb{R}^2)} \le K e^{-\gamma t} \left\| \left(i(0,\cdot), v(0,\cdot)\right)\right\|_{C^0([0,1],\mathbb{R}^2)} \text{ for all } t \ge 0). \end{aligned} \tag{32}$$

Definition 2.16. Let the maps $t \mapsto D_i(t)$ meet assumption (2.12) (resp. meet assumption (2.12) and be continuous). System (23) is said to be L^p (resp. C^0) exponentially stable, $p \in [1,\infty]$, if and only if there exist $\gamma, K > 0$ such that, for all solutions given by Theorem 2.13 part (i) (resp.part (ii)), one has:

$$\left(\int_{-\tau_N}^0 \|z(t+\theta)\|^p d\theta\right)^{1/p} \le K e^{-\gamma t} \left(\int_{-\tau_N}^0 \|z(\theta)\|^p d\theta\right)^{1/p}, \ t \ge 0, \qquad \text{if } 1 \le p < \infty, \tag{33}$$

$$\operatorname*{ess\,sup}_{\theta\in[-\tau_N,0]} \|z(t+\theta)\| \le K e^{-\gamma t} \operatorname*{ess\,sup}_{\theta\in[-\tau_N,0]} \|z(\theta)\|, \ t \ge 0, \qquad \text{if } p = \infty, \tag{34}$$

$$(\text{resp. } \sup_{\theta\in[-\tau_N,0]} \|z(t+\theta)\| \le K e^{-\gamma t} \sup_{\theta\in[-\tau_N,0]} \|z(\theta)\|, \ t \ge 0 \ ). \tag{35}$$

Our main concern in this paper is the exponential stability of system (1)-(10), but we shall need the equivalent formulation as a difference delay system of the form (29), which is a particular case of (23). In fact, exponential stability of the two systems are equivalent properties, as asserted by the following proposition.

Proposition 2.17. *System (1)-(10) is L^p exponentially stable (resp. C^0 exponentially stable) if and only if System* (29) *is L^p exponentially stable (resp. C^0 exponentially stable), $1 \le p \le \infty$.*

Proof. This follows at once from (31) expressing solutions of (1)-(10) from solutions of (29) and vice-versa. ☐

3. Results

3.1. Known results in the time-invariant case

The exponential stability of difference delay systems like (23) when the D_i are constant matrices is well understood. Indeed, the following necessary and sufficient condition is classical.

Theorem 3.1 (Henry-Hale Theorem, [13, 15])**.** *If the matrices D_i in system* (23) *do not depend on t, the following properties are equivalent.*

(*i*) *System (23) is L^p exponentially stable for all $p \in [1,+\infty]$.*
(*ii*) *System (23) is C^0 exponentially stable.*
(*iii*) *There exists $\beta < 0$ for which*

$$Id - \sum_{i=1}^N D_i\, e^{-\lambda\tau_i} \text{ is invertible for all } \lambda \in \mathbb{C} \text{ such that } \Re(\lambda) > \beta. \tag{36}$$

Theorem 3.1 is usually stated for C^0 exponential stability only. However, the proof yields L^p exponential stability as well for $1 \le p \le \infty$, see the discussion after [5, eq. (1.11)]. To study the stability of time-invariant networks of 1-dimensional hyperbolic systems, it is standard to convert them into a delay system, much like we did in the previous section, and to apply Theorem 3.1. There is a sizeable literature on this topic, dealing with more general equations with conservation laws than telegrapher's ones, see for instance the textbook [1] and references therein.

For systems of the form (29), if we assume on top of the dissipativity condition (12) that the coupling matrix $\mathbf{A}(t)$ in fact does not depend on t, then Theorem 3.1 applies to yield exponential stability. This is the content of the following proposition, whose (elementary) proof is given in section 4.2 for completeness:

Proposition 3.2. *If the matrix* $\mathbf{A}(t)$ *is constant and condition* (12) *holds, then the constant matrices* D_i *obtained when putting* (29) *into the form* (23) *satisfy* (36) *for some* $\beta < 0$.

3.2. Sufficient stability condition in the time-varying case

Unfortunately, there is no generalization of the Henry-Hale theorem to time-varying difference delay systems of the form (23), even if we assume that the $D_i(t)$ are periodic with the same period, as is the case in the application to electrical networks outlined in the introduction. To the best of our knowledge, there are very few results on the stability of such systems; let us mention two. One is [5, Lemma 3.2]. It gives exponential stability results in Sobolev norms for the class of time-varying delay systems (23) which come from 1-D hyperbolic equations, where the matrices $D_i(\cdot)$ are continuously differentiable and the delays may be time-dependent. Another, extensive reference is [4], which gives a necessary and sufficient condition for L^p exponential stability when $1 \leq p \leq +\infty$ that obviously remains valid for C^0 exponential stability as well. It is stated in terms of the boundedness of sums of products of the $D_i(t_j)$ at delayed time intants t_j, where the number of terms in the sums and products can be arbitrary large. This is akin to an expression of the solution to (23) in terms of the matrices $D_i(.)$ and the initial conditions (see (61) and (62) further below), which looks difficult to bound efficiently in practice because of the tremendous combinatorics and the many cancellations that can occur. In contrast, we only deal here with telegrapher's equations, or with difference delay systems that can be recast as such, but Assumption 2.5 is a much more manageable sufficient condition for exponential stability.

The main result of the paper —see Theorems 3.5 and 3.6 below— asserts L^p exponential stability for all $p \in [1, \infty]$, as well as C^0 exponential stability, for networks of telegrapher's equations with time-varying coupling conditions of the form or (1)-(10) (or (1)-(5)) under Assumption 2.5 (dissipativity at the nodes), and for difference delay systems (23) under conditions that imply that they can be put in the form (29) with the same dissipativity conditions.

It may be interesting to note that the sufficient condition for stability that we give here is independent of the delays when speaking of a difference delay system (Theorem 3.6) or independent of the caracteristics of the lines (constants C_k and L_k) when speaking of networks of telegrapher's equations (Theorem 3.5). Also, these sufficient conditions are not claimed to be necessary.

Let us state these results, preceded by some auxiliary results of independent interest. The proofs not given right after the theorems can be found in Sections 4.3 through 4.6.

The first step is to establish L^2 exponential stability of System (1)-(10) asserted in the following theorem. We give in Section 4.3 a proof using a natural energy functional as Lyapunov function for the telegrapher equations (1). Condition (12) in Assumption 2.5, which has been termed dissipativity without much explanation so far, expresses dissipativity in the sense of this energy functional. We also sketch, in Section 4.4, a second proof, ellaborating on [5, Lemma 3.2], which is exclusively based on the time-varying delay system (29); see the remark at the end of Section 4.4 for a comparison of the two proofs.

Theorem 3.3. *Under Assumption* 2.5*, the time-varying network of telegrapher's equations* (1)-(10)-(8) *is* L^2 *exponentially stable.*

In view of Proposition 2.17, we get as a corollary that L^2 exponential stability holds for difference delay systems of the form (29).

To deduce L^p exponential stability, for all p, from Theorem 3.3, we rely on the following result.

Theorem 3.4. *Under Assumption* 2.12*, System* (23) *is* L^p *exponentially stable for some* $p \in [1, \infty]$ *if and only if it is* L^p *exponentially stable for all such* p*. Moreover, if the maps* $t \mapsto D_i(t)$ *are continuous, then this is also if and only if System* (23) *is* C^0 *exponentially stable.*

The only original bit here is that C^0 exponential stability implies L^p exponential stability for all p, because the first assertion of Theorem 3.4 is essentially contained in [4, Corollary 3.29]. We do consider C^0 stability, because it is the natural one in the application to electronic circuits mentioned in the introduction. Although, again, the first assertion is a consequence of [4, Cor. 3.29], we nevertheless give an independent proof in Section 4.5. Indeed, we feel our argument is simpler than in [4] (the latter paper contains of course other results), and of independent interest. Moreover, our proof shows (for better readability it is not stated in the theorem) that if System (23) is L^p (resp. C^0) polynomially stable of degree $m > N$ for some $p \in [1, \infty]$ (*i.e.* if (33) or (34) (resp. (35)) holds with $e^{-\gamma t}$ replaced by $(1+t)^{-m}$), then it is L^p polynomially stable of degree 1 for all such p (and also C^0 polynomially stable of degree 1).

An obvious corollary of Theorem 3.4, based on Proposition 2.17, is that System (1)-(10) (network of telegrapher's equations) is L^p exponentially stable for some $p \in [1, \infty]$ if and only if it is C^0 exponentially stable and also L^q exponentially stable for all $q \in [1, \infty]$. This leads to our main result regarding network of telegrapher's equations:

Theorem 3.5. *Under Assumption* 2.5*, the time-varying network of telegrapher's equations* (1)-(10)-(8) *is* L^p *exponentially stable for* $1 \le p \le \infty$*, and also* C^0 *exponentially stable if the maps* $t \mapsto \mathbf{A}(t)$ *are continuous.*

Proof. This is a straightforward consequence of Theorem 3.3 and the "obvious corollary" to Theorem 3.4 mentioned just before Theorem 3.5. □

A direct consequence of Theorem 3.5 and Proposition 2.17 is that the same stability properties hold for difference delay systems of the special form (29). It is interesting to restate this in terms of general delay systems of the form (23), making additional assumptions to fall under the scope of the previous result. This is the purpose of Theorem 3.6 below, whose proof is given in Section 4.6. Recall that the matrices $\widehat{D}_j(t)$ were defined from the matrices $D_i(t)$ in Remark 2.14 (*cf.* (25)), and that they differ from the D_i only when some of the delays η_i appear with repetition in (23)).

Theorem 3.6. *Under Assumption* 2.12*, if Conditions* (i) *and* (ii) *below are satisfied, then the time-varying difference delay system* (23) *is* L^p *exponentially stable for all* $p \in [1, \infty]$*. Moreover, if the maps* $t \mapsto D_i(t)$ *are continuous, then it is also* C^0 *exponentially stable.*

- (i) *The columns of the matrices* $\widehat{D}_j(t)$ *are disjoint, i.e. there is a partition* $\{1, \dots, d\} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_{\widehat{M}}$ *(with* $i \ne j \Rightarrow \mathcal{I}_i \cap \mathcal{I}_j = \varnothing$*) such that the* k^{th} *column of* $\widehat{D}_j(t)$ *is identically zero if* $k \notin \mathcal{I}_j$.
- (ii) *The sum of the matrices* $D_i(t)$ *is uniformly contractive:*
$$\textit{there is a number } \nu < 1 \textit{ such that } \left\lvert\!\left\lvert\!\left\lvert \sum_{i=1}^{M} D_i(t) \right\rvert\!\right\rvert\!\right\rvert \le \nu \textit{ for almost all positive } t\,.$$

Here, $\lvert\!\lvert\!\lvert\cdot\rvert\!\rvert\!\rvert$ is the spectral norm for matrices associated to the Euclidean norm on $\mathbb{R}^d$, like in section 2.2.

To recap, Theorem 3.5 offers a sufficient condition for exponential stability of networks of coupled telegrapher's equations, relevant to the study of oscillations in circuits with transmission lines as explained in Section 1, while Theorem 3.6 deals with difference delay systems and applies to an admittedly narrow class thereof (the disjoint columns assumption is clearly restrictive), but is still worth stating for it points at a class of time-varying systems for which relatively simple sufficient conditions for exponential stability can be given. These results are apparently first to give fairly manageable sufficient conditions for exponential stability in the time-varying case. Another contribution is the somewhat simpler approach, provided by Theorem 3.4 and its proof, to the fact that all types of L^p exponential stability, $1 \le p \le \infty$, are equivalent for general time-varying difference delay systems.

4. Proofs

4.1. A technical lemma

Here, the superscript $*$ denotes the transpose of a real matrix, and the spectral norm $|||\cdot|||$ defined at the beginning of section 2.2 is with respect to the canonical Euclidean norm $\|x\| = (x^* x)^{1/2}$.

Lemma 4.1. *If Q is a square matrix satisfying $Q + Q^* > \kappa\, Id > 0$, there is a unique square matrix R solution of*

$$(Id + Q)R = Id - Q\,, \tag{37}$$

and it satisfies $|||R||| < (1-\kappa)/(1+\kappa) < 1$. Conversely, if R is a square matrix satisfying $|||R||| < 1$, there is a unique square matrix Q solution of (37) and it satisfies $Q + Q^* \geq \frac{1-|||R|||}{1+|||R|||}\, Id$.

Proof. It is clear that -1 cannot be an eigenvalue of Q if $Q + Q^* > 0$ or an eigenvalue of R if $|||R||| < 1$. This allows to solve for R or Q using the inverse of $Id + Q$ or $Id + R$.

Now suppose that R and Q satisfy (37). Then $(Id + Q)(Id + R) = 2\,Id$, hence both $Id + Q$ and $Id + R$ are invertible and R commutes with Q, (37) can be re-written $R(Id + Q) = Id - Q$ that readily implies

$$\frac{\|R\,(Id + Q)y\|^2}{\|(Id + Q)y\|^2} = 1 - 2\,\frac{y^*(Q + Q^*)y}{\|(Id + Q)y\|^2}$$

for any nonzero y. Since on the one hand, using invertibility of $Id + Q$, $|||R||| < 1$ if and only if the left-hand side is less than 1 for any nonzero y and on the other hand the right-hand side is less than one if and only if $y^*(Q + Q^*)y$ is positive, one deduces that $Q + Q^* > 0$ and $|||R||| < 1$ are equivalent. $\square$

4.2. Proof of Proposition 3.2

From the very definition of D_i in terms of $\mathbf{A}$, $\mathbf{K}$ and P_2 —see discussion after (29)— we get that

$$\sum_{i=1}^{N} D_i\, e^{-\lambda\tau_i} = [Id + \mathbf{AK}]^{-1}[Id - \mathbf{AK}]\, P_2\ \mathrm{diag}(e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_N}, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_N})\,. \tag{38}$$

In view of (12), (28) and the strict positivity of the K_j, it holds if we set $Q = \mathbf{K}^{1/2}\mathbf{AK}^{1/2}$ that $Q + Q^* \geq \tilde{\alpha} Id$ with $\tilde{\alpha} = \alpha \min_{1\leq j\leq N} K_j > 0$, hence Lemma 4.1 gives us:

$$\left|\left|\left|(Id + \mathbf{K}^{1/2}\mathbf{AK}^{1/2})^{-1}(Id - \mathbf{K}^{1/2}\mathbf{AK}^{1/2})\right|\right|\right| < 1. \tag{39}$$

Consider now the $\mathbf{K}$-norm on $\mathbb{R}^{2N}$, defined for $x \in \mathbb{R}^{2N}$ by $\|x\|_{\mathbf{K}} = \|\mathbf{K}^{\frac{1}{2}} x\|$. Clearly, for any $2N \times 2N$ complex matrix B, the corresponding operator norm is $|||B|||_{\mathbf{K}} = \left|\left|\left|\mathbf{K}^{\frac{1}{2}} B \mathbf{K}^{-\frac{1}{2}}\right|\right|\right|$; it is obviously multiplicative.

Since $[Id + \mathbf{AK}]^{-1}[Id - \mathbf{AK}] = \mathbf{K}^{-\frac{1}{2}}(Id + \mathbf{K}^{\frac{1}{2}}\mathbf{AK}^{\frac{1}{2}})^{-1}(Id - \mathbf{K}^{\frac{1}{2}}\mathbf{AK}^{\frac{1}{2}})\mathbf{K}^{\frac{1}{2}}$, equation (39) implies that

$$\left|\left|\left|[Id + \mathbf{AK}]^{-1}[Id - \mathbf{AK}]\right|\right|\right|_{\mathbf{K}} < 1, \tag{40}$$

consequently there is $\beta < 0$ such that

$$\left|\left|\left|[Id + \mathbf{AK}]^{-1}[Id - \mathbf{AK}]\right|\right|\right|_{\mathbf{K}}\, e^{-\beta\,\tau_N} < 1\,. \tag{41}$$

To see that (36) holds for this β, pick $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \beta$ and observe that

$$\left|\left|\left|P_2\,\mathrm{diag}(e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_N}, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_N})\right|\right|\right|_{\mathbf{K}} \leq e^{-\beta\tau_N}$$

by (3), the multiplicativity of the $\mathbf{K}$-norm and the fact that P_2 commutes with $\mathbf{K}^{1/2}$. Hence, using (38) and (41) together with the multiplicativity of the $\mathbf{K}$-norm, we see that $\left\|\left\| \sum_{i=1}^{N} D_i\, e^{-\lambda\tau_i} \right\|\right\|_{\mathbf{K}} < 1$ which implies (36). $\square$

4.3. **Proof of Theorem 3.3 *via* a Lyapunov functional approach**

Let $(v_1(t,x),\dots,v_N(t,x),i_1(t,x),\dots,i_N(t,x)) \in L^1_{loc}([0,\infty),(L^2([0,1]))^{2N})$ be the solution to (1)-(10)-(8) set forth in Part I of Theorem 2.8, with initial condition $i^0_k, v^0_k \in L^2([0,1])$ for $1\le k\le N$. We define the energy functional E_k in the line k and the global energy E by

$$E_k(t) = \frac{1}{2}\int_0^1 \left[C_k v_k^2(t,x) + L_k i_k^2(t,x)\right] dx\,, \qquad E(t) = \sum_{k=1}^{N} E_k(t)\,. \tag{42}$$

Fact. Each function E_k is locally absolutely continuous and its derivative is given by:

$$\frac{d}{dt}\,E_k(t) = -\widehat{\imath}_k(t,1)\widehat{v}_k(t,1) + \widehat{\imath}_k(t,0)\widehat{v}_k(t,0), \qquad \text{a.e. } t. \tag{43}$$

Proof of the Fact. This would be easy if the solution were smooth (differentiating under the integral sign and using (1) would readily yield (43)), but we have only proved so far, according to Remark 2.11, that E_k is continuous $[0,+\infty)\to\mathbb{R}$ for each k. In particular it defines a distribution on $(0,+\infty)$; let us compute the derivative of this distribution by approximation. By Proposition 2.10, points (i)-(ii), the functions v_k, i_k are of the form (20) with $f_k\in L^2_{loc}((-\infty,1])$ and $g_k\in L^2_{loc}([0,\infty))$. Let $\check{f}_k$ and $\check{g}_k$ extend f_k and g_k by zero to the whole of $\mathbb{R}$, and pick $\phi:\mathbb{R}\to\mathbb{R}$ a positive, C^∞-smooth function, supported on $[-1,1]$ and such that $\int_{\mathbb{R}}\phi = 1$. For each $\varepsilon>0$, we set $\phi_\varepsilon(x) := \phi(x/\varepsilon)/\varepsilon$ (hence, $\int_{\mathbb{R}}\phi_\varepsilon = 1$) and define

$$\widetilde{f}_{k,\varepsilon}(s) := \int_{\mathbb{R}} \check{f}(y)\phi_\varepsilon(s-y)\,dy, \qquad \widetilde{g}_{k,\varepsilon}(s) := \int_{\mathbb{R}} \check{g}(y)\phi_\varepsilon(s-y)\,dy, \qquad y\in\mathbb{R}. \tag{44}$$

Thus, $\widetilde{f}_{k,\varepsilon}$ is a C^∞ smooth functions in $L^2_{loc}(\mathbb{R})$ satisfying $\|\widetilde{f}_{k,\varepsilon}\|_{L^2(K)} \le \|f_k\|_{L^2(K+[-\varepsilon,\varepsilon])}$ for any compact $K\subset\mathbb{R}$, and similarly for $\widetilde{g}_{k,\varepsilon}$. Moreover, $\widetilde{f}_{k,\varepsilon}$ and $\widetilde{g}_{k,\varepsilon}$ converge, both pointwise a.e. and in $L^2_{loc}(\mathbb{R})$, respectively to $\check{f}_k$ and $\check{g}_k$, when $\varepsilon\to 0$. Indeed, it is enough to check this on an arbitrary compact set $K\subset\mathbb{R}$, and since ϕ_ε is supported on $[-\varepsilon,\varepsilon]$ we may redefine $\check{f}_k$ and $\check{g}_k$ as being zero outside the compact set $K+[-\varepsilon,\varepsilon]$ without changing the values of $\widetilde{f}_{k,\varepsilon}$ nor $\widetilde{g}_{k,\varepsilon}$ on K. Thus, it is enough to prove the desired pointwise and L^2_{loc} convergence when $\check{f}_k$ and $\check{g}_k$ lie in $L^2(\mathbb{R})$, in which case the result is standard [17, ch. III, thm. 2]. Next, let us put

$$\widetilde{v}_{k,\varepsilon}(t,x) := \widetilde{f}_{k,\varepsilon}(x-\frac{t}{\tau_k}) + \widetilde{g}_{k,\varepsilon}(x+\frac{t}{\tau_k})\,, \quad \widetilde{\imath}_{k,\varepsilon}(t,x) = K_k\left(\widetilde{f}_{k,\varepsilon}(x-\frac{t}{\tau_k}) - \widetilde{g}_{k,\varepsilon}(x+\frac{t}{\tau_k})\right), \tag{45}$$

so that $\widetilde{v}_{k,\varepsilon}$ and $\widetilde{\imath}_{k,\varepsilon}$ lie in $L^2_{loc}(\mathbb{R}^2)$ and are C^∞ smooth solutions to (1) on $\mathbb{R}^2$, by Proposition 2.10 point (iii). Because $(t,x)\mapsto(x-t/\tau_k, x+t/\tau_k)$ is a bi-Lipschitz homeomorphism of $\mathbb{R}^2$, it preserves compact sets and sets of measure zero. Thus, since $\check{f}_k$ and $\check{g}_k$ coincide respectively with f_k and g_k on $[0,+\infty)\times[0,1]$, the properties of $\widetilde{f}_{k,\varepsilon}$ and $\widetilde{g}_{k,\varepsilon}$ indicated after (44) imply that $\widetilde{v}_{k,\varepsilon}$, $\widetilde{\imath}_{k,\varepsilon}$ respectively converge pointwise a.e. to v_k, i_k on $[0,+\infty)\times[0,1]$, in such a way that $\|\widetilde{v}_{k,\varepsilon}(t,.)\|_{L^2([0,1])}$ and $\|\widetilde{\imath}_{k,\varepsilon}(t,.)\|_{L^2([0,1])}$ remain essentially bounded with t. Therefore, by dominated convergence, we get for every C^∞ smooth compactly supported function $\psi:(0,+\infty)\to\mathbb{R}$ that

$$\lim_{\varepsilon\to 0}\int_0^{+\infty}\!\!\int_0^1 [C_k\widetilde{v}^2_{k,\varepsilon}(t,x) + L_k\widetilde{\imath}^2_{k,\varepsilon}(t,x)]\psi(t)\,\mathrm{d}t\,\mathrm{d}x = \int_0^{+\infty}\!\!\int_0^1 [C_k v_k^2(t,x) + L_k i_k^2(t,x)]\psi(t)\,\mathrm{d}t\,\mathrm{d}x.$$

In other words: when $\varepsilon\to 0$, then $\widetilde{E}_{k,\varepsilon}(t) := \int_0^1 [C_k\widetilde{v}^2_{k,\varepsilon}(t,x) + L_k\widetilde{\imath}^2_{k,\varepsilon}(t,x)]dx$ converges to $E_k(t)$, as a distribution on $(0,+\infty)$. Now, since $\widetilde{\imath}_{k,\varepsilon}$ and $\widetilde{v}_{k,\varepsilon}$ are smooth, the derivative of $t\mapsto\widetilde{E}_{k,\varepsilon}(t)$ can be computed in the strong sense by differentiating under the integral sign; since $\widetilde{\imath}_{k,\varepsilon}$ and $\widetilde{v}_{k,\varepsilon}$ are solutions of the telegrapher's equation (1), an elementary integration yields:

$$\frac{d}{dt}\widetilde{E}_{k,\varepsilon}(t) = -\widetilde{\imath}_{k,\varepsilon}(t,1)\widetilde{v}_{k,\varepsilon}(t,1) + \widetilde{\imath}_{k,\varepsilon}(t,0)\widetilde{v}_{k,\varepsilon}(t,0). \tag{46}$$

By (45) and the Schwarz inequality, the properties of $\widetilde{f}_{k,\varepsilon}$ and $\widetilde{g}_{k,\varepsilon}$ indicated after (44) imply that the right hand side of (46) converges pointwise a.e. and in $L^1_{loc}(\mathbb{R})$ to the function

$$F(t) := K_k\left(f_k^2(-\frac{t}{\tau_k}) - g_k^2(\frac{t}{\tau_k}) - f_k^2(1-\frac{t}{\tau_k}) + g_k^2(1+\frac{t}{\tau_k})\right),$$

and since we know that $\frac{d}{dt}\widetilde{E}_{k,\varepsilon}$ converges to $\frac{d}{dt}E_k$ as a distribution we conclude that $\frac{d}{dt}E_k = F$. In particular, since E_k is a distribution in dimension 1 whose derivative is a locally integrable function, [9, thm. 6.74] implies local absolute continuity and we get from what precedes that $\frac{d}{dt}E_k(t) = F(t)$ for a.e. t, which can be rewritten as (43) in view of (21). This ends the proof of the above fact.

Proof of Theorem 3.3. Adding equalities (43) for $1 \le k \le N$ and considering (42) together with the boundary conditions (16) yields the following equation, where one may indifferently use $\frac{1}{2}\left(\mathbf{A}(t) + \mathbf{A}(t)^*\right)$ or $\mathbf{A}(t)$:

$$\frac{d}{dt}\,E(t) = -\begin{pmatrix} -\widehat{\imath}_1(t,0) \\ \vdots \\ -\widehat{\imath}_N(t,0) \\ \widehat{\imath}_1(t,1) \\ \vdots \\ \widehat{\imath}_N(t,1) \end{pmatrix}^* \frac{\mathbf{A}(t)+\mathbf{A}(t)^*}{2} \begin{pmatrix} -\widehat{\imath}_1(t,0) \\ \vdots \\ -\widehat{\imath}_N(t,0) \\ \widehat{\imath}_1(t,1) \\ \vdots \\ \widehat{\imath}_N(t,1) \end{pmatrix}, \qquad \text{a.e. } t, \tag{47}$$

Using the dissipativity condition (12) in (47) readily implies:

$$\frac{d}{dt}\,E(t) \le -\frac{\alpha}{2}\sum_{k=1}^{N}\left[\widehat{\imath}_k^2(t,0) + \widehat{\imath}_k^2(t,1)\right], \qquad \text{a.e. } t. \tag{48}$$

This entails that the global energy E is decreasing. In order to show that it tends to zero exponentially, let us express E in terms of the functions f_k, g_k as follows. Substituting (20) in (42), we get since $L_kK_k^2 = C_k$ that

$$E_k(t) = C_k\left(\int_0^1 g_k^2(x+\frac{t}{\tau_k})dx + \int_0^1 f_k^2(x-\frac{t}{\tau_k})dx\right). \tag{49}$$

Changing variables to $\tau = x\tau_k + t$ in the first integral and to $\tau = (1-x)\tau_k + t$ in the second, we obtain:

$$E_k(t) = C_k\int_t^{t+\tau_k}\left(g_k^2(\frac{\tau}{\tau_k}) + f_k^2(1-\frac{\tau}{\tau_k})\right)d\tau. \tag{50}$$

Thus, if we define $G : (0,+\infty) \to \mathbb{R}$ by $G(\tau) := \sum_{k=1}^N C_k\big(g_k^2(\tau/\tau_k) + f_k^2(1-\tau/\tau_k)\big)$, we deduce from (50) that

$$E(t) \le \int_t^{t+\tau_N} G(\tau)\,d\tau. \tag{51}$$

In another connection, we get from (20) that $G(\tau)$ can be expressed as a non-negative quadratic form in the $4N$ variables $\widehat{v}_k(\tau,0)$, $\widehat{v}_k(\tau,1)$, $\widehat{\imath}_k(\tau,0)$, $\widehat{\imath}_k(\tau,1)$, for $1 \le k \le N$, with constant coefficients. Hence, using (16) to substitute the $\widehat{v}_k$'s for the $\widehat{\imath}_k$'s, the same $G(\tau)$ can be expressed as a non-negative quadratic form in the $2N$ variables $\widehat{\imath}_k(\tau,0)$, $\widehat{\imath}_k(\tau,1)$, for $1 \le k \le N$, with time-varying essentially bounded coefficients (Assumption 2.5) depending on the matrix $\mathbf{A}(\tau)$ and the constants K_k. This implies:

$$G(\tau) \;\le\; \widetilde{a} \sum_{k=1}^{N} \left(\widehat{\imath}_k^2(\tau,0) + \widehat{\imath}_k^2(\tau,1)\right), \qquad \text{a.e. } \tau > 0. \tag{52}$$

with a positive constant $\widetilde{a}$ that depends only on the coefficients K_k and the bounds on the coefficients of $\mathbf{A}(.)$. Using this inequality in (48) yields

$$\frac{d}{d\tau} E(\tau) \le -\frac{\alpha}{2\,\widetilde{a}}\, G(\tau), \qquad \text{a.e. } \tau > 0. \tag{53}$$

Integrating (53) between t and $t+\tau_N$ we gather, in view of (51), that

$$0 \le E(t+\tau_N) \le (1-\frac{\alpha}{2\,\widetilde{a}})\, E(t), \qquad t > 0. \tag{54}$$

Comparing the expression of $E(t)$ in (42), this last inequality readily implies that system (1)-(10) is L^2 exponentially stable. □

4.4. **Sketch of an alternative proof of Theorem 3.3 *via* time-delay systems exclusively**

First note that, from a straighforward generalization of Equation (40) using the fact that P_2 and $\mathbf{K}$ commute, there is a $\gamma \in (0,1)$ independent of $t \in \mathbb{R}$ such that :

$$\left|\!\left|\!\left| [Id + \mathbf{A}(t)\mathbf{K}]^{-1}[Id - \mathbf{A}(t)\mathbf{K}]\, P_2 \right|\!\right|\!\right|_{\mathbf{K}} \le \gamma < 1. \tag{55}$$

By applying $\|\cdot\|^2_{\mathbf{K}}$ to each side of Equation (29), using (55) above and integrating the resulting inequality between t and t_2 $(-\tau_N < t < t_2)$, one gets after simple algebraic manipulation the following inequality, valid for any $t > 0$ and $t_2 > t$:

$$\int_t^{t_2} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s \le \frac{1}{1-\gamma^2} \int_t^{t+\tau_N} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s\,, \tag{56}$$

in which one may then take $t_2 = +\infty$. This implies L^2 exponential stability of system (29) (one first proves that, for T large enough, $\int_t^{t+T} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s$ converges exponentially to zero) and thus L^2 exponential stability of system (1)-(10) *via* the equivalence between stability of the difference delay system and of the PDE network, see Proposition 2.17.

Remark 4.2. The above proof expounds that of [5, Lemma 3.2], but in essence is not so different from the previous one. Indeed, the quantity $\int_t^{t+T} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s$ acts as a Lyapunov function for (29), although it is not proved be non-increasing with respect to continuous time, while E, that has a similar expression in terms of the delay system (see (49)), is a Lyapunov function in the usual sense for the network of telegrapher's equations, see (48).

4.5. **Proof of Theorem 3.4**

Before proceeding with the proof, we take a closer look at the structure of solutions to System (23).

Given the ordered collection of delays $0 < \tau_1 \le \tau_2 \le \ldots \le \tau_N$, we define the following subsets of $\mathbb{R}$:

$$\Sigma = \{\sum_{i=1}^{N} q_i \tau_i\,,\; (q_1,\ldots,q_N) \in \mathbb{N}^N\} \;\text{ and }\; \Sigma_t = [0,t] \cap \Sigma \quad \text{for } t \text{ in } [0,+\infty). \tag{57}$$

Call $Q(t) \in \mathbb{N}$ the cardinality of Σ_t. Clearly, $Q(t)$ is no larger than the number of N-tuples $(q_1, \ldots, q_N) \in \mathbb{N}^N$ satisfying $\sum_{i=1}^N q_i \leq t/\tau_1$, and the latter is bounded from above by $(1 + [[t/\tau_1]])^N$, where $[[r]]$ indicates the integer part of the real number r. Hence, we have that

$$Q(t) \leq \left(1 + \frac{t}{\tau_1}\right)^N, \qquad t \in [0, +\infty). \tag{58}$$

We enumerate the elements of Σ as a sequence $0 = \sigma_1 < \sigma_2 < \sigma_3 \cdots$, so that Σ_t is described as:

$$\Sigma_t = \{\sigma_1, \sigma_2, \ldots, \sigma_{Q(t)}\}, \qquad t \in [0, +\infty). \tag{59}$$

Our proof of Theorem 3.4 will dwell on the following observation.

Fact. There is a collection of maps $(M_q)_{q\in\mathbb{N}}$ from $\mathbb{R}$ into $\mathbb{R}^{d\times d}$ enjoying properties (i), (ii), (iii) below.

- (i) The map M_q lies in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^{d\times d})$,
- (ii) M_q satisfies
$$t \notin (\sigma_q - \tau_N, \sigma_q] \;\Rightarrow\; M_q(t) = 0, \tag{60}$$
- (iii) the solution $t \mapsto z(t)$ of (23) with initial condition $z(t') = \phi(t')$, $t' \in [-\tau_N, 0]$, is given by

$$z(t) = \sum_{q=1}^{Q(t+\tau_N)} M_q(t)\, \phi(t - \sigma_q) = \sum_{q=1}^{+\infty} M_q(t)\, \phi(t - \sigma_q), \qquad t \geq 0. \tag{61}$$

Formula (61) applies equally well to continuous solutions and to solutions in $L^p_{loc}([0,\infty), \mathbb{R}^d)$, but in the latter case the equality is understood for almost every t. Note that (60) ensures that the two sums in (61) are equal, and also that they do not depend on the values $\phi(y)$ for $y \notin [-\tau_N, 0]$ (which are not defined).

Proof of the Fact. For $0 \leq t < \tau_1$, Equation (23) is of the form (61), with $M_q(t) = D_i(t)$ if $\sigma_q = \tau_i$ and $M_q(t) = 0$ otherwise. If we assume inductively such a formula for $0 \leq t < \sigma_{q_0}$ and substitute it in the right hand side of (23) when $\sigma_{q_0} \leq t < \sigma_{q_0+1}$ to express $z(t)$ as a linear combination of the $\phi(t - \sigma_q)$ for $-\tau_N \leq t - \sigma_q < 0$, a moment's thinking will convince the reader that we get a formula of the same type over the interval $\sigma_{q_0} \leq t < \sigma_{q_0+1}$ by defining $M_q(t)$ as the sum of the coefficients corresponding, after the above substitution, to one and the same $\phi(t - \sigma_q)$ (the latter may arise as many times as there are decompositions $\sigma_q = \tau_i + \sigma_{q'}$ with $i \in \{1, \cdots, N\}$ and $\sigma_{q'} \in \Sigma_{\sigma_{q_0}+\tau_N}$. Such coefficients are of the form $D_i(t) M_{q'}(t)$, and therefore properties (i), (ii) and (iii) are obviously met.

Although we will not need this, it is instructive to derive an explicit expression for M_q that should be compared with [4, thm. 3.14] or, in the continuous case, with [13, ch. 9, eqns. (1.4)-(1.5)]. Namely, we can take $M_q(t)$ to be the sum of all terms

$$\mathbb{1}_{(\rho_{s-1}, \rho_s]}(t)\, D_{k_1}(t) D_{k_2}(t - \rho_1) D_{k_3}(t - \rho_2) \cdots D_{k_s}\big(t - \rho_{s-1}\big) \tag{62}$$

for all s in $\mathbb{N} \setminus \{0\}$ and all s-tuples $(k_1, \ldots, k_s) \in \{1, \cdots, N\}^s$ such that $\sum_{j=1}^s \tau_{k_j} = \sigma_q$, where the numbers ρ_j are defined by $\rho_0 = 0$ and $\rho_j = \sum_{i=1}^j \tau_{k_i}$ for $j \geq 1$ (in particular $\rho_s = \sigma_q$), and $\mathbb{1}_{(\rho_{s-1}, \rho_s]}$ is the characteristic function of the interval $(\rho_{s-1}, \rho_s]$. These maps M_q satisfies (60) because $(\rho_{s-1}, \rho_s] = (\sigma_q - \tau_{k_s}, \sigma_q]$ is a subset of $(\sigma_q - \tau_N, \sigma_q]$, and formula (61) is easily checked from (23), by induction on j such that $t \in (\sigma_{j-1}, \sigma_j]$.

Proof of Theorem 3.4. Assume first that $1 \leq p < \infty$. If System (23) is L^p exponentially stable, there is by definition $\gamma > 0$ and $C_0 > 0$ such that, for all $\phi \in L^p([-\tau_N, 0], \mathbb{R}^d)$ and all $t > 0$, one has

$$\Big(\int_{t-\tau_N}^t \|z(u)\|^p du\Big)^{1/p} \leq C_0\, e^{-\gamma t} \|\phi\|_{L^p([-\tau_N,0],\mathbb{R}^d)} \tag{63}$$

for $z(.)$ the unique solution of (23) with initial condition ϕ given by Theorem 2.13. Pick $t_\star \in (-\tau_N, 0)$, $v \in \mathbb{R}^d$, $\varepsilon > 0$, and define a function $\phi_{t_\star,v,\varepsilon} \in L^p([-\tau_N, 0], \mathbb{R}^d)$ by

$$\phi_{t_\star,v,\varepsilon}(\theta) = \frac{1}{\varepsilon^{1/p}} \mathbb{1}_{(t_\star-\varepsilon,t_\star)}(\theta)\, v\,, \qquad \theta \in [-\tau_N, 0]. \tag{64}$$

Let $z_{t_\star,v,\varepsilon}(.)$ be the solution to (23) with initial condition $\phi_{t_\star,v,\varepsilon}$ on $[-\tau_N, 0]$. By (63), it holds that

$$\Big(\int_{t-\tau_N}^{t} \|z_{t_\star,v,\varepsilon}(u)\|^p du\Big)^{1/p} \leq C_0\, e^{-\gamma t}\|v\|\,, \qquad t > 0, \tag{65}$$

and from (61) we get for all $u > 0$ that

$$z_{t_\star,v,\varepsilon}(u) \;=\; \frac{1}{\varepsilon^{1/p}} \left(\sum_{q=1}^{+\infty} \mathbb{1}_{(t_\star-\varepsilon,t_\star)}(u-\sigma_q)\; M_q(u)\right) v. \tag{66}$$

Let us fix $t > 0$ for a while. By (60), the only terms in the sum on the right of (66) which may not be zero for a.e. $u \in (t-\tau_N, t)$ are such that $E(q, t_\star, \varepsilon) := (\sigma_q + t_\star - \varepsilon, \sigma_q + t_\star) \cap (\sigma_q - \tau_N, \sigma_q] \cap (t-\tau_N, t)$ has strictly positive measure. The set of integers q for which this holds for some $t_\star \in (-\tau_N, 0)$ and some $\varepsilon > 0$ consists exactly of those q such that $t - \tau_N < \sigma_q < t + \tau_N$. If we pick one of them and, say, $\sigma_q \geq t$, it is easy to check that for ε small enough $E(q, t_\star, \varepsilon) = (t_\star + \sigma_q - \varepsilon, t_\star + \sigma_q)$ when $t_\star \leq t - \sigma_q$ and $E(q, t_\star, \varepsilon) = \varnothing$ when $t_\star > t - \sigma_q$. If on the contrary $\sigma_q < t$, then $E(q, t_\star, \varepsilon) = (t_\star + \sigma_q - \varepsilon, t_\star + \sigma_q)$ when $t_\star > t - \sigma_q - \tau_N$ and $E(q, t_\star, \varepsilon) = \varnothing$ when $t_\star \leq t - \sigma_q - \tau_N$. Altogether, since there are finitely many q under examination (*i.e.* at most $Q(t + \tau_N)$), we can take $\varepsilon > 0$ so small that all intervals $E(q, t_\star, \varepsilon)$ are disjoint, and then we deduce from (66) and the previous discussion that

$$\int_{t-\tau_N}^{t} \|z_{t_\star,v,\varepsilon}(u)\|^p du = \sum_{\{q:\; t-\tau_N-t_\star<\sigma_q\leq t-t_\star\}} \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} \|M_q(t_\star + \sigma_q + \theta)\, v\|^p \mathrm{d}\theta. \tag{67}$$

Observe next that a.e. $t_\star \in (-\tau_N, 0)$ is a Lebesgue point of (each entry of) $s \mapsto M_q(s + \sigma_q)$ for all $q \in \mathbb{N}$, and let E denote the set of such points. By the triangle inequality, E *a fortiori* consists of Lebesgue points of $s \mapsto \|M_q(s + \sigma_q)v\|$, and since $M_q \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^{d\times d})$ it also consists of Lebesgue points of $s \mapsto \|M_q(s + \sigma_q)v\|^p$, by the smoothness of $x \to x^p$ for $x > 0$. Thus, from (65) and (67), we deduce on letting $\varepsilon \to 0$ that

$$t - \tau_N - t_\star < \sigma_q \leq t - t_\star \;\Rightarrow\; \|M_q(t_\star + \sigma_q)v\| \leq C_0 e^{-\gamma t}\|v\|, \qquad t_\star \in E. \tag{68}$$

Now, choose $\sigma_q \in \Sigma$ and $t_\star \in E$. We can find $t > 0$ such that $t - \tau_N < \sigma_q + t_\star < t$ and then, applying what precedes with this t and this $t_\star$, we obtain in view of (68) that

$$\|M_q(t_\star + \sigma_q)v\| \leq C_0 e^{-\gamma t}\|v\| \leq C_0 e^{-\gamma(t_\star+\sigma_q)}\|v\|. \tag{69}$$

As E has full measure in $(-\tau_N, 0]$ and $v \in \mathbb{R}^d$ is arbitrary, we conclude from (69) and (60) that

$$|||M_q(s)||| \leq C_0\, e^{-\gamma s}, \qquad \text{a.e. } s > 0, \tag{70}$$

where $|||\cdot|||$ is the spectral norm for matrices on Euclidean space. Because the number of summands in the middle term of (61) is $Q(t + \tau_N)$ which is bounded above by Ct^N for some constant C, as asserted in (58), the inequality (70) implies that to any $\gamma' \in (0, \gamma)$ there is a constant $C_1 > 0$ for which

$$\|z\|_{L^\lambda((t-\tau_N,t),\mathbb{R}^d)} \leq C_1\, e^{-\gamma' t}\|\phi\|_{L^\lambda((-\tau_N,0),\mathbb{R}^d)}, \qquad t \geq 0, \qquad 1 \leq \lambda < \infty, \tag{71}$$

$$\text{and also} \quad \|z\|_{L^\infty((t-\tau_N,t),\mathbb{R}^d)} \leq C_1 e^{-\gamma' t}\|\phi\|_{L^\infty((-\tau_N,0),\mathbb{R}^d)}, \qquad t \geq 0. \tag{72}$$

Since (72) readily implies C^0 exponential stability when the maps $D_i(.)$ are continuous, this achieves the proof when $1 \le p < \infty$.

Assume now that $p = \infty$, so that (63) gets replaced by $\|z\|_{L^\infty((t-\tau_N,t),\mathbb{R}^d)} \le C_0 e^{-\gamma t}\|\phi\|_{L^\infty((-\tau_N,0),\mathbb{R}^d)}$ for all $t > 0$. The goal is again to prove (70) from which the result follows, as we just saw. For this, we argue much like we did before, defining $\phi_{t_\star,v,\varepsilon}$ as in (64) except that we do not divide by $\varepsilon^{1/p}$. Then, (65) becomes

$$\operatorname*{ess.\,sup}_{\alpha\in(t-\tau_N,t)} \|z_{t_\star,v,\varepsilon}(\alpha)\| \le C_0 e^{-\gamma t}\|v\|, \tag{73}$$

and the discussion that led us to (67) now yields for $\varepsilon > 0$ small enough:

$$\operatorname*{ess.\,sup}_{\alpha\in(t-\tau_N,t)} \|z_{t_\star,v,\varepsilon}(\alpha)\| = \max_{\{q:\ t-\tau_N-t_\star<\sigma_q\le t-t_\star\}} \ \operatorname*{ess.\,sup}_{\theta\in(-\varepsilon,0)} \|M_q(t_\star+\sigma_q+\theta)\,v\|. \tag{74}$$

We need now to replace Lebesgue points by points of approximate continuity. Recall that a function $f : \mathbb{R} \to \mathbb{R}^m$ is approximately continuous at x if, for every $\epsilon > 0$,

$$\lim_{r\to 0^+} \frac{1}{2\,r}\,\mathcal{H}^1\Big((x-r,x+r)\cap\{y : \|f(y)-f(x)\| > \epsilon\}\Big) = 0$$

where $\mathcal{H}^1$ is the Lebesgue measure on $\mathbb{R}$, and that a measurable f is approximately continuous at almost every point [10, thm. 1.37]. Thus, if we define $E \subset (-\tau_N, 0)$ to be the set of approximate continuity points of all maps $s \mapsto M_q(s+\sigma_q)$ (say, extended by zero off $(-\tau_N,0)$) as q ranges over $\mathbb{N}$, then E has full measure in $(-\tau_N, 0)$ and letting $\varepsilon \to 0$ in (74) we find that (68) holds. From the latter we obtain (70) by the same reasoning as before, thereby completing the proof when $p = \infty$.

It remains to handle the case where the $D_i(t)$ are continuous and System (23) is C^0 exponentially stable. Then, the previous argument needs adjustment because $\phi_{t_\star,v,\varepsilon} \notin \mathcal{C}$. However, it is easy to construct a sequence of continuous functions $\varphi_k : [-\tau_N, 0] \to [0, 1]$, with $\varphi_k(0) = \varphi_k(-\tau_j) = 0$ for $1 \le j \le N$, such that φ_k converges pointwise a.e. to $\mathbb{1}_{(t_\star-\varepsilon,t_\star)}$ when $k \to +\infty$ (for instance, we may take piecewise linear φ_k). Then, the $\phi_k(\theta) := \varphi_k(\theta)v$ lie in $\mathcal{C}$, and if z_k denotes the solution to System (23) with initial condition ϕ_k, we get by assumption that $\sup_{\alpha\in(t-\tau_N,t)} \|z_k(\alpha)\| \le C_0 e^{-\gamma t}\|v\|$. As ϕ_k converges pointwise a.e. to $\phi_{t_\star,v,\varepsilon}$ on $[-\tau_N, 0]$, we see from (61) that z_k converges to $z_{t_\star,v,\varepsilon}$ pointwise a.e. on $\mathbb{R}$. Thus, letting $k \to +\infty$, we deduce that (73) holds and we conclude as before. □

4.6. **Proof of Theorem 3.6**

First assume that each set $\mathcal{I}_j$ has even cardinality $2\,n_j$, and put $N = \sum_{j=1}^{\widehat{M}} n_j$ so that $d = 2N$. Let P_3 be the permutation matrix sending $\mathcal{I}_1$ to $\{1,\dots,n_1\}\cup\{N+1,\dots,N+n_1\}$ and, more generally, $\mathcal{I}_j$ to $\{1+\sum_{\ell=1}^{j-1} n_\ell,\dots,\sum_{\ell=1}^{j} n_\ell\}\cup\{N+1+\sum_{\ell=1}^{j-1} n_\ell,\dots,N+\sum_{\ell=1}^{j} n_\ell\}$ for each j. Set $\tau_k = \eta_j$ for each k in $\{1+\sum_{\ell=1}^{j-1} n_\ell,\dots,\sum_{\ell=1}^{j} n_\ell\}$. Using P_3 as change of basis and denoting by $(x_1,\dots,x_N,y_1,\dots,y_N)$ the new coordinates, one can (by gathering the matrices with disjoint nonzero columns into a single one) re-write (75) as (29) where $-\left(I+\mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I-\mathbf{A}(t)\,\mathbf{K}\right)P_2$ has been replaced with $\sum_{j=1}^{\widehat{M}} {P_3}^{-1}\,\widehat{D}_j(t)\,P_3$. We want now to find $\mathbf{A}(t)$ and $\mathbf{K}$ so that these two matrices coincide. For this, we fix $\mathbf{K} = Id$ and solve $\left(I+\mathbf{A}(t)\right)^{-1}\left(I-\mathbf{A}(t)\right) = R(t)$ with respect to $\mathbf{A}(t)$, where $R(t) = -{P_3}^{-1}\left(\sum_{j=1}^{\widehat{M}} \widehat{D}_j(t)\right)P_3{P_2}^{-1} = -{P_3}^{-1}\left(\sum_{i=1}^{M} D_i(t)\right)P_3{P_2}^{-1}$ (the last equality is clear from the definition of $\widehat{D}_i(t)$ in (25)). Assumption (ii) implies $|||R(t)||| \le \nu < 1$ because P_2 and P_3 are orthogonal matrices, hence, according to Lemma 4.1, setting $\mathbf{A}(t) = (Id-R(t))(Id+R(t))^{-1}$ solves the above and satisfies Assumption 2.5 with $\alpha = (1-\nu)/(1+\nu)$; (28) is satisfied too with $\mathbf{K} = Id$, setting all the numbers K_k to 1. By virtue of Proposition 2.17 and Theorem 3.5, the difference delay equation (29) with these

$\mathbf{A}(.)$ and τ_k is L^p exponentially stable for all $p \in [1, \infty]$, as well as C^0 exponentially stable if the maps $D_i(.)$ (hence $\mathbf{A}(.)$) are continuous. This proves the result if all the sets $\mathcal{I}_j$ has even cardinality.

If some of the sets $\mathcal{I}_j$ have odd cardinality, define $d' > d$ so that $d' - d$ is the number of such sets $\mathcal{I}_j$. By adjoining to each such $\mathcal{I}_j$ one element of $\{d+1, \ldots, d'\}$, one constructs a partition $\widetilde{\mathcal{I}}_1, \ldots, \widetilde{\mathcal{I}}_{\widehat{M}}$ of $\{1, \ldots, d'\}$ such that, for each j, $\widetilde{\mathcal{I}}_j$ has even cardinality and contains $\mathcal{I}_j$. Constructing some $d' \times d'$ matrices $\widetilde{D}_j(t)$ by adding $d' - d$ zero last lines and $d' - d$ zero last columns to $\widehat{D}_j(t)$, the following difference delay system (with state $\widetilde{z}$ in $\mathbb{R}^{d'}$):

$$\widetilde{z}(t) = \sum_{j=1}^{\widehat{M}} \widetilde{D}_j(t) \widetilde{z}(t - \eta_j) \tag{75}$$

satisfies the assumptions of the theorem: (i) with the sets $\widetilde{\mathcal{I}}_j$ instead of the original sets $\mathcal{I}_j$ and (ii) because adding zero lines and columns to a matrix does not increase its norm, hence the first part of the proof gives exponential stability, that yields exponential stability of the original system because, since the last $d' - d$ columns are zero, the evolution of the d first entries of z does not depend on the last ones. $\square$

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References

[1] G. Bastin and J.-M. Coron, *Stability and boundary stabilization of 1-D hyperbolic systems*, vol. 88 of Progress in Nonlinear Differential Equations and their Applications, Springer, 2016.

[2] R. K. Brayton, *Small-signal stability criterion for electrical networks containing lossless transmission lines*, IBM Journal of Research and Development, 12 (1968), pp. 431–440.

[3] A. Bressan, *Hyperbolic systems of conservation laws*, vol. 20 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[4] Y. Chitour, G. Mazanti, and M. Sigalotti, *Stability of non-autonomous difference equations with applications to transport and wave propagation on networks*, Netw. Heterog. Media, 11 (2016), pp. 563–601.

[5] J.-M. Coron and H.-M. Nguyen, *Dissipative boundary conditions for nonlinear 1-d hyperbolic systems: sharp conditions through an approach via time-delay systems*, SIAM J. Math. Anal., 47 (2015), pp. 2220–2240.

[6] ———, *Optimal time for the controllability of linear hyperbolic systems in one-dimensional space*, SIAM J. Control Optim., 57 (2019), pp. 1127–1156.

[7] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, *Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping*, SIAM Journal on Control and Optimization, 51 (2013), pp. 2005–2035.

[8] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer Berlin Heidelberg, 2010.

[9] F. Demengel and G. Demengel, *Espaces fonctionnels, Utilisation dans la résolution des équations aux dérivées partielles*, EDP Sciences, 2007.

[10] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[11] E. Fridman and G. Tsodik, *H_∞ control of distributed and discrete delay systems via discretized Lyapunov functional*, European Journal of Control, 15 (2009), pp. 84 – 94.

[12] S. Fueyo, *PhD. thesis*. Under preparation.

[13] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, vol. 99 of Applied Mathematical Sciences, Springer-Verlag, New York, 1993.

[14] ———, *Strong stabilization of neutral functional differential equations*, IMA J. Math. Control Inform., 19 (2002), pp. 5–23. Special issue on analysis and design of delay and propagation systems.

[15] D. Henry, *Linear autonomous neutral functional differential equations*, J. Differential Equations, 15 (1974), pp. 106–128.

[16] V. Răsvan and S.-I. Niculescu, *Oscillations in lossless propagation models: a Liapunov–Krasovskii approach*, IMA J. Math. Control Inform., 19 (2002), pp. 157–172.

[17] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[18] A. Suarez, *Analysis and Design of Autonomous Microwave Circuits*, Wiley-IEEE Press, 2009.