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DECOMPOSITION OF L^2-VECTOR FIELDS ON LIPSCHITZ SURFACES: CHARACTERIZATION VIA NULL-SPACES OF THE SCALAR POTENTIAL*

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Abstract. For $\partial\Omega$ the boundary of a bounded and connected strongly Lipschitz domain in $\mathbb{R}^d$ with $d \geq 3$, we prove that any field $f \in L^2(\partial\Omega;\mathbb{R}^d)$ decomposes, in a unique way, as the sum of three invisible vector fields—fields whose magnetic potential vanishes in one or both components of $\mathbb{R}^d \setminus \partial\Omega$. Moreover, this decomposition is orthogonal if and only if $\partial\Omega$ is a sphere. We also show that any f in $L^2(\partial\Omega;\mathbb{R}^d)$ is uniquely the sum of two invisible fields and a Hardy function, in which case the sum is orthogonal regardless of $\partial\Omega$; we express the corresponding orthogonal projections in terms of layer potentials. When $\partial\Omega$ is a sphere, both decompositions coincide and match what has been called the Hardy–Hodge decomposition in the literature.

Key words. Hardy–Hodge decomposition, Hardy spaces, potential theory, orthogonal sum decomposition, harmonic functions, layer potentials

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1. Introduction.

Orthogonal sum decompositions and the method of orthogonal projections are important constructive tools in harmonic analysis. Introduced into potential theory by Weyl [1] and further developed by Vishik [2, 3] and Gårding [4], this method now serves as a powerful computational approach to elliptic and parabolic boundary value problems (see, for example, [5, 6]).

On a sphere, there is an orthogonal decomposition for square integrable vector fields which is of particular interest to magnetic inverse problems. Namely, the decomposition of a magnetization (modeled as a square integrable field) into contributions that do not create a magnetic field inside or outside of the sphere. This decomposition turns out to coincide with another one, connected to Hodge theory and referred to below as the Hardy–Hodge decomposition. Hereby a vector field is expressed as the sum of two harmonic gradients (one from each side of the sphere) and a tangent divergence free term. In this paper, we generalize these two decompositions to compact Lipschitz surfaces, though as we shall see they differ when the surface is no longer a sphere.

The first decomposition naturally occurs in the following setting. Consider an object whose surface is magnetized. Such a surface magnetization can be of two types: either it will be "visible," meaning that it will create a magnetic field inside and outside of the object; or it will be "invisible," meaning that its magnetic field will vanish identically inside or outside. Inverse problems in magneto-statics are nonunique precisely because invisible magnetizations exist, and for a generic surface it

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is important to understand when a magnetization is invisible. In this paper we answer this question for a fairly general class of Lipschitz surfaces.

To make the statement more precise, let us consider a domain Ω with boundary $\partial\Omega$, and let us model magnetizations supported on $\partial\Omega$ by L^2 (i.e., square integrable) vector fields. We denote the space of L^2-fields that are invisible everywhere by $D(\partial\Omega) = \mathcal{N}(\mathcal{P}_i) \cap \mathcal{N}(\mathcal{P}_c)$ (for the notation, see Definition 3.1 and the remark that follows it). We also introduce another two spaces which are orthogonal to $D(\partial\Omega)$: the space of L^2-fields invisible outside of Ω, denoted $O(\partial\Omega) = \mathcal{N}(\mathcal{P}_c) \cap D(\partial\Omega)^\perp$; and the space of L^2-fields invisible inside of Ω, denoted $I(\partial\Omega) = \mathcal{N}(\mathcal{P}_i) \cap D(\partial\Omega)^\perp$.

When Ω is a ball, so that $\partial\Omega$ is a sphere, it is known that $I(\partial\Omega)$ and $O(\partial\Omega)$ are mutually orthogonal and that the space of $\mathbb{R}^3$-valued square integrable fields splits into an orthogonal direct sum

$$L^2(\partial\Omega; \mathbb{R}^3) = I(\partial\Omega) + O(\partial\Omega) + D(\partial\Omega). \tag{1}$$

In other words every L^2-field on the sphere is the sum of three invisible magnetizations, one that is invisible inside, one that is invisible outside, and one that is invisible everywhere. Moreover, the nonzero members of $I(\partial\Omega)$ (resp., $O(\partial\Omega)$) are precisely those magnetizations that one can "see" from the outside (resp., inside). Because of this, decomposition (1) becomes an important tool to solve magnetic inverse problems on a sphere or a plane (a sphere with infinite radius). For example, in medical imaging it is used to process EEG/MEG measurements on spherical human head models [7], in scanning magnetic microscopy it arises in the study of planetary rock samples [8, 9], it has been used to separate magnetic fields measured on satellite orbits with respect to their sources [10, 11, 12, 13], and to invert magnetic fields for spherical source currents [14], as well as for the lithospheric magnetization [15, 16, 17, 18].

The second decomposition mentioned above, the Hardy–Hodge decomposition, goes as follows:

$$L^2(\partial\Omega; \mathbb{R}^d) = H^2_+(\partial\Omega) + H^2_-(\partial\Omega) + D_f(\partial\Omega); \tag{2}$$

see [9, 15, 19] for the case of a plane or a sphere in various smoothness classes. In (2), $D_f(\partial\Omega)$ denotes the space of tangent divergence-free fields on $\partial\Omega$ while $H^2_+(\partial\Omega)$ and $H^2_-(\partial\Omega)$ are the harmonic Hardy spaces, initially introduced in [20] on half-spaces (see also [21] as well as [22] for spheres) and later studied over C^1-hypersurfaces in [23, 24], and on Lipschitz surfaces in [25, 26]. In a nutshell, these spaces define vector fields whose harmonic extension is a gradient, inside or outside of the sphere. From the point of view of mathematical analysis, the Hardy–Hodge decomposition is an interesting tool in itself. In fact, it features the analogue of the Riesz transforms which are the standard Calderon–Zygmund operators that gave rise to the elliptic regularity theory [21, 22]. On the sphere, it holds that $I(\partial\Omega) = H^2_+(\partial\Omega)$ and $O(\partial\Omega) = H^2_-(\partial\Omega)$, which makes (2) adequate for various inverse problems.

If the surface $\partial\Omega$ is not a sphere, the decomposition (1) seems new. Also, the Hardy–Hodge decomposition (2) is apparently unpublished when $\partial\Omega$ is merely Lipschitz (see [27] for a version in L^p). Consequently, the relation between the spaces $I(\partial\Omega)$, $O(\partial\Omega)$ and $H^2_+(\partial\Omega)$, $H^2_-(\partial\Omega)$ has not been studied either. In this paper, we address these questions for Lipschitz surfaces and vector fields of L^2-class. This lays the groundwork to extend existing analytical techniques and applications known on the sphere to Lipschitz surfaces.

Our main result proves the following statement that we separated into several theorems for the sake of clarity.

MAIN RESULT. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d \geq 3)$ with connected boundary $\partial\Omega$. The space $L^2(\partial\Omega;\mathbb{R}^d)$ of square integrable $\mathbb{R}^d$-valued vector fields on $\partial\Omega$ decomposes into the following orthogonal direct sums:*

$$L^2(\partial\Omega;\mathbb{R}^d) = I(\partial\Omega) \oplus H^2_-(\partial\Omega) \oplus D(\partial\Omega) \qquad (\textit{Corollary } 3.6),$$
$$L^2(\partial\Omega;\mathbb{R}^d) = H^2_+(\partial\Omega) \oplus O(\partial\Omega) \oplus D(\partial\Omega) \qquad (\textit{Corollary } 3.17).$$

The corresponding orthogonal projections are computed in Theorems 3.8 *and* 3.18*. Moreover, $L^2(\partial\Omega;\mathbb{R}^d)$ also decomposes into the topological direct sums*

$$L^2(\partial\Omega;\mathbb{R}^d) = H^2_+(\partial\Omega) + H^2_-(\partial\Omega) + D_f(\partial\Omega) \qquad (\textit{Theorem } 4.1),$$
$$L^2(\partial\Omega;\mathbb{R}^d) = I(\partial\Omega) + O(\partial\Omega) + D(\partial\Omega) \qquad (\textit{Theorem } 4.3).$$

These topological direct sums are orthogonal if and only if $\partial\Omega$ is a sphere (Theorem 5.1 *and Corollary* 5.2*), in which case $H^2_+(\partial\Omega) = I(\partial\Omega)$ and $H^2_-(\partial\Omega) = O(\partial\Omega)$ (Corollary* 5.3*). Furthermore, regardless of $\partial\Omega$, provided it is Lipschitz, it holds that $D(\partial\Omega) = D_f(\partial\Omega)$ (Lemma* 4.2*), and in the above sums this term is always orthogonal to the other two.*

In section 2, we set up notational conventions and discuss layer potentials. The latter are well studied in the literature (for example, [25, 26, 28]) and lie at the core of all arguments in this paper. The orthogonal decompositions are treated in section 3 and the skew-orthogonal decompositions in section 4. The case of a sphere, where all mentioned decompositions coincide, is treated in section 5.

2. Preliminaries and notation.

2.1. Conventions. In this paper we reserve the symbol Ω to denote a bounded strongly Lipschitz domain in the d-dimensional Euclidean space $\mathbb{R}^d$ $(d \geq 3)$. Throughout, we assign the symbol $\partial\Omega$ to denote the boundary of Ω, which is a connected and closed hypersurface, locally given as the graph of a Lipschitz function (see Appendix A for more details). We call Ω the inside of $\partial\Omega$ and $\bar{\Omega}^c = \mathbb{R}^d\backslash\overline{\Omega}$ the outside. The surface measure on $\partial\Omega$ will be denoted by σ; it is the restriction to $\partial\Omega$ of the $(d-1)$-Hausdorff measure; since $\partial\Omega$ is compact, σ is finite. Statements made almost everywhere (a.e.) on $\partial\Omega$ are always understood with respect to σ.

On $\mathbb{R}^d$, we denote the Euclidean scalar product by $\langle x,y\rangle_{\mathbb{R}^d}$ and the Euclidean norm by $|x| = \sqrt{\langle x,x\rangle_{\mathbb{R}^d}}$. If f is a differentiable function defined on an open region in $\mathbb{R}^d$, then ∇f denotes the Euclidean gradient of f. We denote the Euclidean Laplacian by Δ that reads in coordinates $\Delta f = \sum_{i=1}^d \partial^2 f/\partial x_j^2$.

The space $L^2(\partial\Omega)$ $\big(= L^2(\partial\Omega,\sigma)\big)$ is the space of scalar-valued square integrable functions on $\partial\Omega$. The subspace $L^2_{\mathrm{ZM}}(\partial\Omega) \subset L^2(\partial\Omega)$ consists of functions with zero mean. We call an $\mathbb{R}^d$-valued function on $\partial\Omega$ a field, and we denote the space of square integrable fields by $L^2(\partial\Omega;\mathbb{R}^d)$ $\big(= L^2\big(\partial\Omega;\mathbb{R}^d,\sigma\big)\big)$. If f and g are two fields in $L^2(\partial\Omega;\mathbb{R}^d)$, their scalar product is $\langle f,g\rangle = \int_{\partial\Omega}\langle f(q),g(q)\rangle_{\mathbb{R}^d}\, d\sigma(q)$ and the norm of f in $L^2(\partial\Omega;\mathbb{R}^d)$ is $\|f\| = \sqrt{\langle f,f\rangle}$.

The tangent space to $\partial\Omega$ at x is denoted by $\mathrm{T}_x(\partial\Omega)$; it is well defined a.e. and so is the outer unit normal field η (see Appendix A for more details). We identify $\mathrm{T}_x(\partial\Omega)$ with a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. If f is a field in $L^2(\partial\Omega;\mathbb{R}^d)$ and for a.e. x the vector $f(x)$ lies in $\mathrm{T}_x(\partial\Omega)$, we call f a tangent field. We denote the space of all tangent fields by $T(\partial\Omega)$; it is a closed subspace of $L^2(\partial\Omega;\mathbb{R}^d)$. We will often split a field $f \in L^2(\partial\Omega;\mathbb{R}^d)$ into a normal part $f_\eta \in L^2(\partial\Omega)$ and the tangent part $f_{\mathrm{T}} \in T(\partial\Omega)$, and write $f = \eta f_\eta + f_{\mathrm{T}}$.

A Lipschitz function $f : \partial\Omega \to \mathbb{R}$ is differentiable a.e. on $\partial\Omega$. Therefore, it has a well defined tangential gradient $\nabla_{\mathrm{T}} f(x) \in \mathrm{T}_x(\partial\Omega)$ at almost every x. We define the Sobolev space $W^{1,2}(\partial\Omega)$ on $\partial\Omega$ to be the completion of Lipschitz functions for the norm $(\|f\|^2 + \|\nabla_{\mathrm{T}} f\|^2)^{1/2}$. If f is in $W^{1,2}(\partial\Omega)$, then f is in $L^2(\partial\Omega)$ and $\nabla_{\mathrm{T}} f$ is a tangent field in $T(\partial\Omega)$ (see Appendix A for details).

If A is a bounded operator, then $\mathcal{N}(A)$ denotes its null-space and $\mathcal{R}(A)$ its range. For N a subspace of a Hilbert space, we write $N^\perp$ to denote the orthogonal complement of N. When saying that A is invertible, we always mean that A has a bounded inverse.

Regular family of cones. We will use a regular family of cones. Even though this family does not explicitly appear in what follows, it is fundamental to the very definition of boundary values on $\partial\Omega$ and the limiting behavior of potentials that we invoke repeatedly. The existence of a regular family of cones is folklore, but it is hard to locate a proof; see [27].

More precisely, for $\theta \in (0, \pi/2)$ and $y, z \in \mathbb{R}^d$ with $|z| = 1$, we put $C_{\theta,z}(y)$ for the open, right circular, positive cone with vertex at y, axis directed by z, and aperture angle 2θ; the cone being truncated to some fixed suitable length. We do not make the length explicit in the notation, for it plays no role provided that it is small enough (how small depends on Ω). Covering $\partial\Omega$ with finitely many open sets O_j whose intersection with $\partial\Omega$ is the graph of a Lipschitz function, to each point in $\partial\Omega \cap O_j$ we can attach two "natural" cones with fixed aperture such that (i) their direction is that of the graph or opposite to it and (ii) their length is small enough that one of them lies in Ω and the other in Ω^c.

Let $\mathbb{S}^{d-1}$ denote the $(d-1)$-dimensional unit sphere, and let $Z\colon \partial\Omega \to \mathbb{S}^{d-1}$ be a continuous function with the following property: For some fixed $\theta_1 < \theta < \theta_2$ independent of q, we require that within some O_j containing q the cone $C_{\theta,\pm Z(q)}(q)$, truncated to suitable length independent of q, contains a natural cone of aperture $2\theta_1$ and is contained in another natural cone of aperture $2\theta_2$. We may assume, replacing Z by $-Z$ if necessary, that $C_{\theta,Z(q)}(q) \subset \Omega$ and $C_{\theta,-Z(q)}(q) \subset \Omega^c$.

A regular family of cones for Ω (resp., Ω^o) is a map that associates to every $q \in \partial\Omega$ a cone $C_{\theta,Z(q)}(q) \subset \Omega$ (resp., $C_{\theta,-Z(q)}(q) \subset \Omega^o$) with Z as above; compare the definition in [25]. Hereafter, we fix such a family once and for all, and we write $\Gamma_i(q)$ for the inner cone at q in this family, $\Gamma_c(q)$ for the outer cone at q, and $\Gamma(q) = \Gamma_i(q) \cup \Gamma_c(q)$ for the double cone.

Associated to the regular family of cones is a nontangential maximal function defined as follows. If $g_i\colon \Omega \to \mathbb{R}^d$ (resp., $g_c\colon \bar{\Omega}^c \to \mathbb{R}^d$) is a function defined on Ω (resp., $\bar{\Omega}^c$), we denote the nontangential maximal functions of g at $p \in \partial\Omega$ as

$$g_i^M(p) = \sup\{|g(x)| \,:\, x \in \Gamma_i(p)\}, \tag{3}$$

$$(\text{resp., } g_c^M(p) = \sup\{|g(x)| \,:\, x \in \Gamma_c(p)\}). \tag{4}$$

When g is defined on $\Omega \cup \bar{\Omega}^c$, we set

$$g^M(p) = \sup\{|g(x)| \,:\, x \in \Gamma(p)\}. \tag{5}$$

A function h on an open set $O \subset \mathbb{R}^d$ is harmonic if it satisfies $\Delta h = 0$. As soon as Δh exists in the distributional sense, h is infinitely differentiable. It follows from [29, sect. 5, Thm.] and [30, Thm. 1] that every harmonic function on Ω (resp., $\bar{\Omega}^c$) whose maximal function is in $L^2(\partial\Omega)$ has a nontangential limit a.e. on $\partial\Omega$, and that limit function is in $L^2(\partial\Omega)$.

2.2. Potentials. In this section, we summarize known results about the single and the double layer potentials. Most of the statements and references or proofs for them can be found in [25]. In the following we denote the surface area of a d-dimensional sphere by ω_d.

Single layer potential. The single layer potential of $f \in L^2(\partial\Omega)$ is

$$\mathscr{S}f(x) \doteq \frac{-1}{\omega_d(d-2)} \int_{\partial\Omega} \frac{1}{|x-q|^{d-2}} f(q)\, d\sigma(q) \qquad \left(x \in \mathbb{R}^d \backslash \partial\Omega\right). \tag{6}$$

It is harmonic on $\mathbb{R}^d \backslash \partial\Omega$. The result below follows from the combination of [31, Thm. 1] and [32, Chap. 15, Thm. 1]; see also [25, Thm. 08.D and Lem. 1.3].

LEMMA 2.1. *For a.e. $p \in \partial\Omega$, the limit*

$$\lim_{\substack{x\to p\\ x\in\Gamma(p)}} \mathscr{S}f(x) = \text{p.v. } \frac{-1}{\omega_d(d-2)} \int_{\partial\Omega} \frac{1}{|x-q|^{d-2}} f(q)\, d\sigma(q) \doteq Sf(p) \tag{7}$$

exists, and the limit function is in $W^{1,2}(\partial\Omega)$. Moreover, $\|(\mathscr{S}f)^M\| \le C\|f\|$ for some constant $C = C(\partial\Omega)$.

The operator $S\colon L^2(\partial\Omega) \to W^{1,2}(\partial\Omega)$, defined by the (weakly) singular integral in (7), is a bounded linear operator [25, Lem. 1.8]; when no confusion is possible, we also call it the single layer potential.

The following property of S will be important in what follows.

THEOREM 2.2 (see [25, Thm 3.3]). *The operator $S\colon L^2(\partial\Omega) \to W^{1,2}(\partial\Omega)$ is invertible.*

Remark 2.3. The previous results hold for a more general range of exponents, but for the purpose of this paper statements in L^2 will suffice.

The gradient of $\mathscr{S}f$. The Euclidean gradient of the single layer potential is

$$\nabla\mathscr{S}f(x) \doteq \frac{1}{\omega_d} \int_{\partial\Omega} \frac{x-q}{|x-q|^{d}} f(q)\, d\sigma(q) \qquad \left(x \in \mathbb{R}^d \backslash \partial\Omega\right). \tag{8}$$

Each of the vector components of $\nabla\mathscr{S}f$ is a harmonic function on $\mathbb{R}^d \backslash \partial\Omega$, and it holds that $\|\left(\nabla\mathscr{S}f\right)^M\| \le C\|f\|$ for some constant $C = C(\Omega)$. This last fact follows from [32, Chap. 15, Thm. 1] (see also [25, Lem. 1.3]). Thus, $\nabla\mathscr{S}f$ has nontangential limits a.e. on $\partial\Omega$ from each side. We shall describe their tangential and normal components separately. We begin with the tangential component, which is the same from either side (compare [25, Thm. 1.6]).

LEMMA 2.4. *For every tangent field $\tau \in T(\partial\Omega)$ the limit*

$$\begin{aligned}\lim_{\substack{x\to p\\ x\in\Gamma(p)}} \langle \tau(p), \nabla\mathscr{S}f(x)\rangle_{\mathbb{R}^d} &= \text{p.v. } \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle \tau(p), p-q\rangle_{\mathbb{R}^d}}{|p-q|^{d}} f(q)\, d\sigma(q)\\ &= \langle \tau(p), \nabla_{\mathrm{T}} Sf(p)\rangle_{\mathbb{R}^d}\end{aligned} \tag{9}$$

exists at a.e. $p \in \partial\Omega$, and defines the tangent field $\nabla_{\mathrm{T}} Sf \in T(\partial\Omega)$. Moreover, $\|\nabla_{\mathrm{T}} Sf\| \le C\|f\|$.

Proof. See Lemma A.1 in Appendix A. ∎

We can define the operator $\nabla_{\mathrm{T}} S\colon L^2(\partial\Omega) \to T(\partial\Omega)$ such that $(\nabla_{\mathrm{T}} S)f = \nabla_{\mathrm{T}}(Sf)$. By the above lemma, $\nabla_{\mathrm{T}} S$ is bounded and linear.

The scalar potential. The scalar potential of a field $f \in L^2(\partial\Omega;\mathbb{R}^d)$ is

$$\mathcal{P}f\,(x) \doteq \frac{1}{\omega_d}\int_{\partial\Omega}\frac{\langle q-x, f(q)\rangle_{\mathbb{R}^d}}{|q-x|^d}\,d\sigma(q) \qquad \left(x\in\mathbb{R}^d\backslash\partial\Omega\right). \tag{10}$$

It is harmonic on $\mathbb{R}^d\backslash\partial\Omega$.

By definition, the scalar potential is a constant multiple of the single layer potential of the divergence of f, and it behaves much like the Euclidean gradient of a single layer potential except that it acts on fields rather than functions. Note that the integral kernel of $\mathcal{P}$ differs from the integral kernel of $\nabla\mathscr{S}$ by a minus sign. As the next lemma shows, it is convenient to define the scalar potential in this way.

LEMMA 2.5. *For every tangent field $\tau \in T(\partial\Omega)$, the limit*

$$\lim_{\substack{x\to p\\ x\in\Gamma(p)}}\mathcal{P}\tau\,(x) = \text{p.v. }\frac{1}{\omega_d}\int_{\partial\Omega}\frac{\langle q-p,\tau(q)\rangle_{\mathbb{R}^d}}{|p-q|^d}\,d\sigma(q) \doteq (\nabla_{\mathrm{T}}S)^{\star}\,\tau\,(p) \tag{11}$$

exists at a.e. $p\in\partial\Omega$, and the limit function is in $L^2(\partial\Omega)$.

The operator $(\nabla_{\mathrm{T}}S)^{\star} : T(\partial\Omega)\to L^2(\partial\Omega)$, defined by (11), is linear and bounded, and one can see from (9) that $(\nabla_{\mathrm{T}}S)^{\star}$ is indeed the L^2-adjoint of $\nabla_{\mathrm{T}}S$.

The limits in (9) and (11) are independent of the components of the cone $\Gamma(p)$. That is, the tangent part of $\nabla\mathscr{S}$ transitions continuously.

Remark 2.6. The operator $(\nabla_{\mathrm{T}}S)^{\star}$ is closely connected with the divergence operator: For $f\in T(\partial\Omega)$, its divergence $\operatorname{div}_{\mathrm{T}}f$ is a continuous functional on $W^{1,2}(\partial\Omega)$ acting on h by the rule $\langle \operatorname{div}_{\mathrm{T}}f, h\rangle = -\langle f,\nabla_{\mathrm{T}}h\rangle$. Now, for $g\in L^2(\partial\Omega)$, we get that

$$\left\langle (\nabla_{\mathrm{T}}S)^{\star}\,f, g\right\rangle = \langle f,\nabla_{\mathrm{T}}Sg\rangle = -\left\langle \operatorname{div}_{\mathrm{T}}f, Sg\right\rangle.$$

If we identify the dual space of $L^2(\partial\Omega)$ with $L^2(\partial\Omega)$ itself and the dual space of $W^{1,2}(\partial\Omega)$ with the Sobolev space $W^{-1,2}(\partial\Omega)$ of negative exponent,[1] it holds that $(\nabla_{\mathrm{T}}S)^{\star} = S^{*}\operatorname{div}_{\mathrm{T}}$, where $S^{*}\colon W^{-1,2}(\partial\Omega)\to L^2(\partial\Omega)$. In particular, since S is invertible by Theorem 2.2, so is S^{*} and we conclude that $\mathcal{N}((\nabla_{\mathrm{T}}S)^{\star})$ consists exactly of divergence-free vector fields in $T(\partial\Omega)$.

Double layer potential. Let η denote the outward unit normal vector on $\partial\Omega$. For $f_\eta\in L^2(\partial\Omega)$, the double layer potential of f_η is

$$\mathscr{K}f_\eta\,(x) = \frac{1}{\omega_d}\int_{\partial\Omega}\frac{\langle q-x,\eta(q)\rangle_{\mathbb{R}^d}}{|q-x|^d}f_\eta(q)\,d\sigma(q) = \mathcal{P}(\eta f_\eta)\,(x) \qquad \left(x\in\mathbb{R}^d\backslash\partial\Omega\right),$$

and the (boundary) double layer potential is

$$Kf_\eta\,(p) = \text{p.v. }\frac{1}{\omega_d}\int_{\partial\Omega}\frac{\langle q-p,\eta(q)\rangle_{\mathbb{R}^d}}{|q-p|^d}f_\eta(q)\,d\sigma(q), \tag{12}$$

defined for a.e. p on $\partial\Omega$. The operator $K\colon L^2(\partial\Omega)\to L^2(\partial\Omega)$ is linear and bounded and we denote its L^2-adjoint by $K^{\star}$. These operators define the boundary behavior of $\mathscr{K}$ by the following lemma [25, Thms. 1.10 and 1.11].

[1] This goes as in the Euclidean case; see [33, sect. 3.13].

LEMMA 2.7. *For $f \in L^2(\partial\Omega)$ and for a.e. $p \in \partial\Omega$ it holds that*

$$\lim_{\substack{x \to p \\ x \in \Gamma_i(p)}} \mathscr{K} f(x) = \left(\frac{1}{2} + K\right) f(p), \tag{13}$$

$$\lim_{\substack{x \to p \\ x \in \Gamma_c(p)}} \mathscr{K} f(x) = -\left(\frac{1}{2} - K\right) f(p) \tag{14}$$

and

$$\lim_{\substack{x \to p \\ x \in \Gamma_i(p)}} \langle \eta(p), \nabla \mathscr{S} f(x) \rangle_{\mathbb{R}^d} = -\left(\frac{1}{2} - K^\star\right) f(p), \tag{15}$$

$$\lim_{\substack{x \to p \\ x \in \Gamma_c(p)}} \langle \eta(p), \nabla \mathscr{S} f(x) \rangle_{\mathbb{R}^d} = \left(\frac{1}{2} + K^\star\right) f(p). \tag{16}$$

The next two results will be often used in what follows.

LEMMA 2.8 (see [25, Thm. 3.3]). *Let $L^2_{\mathrm{ZM}}(\partial\Omega)$ be the space of square integrable functions with zero mean. The operators*

$$\left(\frac{1}{2} + K\right) : L^2(\partial\Omega) \to L^2(\partial\Omega) \qquad \textit{and} \qquad \left(\frac{1}{2} - K^\star\right) : L^2_{\mathrm{ZM}}(\partial\Omega) \to L^2_{\mathrm{ZM}}(\partial\Omega)$$

are invertible.

LEMMA 2.9. *There exists a unique function $\nu_{\mathrm{o}} \in L^2_{\mathrm{ZM}}(\partial\Omega)$, possibly zero, such that $S(1-\nu_{\mathrm{o}})$ is a nonzero constant with*

$$\left(\frac{1}{2} - K^\star\right)(1-\nu_{\mathrm{o}}) = 0 \qquad \textit{and} \qquad \nabla_{\mathrm{T}} S(1-\nu_{\mathrm{o}}) = 0. \tag{17}$$

Further, there exists a constant $C > 0$ that depends on Ω, such that for every $f \in L^2(\partial\Omega)$ it holds that

$$\left\| \left(\frac{1}{2} - K^\star\right) f \right\| \leq C \, \|\nabla_{\mathrm{T}} S f\|. \tag{18}$$

The first statement of Lemma 2.9 was shown in the proof of [25, Thm. 3.3. (ii)]. The second was established in the proof of [25, Thm. 2.1]. Note that the equalities $\nabla_{\mathrm{T}} S1 = 0 = \left(\frac{1}{2} - K^\star\right) 1$ hold if and only if $\nu_{\mathrm{o}} = 0$. This fact will be used at several places below.

Remark 2.10. The null-space of $\nabla_{\mathrm{T}} S$ coincides with the null-space of $\left(\frac{1}{2} - K^*\right)$. To see this observe that the first identity in (17) means that $\mathscr{S}(1-\nu_0)$ is the solution to a harmonic Neumann problem in Ω with nontangential maximal function of its gradient in $L^2(\partial\Omega)$ and zero boundary data, while the second identity says that $\mathscr{S}(1-\nu_0)$ is the solution to a harmonic Dirichlet problem in Ω with nontangential maximal function in $L^2(\partial\Omega)$ and constant boundary data. In both cases, this means that $\mathscr{S}(1-\nu_0)$ is constant on Ω.

Remark 2.11. Equation (17) entails that the Newtonian equilibrium measure μ of $\overline{\Omega}$—a well studied object in potential theory [34]—is given by $d\mu = (1/\sigma(\partial\Omega))(1-\nu_{\mathrm{o}})\, d\sigma$. Gruber's conjecture asserts that ν_{o} is zero if and only if $\partial\Omega$ is a sphere; and this is known to hold when Ω is convex [35, Thm. 4.12]. Even though a discussion of Gruber's conjecture is beyond the scope of the present paper, our analysis will stress a link between the equilibrium measure and the properties of projectors onto Hardy spaces.

3. Orthogonal decomposition of fields. In this section, we present two orthogonal decompositions of the space $L^2(\partial\Omega;\mathbb{R}^d)$ based on the null-space of the scalar potential. Hereafter, we will denote the space of harmonic functions in an open region $N \subset \mathbb{R}^d$ by $\mathcal{H}(N)$.

We begin with a definition of inner and outer scalar potentials.

DEFINITION 3.1. *Let $g|_\Omega$ denote the restriction of the function g to the region Ω, and recall from* (10) *the scalar potential $\mathcal{P}$. We define the inner scalar potential as*

$$\begin{aligned} \mathcal{P}_i\colon L^2(\partial\Omega;\mathbb{R}^d) &\to \mathcal{H}(\Omega), \\ f &\mapsto (\mathcal{P}f)|_\Omega \end{aligned} \tag{19}$$

and the outer scalar potential as

$$\begin{aligned} \mathcal{P}_c\colon L^2(\partial\Omega;\mathbb{R}^d) &\to \mathcal{H}(\bar{\Omega}^c), \\ f &\mapsto (\mathcal{P}f)|_{\bar{\Omega}^c}. \end{aligned} \tag{20}$$

Both operators are linear.

The operators $\mathcal{P}_i$ and $\mathcal{P}_c$ have nontrivial null-spaces, leading to orthogonal direct sum decompositions of $L^2(\partial\Omega;\mathbb{R}^d)$ as $\mathcal{N}(\mathcal{P}_i)\oplus\mathcal{N}(\mathcal{P}_i)^\perp$ and $\mathcal{N}(\mathcal{P}_c)\oplus\mathcal{N}(\mathcal{P}_c)^\perp$.

In the following we call $D(\partial\Omega) := \mathcal{N}(\mathcal{P}_i)\cap\mathcal{N}(\mathcal{P}_c)$ the space of invisible vector fields; $I(\partial\Omega) := \mathcal{N}(\mathcal{P}_i)\cap D(\partial\Omega)^\perp$ the space of invisible vector fields inside of Ω; and $O(\partial\Omega) := \mathcal{N}(\mathcal{P}_c)\cap D(\partial\Omega)^\perp$ the space of invisible vector fields outside of Ω.

When $\partial\Omega$ is a sphere, the spaces $\mathcal{N}(\mathcal{P}_i)$ and $\mathcal{N}(\mathcal{P}_c)$ relate to nontangential limits of harmonic gradients—the Hardy spaces. Below, we recall the definition of Hardy spaces.

Hardy spaces. The inner Hardy space, $H^2_+(\Omega)$, is the space of gradients of harmonic functions in Ω whose nontangential maximal function is in $L^2(\partial\Omega)$:

$$H^2_+(\Omega) = \left\{\nabla g \,:\, g\in\mathcal{H}(\Omega),\, \|(\nabla g)^M_i\| < \infty\right\}. \tag{21}$$

The outer Hardy space $H^2_-(\Omega)$ is defined similarly on the outside of $\partial\Omega$:

$$H^2_-(\Omega) = \left\{\nabla g \,:\, g\in\mathcal{H}(\bar{\Omega}^c),\, \|(\nabla g)^M_c\| < \infty,\, \lim_{x\to\infty}|g(x)| = 0\right\}. \tag{22}$$

For each $\nabla g \in H^2_+(\Omega)$ there is a square integrable boundary field $f\in L^2(\partial\Omega;\mathbb{R}^d)$, such that ∇g converges nontangentially from inside to f, a.e. on $\partial\Omega$ (see, for example, [28, Thm. 7.9]). The space of boundary fields obtained in this way is denoted by $H^2_+(\partial\Omega)$. The same limiting procedure for $H^2_-(\Omega)$-functions leads to the space $H^2_-(\partial\Omega)$. We still call the space $H^2_+(\partial\Omega)$ (or $H^2_-(\partial\Omega)$) the inner (or outer) Hardy space, with no fear of confusion.

If $\partial\Omega$ is a sphere, then $H^2_+(\partial\Omega)$ and $H^2_-(\partial\Omega)$ are orthogonal. Denoting by $D_f(\partial\Omega) \subset T(\partial\Omega)$ the space of divergence-free tangent fields on $\partial\Omega$, we have that $\mathcal{N}(\mathcal{P}_i) = H^2_+(\partial\Omega)\oplus D_f(\partial\Omega)$ and $\mathcal{N}(\mathcal{P}_i)^\perp = H^2_-(\partial\Omega)$. This is the Hardy–Hodge decomposition on a sphere—an orthogonal direct sum decomposition of $L^2(\partial\Omega;\mathbb{R}^d)$ as $L^2(\partial\Omega;\mathbb{R}^d) = H^2_+(\partial\Omega)\oplus H^2_-(\partial\Omega)\oplus D_f(\partial\Omega)$. Moreover, if we use the null-space of $\mathcal{P}_c$ instead of the null-space of $\mathcal{P}_i$, the roles of $H^2_+(\partial\Omega)$ and $H^2_-(\partial\Omega)$ get swapped.

On Lipschitz domains, the Hardy–Hodge decomposition still exists, but the inner and outer Hardy spaces are no longer orthogonal to each other when $\partial\Omega$ is not a sphere (see Corollary 5.2).

3.1. Inner decomposition. Below we derive an orthogonal direct sum for $L^2(\partial\Omega;\mathbb{R}^d)$, based on the null-space of the inner scalar potential. For a field f, recall its normal component f_η and its tangent component f_{T} such that $f = \eta f_\eta + f_{\mathrm{T}}$.

DEFINITION 3.2. *Define the operator,* $B_i : L^2(\partial\Omega;\mathbb{R}^d) \to L^2(\partial\Omega)$*, as*

$$B_i f \doteq \left(\frac{1}{2} + K\right) f_\eta + (\nabla_{\mathrm{T}} S)^\star f_{\mathrm{T}}; \tag{23}$$

it is linear and bounded. We call B_i *the inner boundary operator.*

THEOREM 3.3. *The inner scalar potential of* $f \in L^2(\partial\Omega;\mathbb{R}^d)$ *can be expressed as*

$$\mathcal{P}_i f(x) = \mathscr{K}\left(\frac{1}{2} + K\right)^{-1} B_i f(x) \qquad (x \in \Omega). \tag{24}$$

Moreover, the the null-space of $\mathcal{P}_i$ *is given by*

$$\mathcal{N}(\mathcal{P}_i) = \mathcal{N}(B_i) = \left\{ f \in L^2(\partial\Omega;\mathbb{R}^d) \, : \left(\frac{1}{2} + K\right) f_\eta = -(\nabla_{\mathrm{T}} S)^\star f_{\mathrm{T}} \right\}. \tag{25}$$

Proof. Let f be in $L^2(\partial\Omega;\mathbb{R}^d)$. For $x \in \Omega$, we get that

$$\mathcal{P}_i f(x) = \mathcal{P}_i(\eta f_\eta)(x) + \mathcal{P}_i f_{\mathrm{T}}(x) = \mathscr{K} f_\eta(x) + \mathcal{P}_i f_{\mathrm{T}}(x). \tag{26}$$

Taking the nontangential limit on $\partial\Omega$ from Ω on both sides of the above equation and using (11) and (13) yields

$$\lim_{x\to q} \mathcal{P}_i f(x) = \left(\frac{1}{2} + K\right) f_\eta(q) + (\nabla_{\mathrm{T}} S)^\star f_{\mathrm{T}}(q) = B_i f(q). \tag{27}$$

By Lemma 2.8 the operator $\left(\frac{1}{2} + K\right)^{-1}$ is well defined. Hence,

$$\mathcal{P}_i f - \mathscr{K}\left(\frac{1}{2} + K\right)^{-1} B_i f \tag{28}$$

is harmonic in Ω, converges nontangentially to zero a.e. on $\partial\Omega$, and has L^2-bounded nontangential maximal function. By uniqueness of the Dirichlet problem (see, for example, [25, Cor. 3.2]) that function is zero; this proves (24). It follows that $\mathcal{N}(\mathcal{P}_i) = \mathcal{N}(B_i)$ and the second equality in (25) is immediate from the definition of B_i. ∎

COROLLARY 3.4. *For* $f \in \mathcal{N}(\mathcal{P}_i)$*, the normal component of* f *is uniquely determined by the tangent part of* f *through the relation*

$$f_\eta = -\left(\frac{1}{2} + K\right)^{-1} (\nabla_{\mathrm{T}} S)^\star f_{\mathrm{T}}. \tag{29}$$

Theorem 3.3 shows that the operator B_i determines the null-space for $\mathcal{P}_i$. The following lemma shows that the adjoint $B_i^\star$ of B_i determines the orthogonal complement of the null-space of $\mathcal{P}_i$.

LEMMA 3.5. *The operator* $B_i^\star : L^2(\partial\Omega) \to L^2(\partial\Omega;\mathbb{R}^d)$ *can be written as*

$$B_i^\star g = \eta\left(\frac{1}{2} + K^\star\right) g + \nabla_{\mathrm{T}} S g. \tag{30}$$

The range of $B_i^\star$ *satisfies* $\mathcal{R}(B_i^\star) = H^2_-(\partial\Omega) = \mathcal{N}(\mathcal{P}_i)^\perp$.

Proof. Let $f = \eta f_\eta + f_{\mathrm{T}}$ be in $L^2(\partial\Omega;\mathbb{R}^d)$. For every $g \in L^2(\partial\Omega)$, we get that

$$\langle B_i f, g\rangle = \left\langle \left(\frac{1}{2} + K\right) f_\eta + (\nabla_{\mathrm{T}} S)^\star f_T, g \right\rangle \tag{31}$$

$$= \left\langle f_\eta, \left(\frac{1}{2} + K^\star\right) g \right\rangle + \left\langle f_T, \nabla_{\mathrm{T}} S g \right\rangle = \langle f, B_i^\star g\rangle . \tag{32}$$

Hence, using (16) we have that

$$B_i^\star g\,(p) = \eta \left(\frac{1}{2} + K^\star\right) g\,(p) + \nabla_{\mathrm{T}} S g\,(p) = \lim_{x\to p} \nabla \mathscr{S} g\,(x) \tag{33}$$

when x approaches p within $\Gamma_c(p)$. Since $\mathscr{S}g$ is harmonic in $\bar{\Omega}^c$ and has L^2-bounded maximal function, the right-hand side of (33) defines a function in $H^2_-(\partial\Omega)$.

To see that $B_i^\star$ is onto, consider $f \in H^2_-(\partial\Omega)$. By the well posedness of the Neumann problem, there exists a unique function φ that is harmonic in $\bar{\Omega}^c$, vanishes at infinity, and whose normal derivative equals f_η a.e. on $\partial\Omega$, while $\|(\nabla\varphi)^M\|$ is finite. On the other hand, since $\frac{1}{2} + K^\star$ is invertible because its adjoint is by Lemma 2.8, we get from (16) that $\varphi = \mathscr{S}\left(\frac{1}{2} + K^\star\right)^{-1} f_\eta$, and so the function $g = \left(\frac{1}{2} + K^\star\right)^{-1} f_\eta \in L^2(\partial\Omega)$ is such that

$$f(p) = \lim_{x\to p} \nabla\varphi\,(p) = \lim_{x\to p} \nabla\mathscr{S} g\,(x)\,,$$

where the nontangential limits are taken from $\bar{\Omega}^c$. Consequently, $B_i^\star$ is onto and $\mathcal{R}(B_i^\star) = H^2_-(\partial\Omega)$.

For the last statement, observe by general properties of operators on Hilbert space that $\mathcal{N}(B_i) = \mathcal{R}(B_i^\star)^\perp = H^2_-(\partial\Omega)^\perp$. The conclusion now follows since $H^2_-(\partial\Omega)$ is closed. ∎

COROLLARY 3.6. *Setting $I(\partial\Omega) = \mathcal{N}(\mathcal{P}_i) \cap D(\partial\Omega)^\perp$, the space $L^2(\partial\Omega;\mathbb{R}^d)$ splits into an orthogonal sum as*

$$L^2(\partial\Omega;\mathbb{R}^d) = H^2_-(\partial\Omega) \oplus I(\partial\Omega) \oplus D(\partial\Omega). \tag{34}$$

3.2. Orthogonal projections for the inner decomposition. The goal of this section is the statement of the orthogonal projections onto $H^2_-(\partial\Omega)$, $I(\partial\Omega)$, and $D(\partial\Omega)$, and their explicit expression in terms of layer potentials. For the rest of the paper, we make the following definitions.

DEFINITION 3.7. *We define a space of tangent fields with zero mean as*

$$T_{\mathrm{ZM}}(\partial\Omega) = \left\{ f \in T(\partial\Omega) \,:\, (\nabla_{\mathrm{T}} S)^\star f \in L^2_{\mathrm{ZM}}(\partial\Omega) \right\},$$

and the space of tangent gradient fields as

$$G(\partial\Omega) = \left\{ \nabla_{\mathrm{T}}\varphi \,:\, \varphi \in W^{1,2}(\partial\Omega) \right\},$$

as well as the space of tangent divergence-free vector fields (see Remark 2.6):

$$D_f(\partial\Omega) = \left\{ f \in T(\partial\Omega) \,:\, (\nabla_{\mathrm{T}} S)^\star f = 0 \right\}.$$

Since S is onto $W^{1,2}(\partial\Omega)$, we can equivalently write the space of gradient fields as $G(\partial\Omega) = \left\{ \nabla_{\mathrm{T}} S f \,:\, f \in L^2(\partial\Omega) \right\}$. Then, it is immediate that $D_f(\partial\Omega)$ is the orthogonal complement of $G(\partial\Omega)$. The spaces $T_{\mathrm{ZM}}(\partial\Omega)$ and $D_f(\partial\Omega)$ are closed, since $(\nabla_{\mathrm{T}} S)^\star$ is *continuous, and therefore, there exist orthogonal projections* $P_{\mathrm{ZM}}\colon T(\partial\Omega)\to T_{\mathrm{ZM}}(\partial\Omega)$, $P_{\mathrm{G}}\colon T(\partial\Omega)\to T(\partial\Omega)$, *and* $P_{\mathrm{D}}: T(\partial\Omega)\to D_f(\partial\Omega)$. *These projections naturally extend to the entire* $L^2(\partial\Omega;\mathbb{R}^d)$ *such that for* $f=\eta f_\eta+f_{\mathrm{T}}\in L^2(\partial\Omega;\mathbb{R}^d)$,

$$P_{\mathrm{D}}f=P_{\mathrm{D}}f_{\mathrm{T}},\qquad\qquad P_{\mathrm{G}}f=\eta f_\eta+P_{\mathrm{G}}f_{\mathrm{T}}.$$

We will prove in Lemma 4.2 that $D(\partial\Omega)=D_f(\partial\Omega)$, which justifies the notation P_{D} rather than P_{D_f} for the projection onto the space of divergence-free fields.

THEOREM 3.8. *The operator*

$$P_- = B_i^\star\left(B_i B_i^\star\right)^{-1}B_i \tag{35}$$

defines the orthogonal projection from $L^2(\partial\Omega;\mathbb{R}^d)$ *onto* $H^2_-(\partial\Omega)$. *Let* P_{D} *be as in Definition* 3.7. *Then,*

$$P_i \doteq \mathbb{1}-P_{\mathrm{D}}-P_- \tag{36}$$

is the orthogonal projection from $L^2(\partial\Omega;\mathbb{R}^d)$ *onto* $I(\partial\Omega)=\mathcal{N}(\mathcal{P}_i)\cap D(\partial\Omega)^\perp$.

The proof of the theorem is immediate once it is shown that the operator $B_iB_i^\star$ is invertible, thereby implying that the P_- introduced in (35) is a well defined orthogonal projection.

LEMMA 3.9. *Let* B_i *be the inner boundary operator from Definition* 3.2, *and let* $B_i^\star$ *denote its adjoint from Lemma* 3.5*; then* $B_iB_i^\star\colon L^2(\partial\Omega)\to L^2(\partial\Omega)$ *is invertible.*

Proof. Define a symmetric continuous bilinear form, M, such that

$$M(f,g)\doteq\langle B_iB_i^\star f,g\rangle\qquad\left(f,g\in L^2(\partial\Omega)\right). \tag{37}$$

Since $\left(\frac{1}{2}+K^\star\right)$ is invertible, there exists a positive constant C such that

$$M(f,f)=\|B_i^\star f\|^2\geq\left\|\left(\frac{1}{2}+K^\star\right)f\right\|^2\geq C\|f\|^2. \tag{38}$$

Thus, M is coercive and $B_iB_i^\star$ is invertible by the Lax–Milgram theorem. ∎

Proof of Theorem 3.8. Let P_{D} be as above, and recall the operators introduced in the theorem $P_-=B_i^\star\left(B_iB_i^\star\right)^{-1}B_i$ and $P_i=\mathbb{1}-P_{\mathrm{D}}-P_-$. Since we know from Lemma 3.9 that $B_iB_i^\star$ is invertible it is clear that P_- is an orthogonal projection onto the image of $B_i^\star$. By Lemma 3.5 the image of $B_i^\star$ is the space $H^2_-(\partial\Omega)$. This proves the first statement in the theorem.

For the second statement we need the identity $D_f(\partial\Omega)=D(\partial\Omega)$, which we prove below in Lemma 4.2. Granted this, the statement is immediate since in this case P_{D} and P_- commute as projections onto orthogonal spaces. This guarantees that the P_i from (36) is actually an orthogonal projection and concludes the proof. ∎

In the remainder of the section, we will consider an explicit representation of the projection P_{D} required in (36) and reexamine the connection between the normal and tangential contributions of vector fields that are invisible inside. We let R_{ZM} denote the canonical projection of $L^2(\partial\Omega)$ onto $L^2_{\mathrm{ZM}}(\partial\Omega)$.

LEMMA 3.10. *The operator* $R_{\mathrm{ZM}}\left(\nabla_{\mathrm{T}}S\right)^\star\nabla_{\mathrm{T}}S\colon L^2_{\mathrm{ZM}}(\partial\Omega)\to L^2_{\mathrm{ZM}}(\partial\Omega)$ *is self-adjoint and invertible.*

Proof. The operator $R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S$ is self-adjoint, since for $f, g \in L^2_{\mathrm{ZM}}(\partial\Omega)$ we have

$$\begin{aligned}\left\langle R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S f, g\right\rangle &= \left\langle (\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S f, g\right\rangle \\ &= \left\langle f, (\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S g\right\rangle = \left\langle f, R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S g\right\rangle . \end{aligned} \tag{39}$$

By the Riesz lemma it defines a symmetric and continuous bilinear form, say M, such that for $f, g \in L^2_{\mathrm{ZM}}(\partial\Omega)$

$$M\,(f,g) \doteq \left\langle R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S f, g\right\rangle . \tag{40}$$

From (18) and Lemma 2.8, it follows that M is coercive, and by the Lax–Milgram theorem $R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S$ has bounded inverse. ∎

LEMMA 3.11. *Let P_{ZM} and P_{D} be the projections from Definition* 3.7*. Then,*

$$P_{\mathrm{D}} = P_{\mathrm{ZM}}\left(\mathbb{1} - \nabla_{\mathrm{T}} S\left(R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S\right)^{-1}(\nabla_{\mathrm{T}} S)^{\star}\right) P_{\mathrm{ZM}}. \tag{41}$$

Proof. Let ν_{o} be as in Lemma 2.9. The projection onto $T_{\mathrm{ZM}}(\partial\Omega)$ can be written as

$$P_{\mathrm{ZM}} f = (\mathbb{1} - P_{\mathrm{S}})\,f \qquad \text{with} \qquad P_{\mathrm{S}} f = \begin{cases} \langle f, \nabla_{\mathrm{T}} S1\rangle\,\frac{\nabla_{\mathrm{T}} S1}{\|\nabla_{\mathrm{T}} S1\|^2} & \text{if } \nu_{\mathrm{o}} \neq 0, \\ 0 & \text{if } \nu_{\mathrm{o}} = 0. \end{cases} \tag{42}$$

To see this, observe that P_{ZM} and P_{S} as defined above are complementary projections. Then, for $f \in T(\partial\Omega)$, we have that

$$\int_{\partial\Omega} (\nabla_{\mathrm{T}} S)^{\star}\,P_{\mathrm{ZM}} f = \int_{\partial\Omega} (\nabla_{\mathrm{T}} S)^{\star}\,(\mathbb{1} - P_{\mathrm{S}}) f\,(q)\;\,d\sigma(q) = \langle f, \nabla_{\mathrm{T}} S1\rangle - \langle f, \nabla_{\mathrm{T}} S1\rangle = 0,$$

and thus $P_{\mathrm{ZM}} f$ is in $T_{\mathrm{ZM}}(\partial\Omega)$. Conversely, if f is in $T_{\mathrm{ZM}}(\partial\Omega)$, then $P_{\mathrm{S}} f = 0$ and $P_{\mathrm{ZM}} f = f$. Hence, $P_{\mathrm{ZM}} T(\partial\Omega) = T_{\mathrm{ZM}}(\partial\Omega)$.

Next, we write the operator P_{D} from (41) as

$$P_{\mathrm{D}} = P_{\mathrm{ZM}}\,(\mathbb{1} - A)\,P_{\mathrm{ZM}} \qquad \text{with} \qquad A = \nabla_{\mathrm{T}} S\left(R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S\right)^{-1}(\nabla_{\mathrm{T}} S)^{\star}\,, \tag{43}$$

and we show that it is the orthogonal projection onto $D_f(\partial\Omega)$.

The operator $P_{\mathrm{ZM}} A P_{\mathrm{ZM}}$ is well defined, because P_{ZM} projects $T(\partial\Omega)$ onto $T_{\mathrm{ZM}}(\partial\Omega)$ and $(\nabla_{\mathrm{T}} S)^{\star}$ maps the latter into $L^2_{\mathrm{ZM}}(\partial\Omega)$, where the operator $\left(R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star}\,\nabla_{\mathrm{T}} S\right)^{-1}$ is well defined by Lemma 3.10. Also, the operator A preserves the space $T_{\mathrm{ZM}}(\partial\Omega)$; that is, for $f \in T_{\mathrm{ZM}}(\partial\Omega)$, we have $Af \in T_{\mathrm{ZM}}(\partial\Omega)$. Indeed, observe that if f is in $T_{\mathrm{ZM}}(\partial\Omega)$, then by (17) and Lemma 3.10 we have

$$\begin{aligned}&\left\langle (\nabla_{\mathrm{T}} S)^{\star}\,Af, 1\right\rangle = \langle Af, \nabla_{\mathrm{T}} S\nu_{\mathrm{o}}\rangle = \left\langle (R_0(\nabla_{\mathrm{T}} S)^{*}\nabla_{\mathrm{T}} S)^{-1}(\nabla_{\mathrm{T}} S)^{*} f, (\nabla_{\mathrm{T}} S)^{*}\nabla_{\mathrm{T}} S\nu_0\right\rangle \\ &\quad = \left\langle (R_0(\nabla_{\mathrm{T}} S)^{*}\nabla_{\mathrm{T}} S)^{-1}(\nabla_{\mathrm{T}} S)^{*} f, R_0(\nabla_{\mathrm{T}} S)^{*}\nabla_{\mathrm{T}} S\nu_0\right\rangle = \langle f, \nabla_{\mathrm{T}} S\nu_{\mathrm{o}}\rangle = \langle f, \nabla_{\mathrm{T}} S1\rangle = 0.\end{aligned}$$

Consequently, Af is in $T_{\mathrm{ZM}}(\partial\Omega)$. By a similar calculation, $AAP_{\mathrm{ZM}} = AP_{\mathrm{ZM}}$ and thus $AP_{\mathrm{ZM}}AP_{\mathrm{ZM}} = AAP_{\mathrm{ZM}} = AP_{\mathrm{ZM}}$.

From the above we see that P_{D} is idempotent, since

$$P_{\mathrm{D}} P_{\mathrm{D}} = P_{\mathrm{ZM}}\,(\mathbb{1} - A)\,P_{\mathrm{ZM}}\,(\mathbb{1} - A)\,P_{\mathrm{ZM}} = P_{\mathrm{ZM}}\,(\mathbb{1} - A)\,P_{\mathrm{ZM}} = P_{\mathrm{D}}. \tag{44}$$

Also, P_{D} is self-adjoint, because $\mathbb{1}$, P_{ZM}, and A are self-adjoint. Thus, P_{D} is an orthogonal projection.

We claim that the range of P_{D} is $D_f(\partial\Omega)$. To support this claim, it is enough to show that $f \in D_f(\partial\Omega)$ holds if and only if $P_{\mathrm{D}}f = f$. Assume that $P_{\mathrm{D}}f = f$ holds. Since the range of P_{D} is in the range of P_{ZM}, it follows that $P_{\mathrm{ZM}}f = f \in T_{\mathrm{ZM}}(\partial\Omega)$, and since A preserves $T_{\mathrm{ZM}}(\partial\Omega)$, we get from (43) that

$$Af = P_{\mathrm{ZM}}AP_{\mathrm{ZM}}f = P_{\mathrm{ZM}}f - P_{\mathrm{D}}f = f - f = 0. \tag{45}$$

This implies $(\nabla_{\mathrm{T}}S)^\star f = 0$, as the remaining operators in the definition of A are injective on functions with zero mean by Lemma 2.9. Therefore, f is in $D_f(\partial\Omega)$.

Conversely, take $f \in D_f(\partial\Omega)$ and write $f = c\nabla_{\mathrm{T}}S1 + P_{\mathrm{ZM}}f \,(= P_{\mathrm{S}}f + P_{\mathrm{ZM}}f)$ for some constant $c \in \mathbb{R}$. Then,

$$0 = \left\langle (\nabla_{\mathrm{T}}S)^\star f, 1\right\rangle = c\left\langle \nabla_{\mathrm{T}}S1, \nabla_{\mathrm{T}}S1\right\rangle + \left\langle P_{\mathrm{ZM}}f, \nabla_{\mathrm{T}}S1\right\rangle = c\|\nabla_{\mathrm{T}}S1\|^2. \tag{46}$$

Hence, either $\nabla_{\mathrm{T}}S1 = 0$ or $\nabla_{\mathrm{T}}S1 \neq 0$, and in the latter case $c = 0$; thus, $P_{\mathrm{ZM}}f = f$ in all cases, so that f is in $T_{\mathrm{ZM}}(\partial\Omega)$, and hence,

$$AP_{\mathrm{ZM}}f = \nabla_{\mathrm{T}}S\left(R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}}S)^\star\,\nabla_{\mathrm{T}}S\right)^{-1}(\nabla_{\mathrm{T}}S)^\star f = 0,$$

ensuing that $P_{\mathrm{D}}f = P_{\mathrm{ZM}}f + P_{\mathrm{ZM}}AP_{\mathrm{ZM}}f = f$. This shows that the range of P_{D} is $D_f(\partial\Omega)$ and completes the proof. □

Since divergence-free fields are orthogonal to gradients, it follows that

$$P_{\mathrm{G}} \doteq \mathbb{1} - P_{\mathrm{D}} = P_{\mathrm{S}} + P_{\mathrm{ZM}}AP_{\mathrm{ZM}} \tag{47}$$

is the orthogonal projection onto $G(\partial\Omega)$, so we can write the Helmholtz–Hodge decomposition explicitly as

$$T(\partial\Omega) = P_{\mathrm{G}}T(\partial\Omega) + P_{\mathrm{D}}T(\partial\Omega) = G(\partial\Omega) + D_f(\partial\Omega). \tag{48}$$

Remark 3.12. The existence of the Helmholtz–Hodge decomposition is trivial for it reduces to the fact that a Hilbert space decomposes as the sum of a closed subspace and its orthogonal complement. However, the point in (48) is the explicit expression in terms of layer potentials. Note that the divergence-free term can split further by Hodge theory on Lipschitz orientable Riemannian manifolds [36].

COROLLARY 3.13. *Let $f = \eta f_\eta + f_{\mathrm{T}}$ be in $\mathcal{N}(\mathcal{P}_i)$. Then, f_η determines f_{T} uniquely up to a divergence-free tangent field.*

Proof. For $f \in \mathcal{N}(\mathcal{P}_i)$, the relation between f_η and f_{T} from Theorem 3.3 reads as

$$\left(\frac{1}{2} + K\right) f_\eta = -\,(\nabla_{\mathrm{T}}S)^\star f_{\mathrm{T}}. \tag{49}$$

The divergence-free component of f_{T} does not contribute to this relation as it satisfies $(\nabla_{\mathrm{T}}S)^\star P_{\mathrm{D}}f_{\mathrm{T}} = 0$. Therefore, we can assume without loss of generality that f_{T} is a gradient field. To prove the corollary, we have to invert the right-hand side of (49). The difficulty is that the left-hand side of (49) may not have zero mean and thus, we cannot directly use Lemma 3.10.

To circumvent this, we split f_{T} into a field with zero mean plus $\nabla_{\mathrm{T}}Sc$ for some constant c. Specifically, since f_{T} is a gradient field, there exist a constant c and a function $\varphi \in L^2_{\mathrm{ZM}}(\partial\Omega)$ such that $f_{\mathrm{T}} = \nabla_{\mathrm{T}}S\,(c + \varphi)$, and then $\nabla_{\mathrm{T}}S\varphi$ is in $T_{\mathrm{ZM}}(\partial\Omega)$ (this is simply the decomposition $f_{\mathrm{T}} = P_{\mathrm{G}}f_{\mathrm{T}} = P_{\mathrm{S}}f_{\mathrm{T}} + P_{\mathrm{ZM}}AP_{\mathrm{ZM}}f_{\mathrm{T}}$, where we set

$\nabla_{\mathrm{T}} S\varphi = P_{\mathrm{ZM}} A P_{\mathrm{ZM}} f_{\mathrm{T}}$, which is possible because both f_{T} and $P_{\mathrm{S}} f_{\mathrm{T}}$ are gradients). Inserting this decomposition into (49) and rearranging terms, we get that

$$(\nabla_{\mathrm{T}} S)^{\star} \nabla_{\mathrm{T}} S\varphi = -\left(\frac{1}{2} + K\right) f_\eta - c\,(\nabla_{\mathrm{T}} S)^{\star} \nabla_{\mathrm{T}} S1. \tag{50}$$

Since the left-hand side is in $L^2_{\mathrm{ZM}}(\partial\Omega)$, the right-hand side also has zero mean. Therefore, either $\nabla_{\mathrm{T}} S1 = 0$ and then Lemma 3.10 achieves the proof, or else

$$c = -\frac{1}{\|\nabla_{\mathrm{T}} S1\|^2} \left\langle \left(\frac{1}{2} + K\right) f_\eta, 1 \right\rangle. \tag{51}$$

In the latter case, applying $(R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star} \nabla_{\mathrm{T}} S)^{-1}$ gives us

$$\varphi = \left(R_{\mathrm{ZM}}\,(\nabla_{\mathrm{T}} S)^{\star} \nabla_{\mathrm{T}} S\right)^{-1} \left[\left\langle \left(\frac{1}{2} + K\right) f_\eta, 1 \right\rangle \frac{(\nabla_{\mathrm{T}} S)^{\star} \nabla_{\mathrm{T}} S1}{\|\nabla_{\mathrm{T}} S1\|^2} - \left(\frac{1}{2} + K\right) f_\eta \right], \tag{52}$$

whence the constant c and the function φ are both determined by f_η. Thus, so is the tangent field f_{T}, as desired. ∎

3.3. Outer decomposition. In this section we derive an orthogonal decomposition of $L^2(\partial\Omega; \mathbb{R}^d)$ based on the null-space of the outer scalar potential, analogous to the considerations in section 3.1 for the inner scalar potential.

DEFINITION 3.14. *Define the operator, $B_c \colon L^2(\partial\Omega; \mathbb{R}^d) \to L^2(\partial\Omega)$, as*

$$B_c f \doteq -\left(\frac{1}{2} - K\right) f_\eta + (\nabla_{\mathrm{T}} S)^{\star} f_{\mathrm{T}}; \tag{53}$$

it is linear and bounded. We call B_c the outer boundary operator.

As in the previous section, we can write the outer potential as a harmonic extension of the outer boundary operator. Contrary to that section, however, we need to take separate care of the constant mode, because the operators $\frac{1}{2} - K$ and $\frac{1}{2} - K^{\star}$ are invertible on $L^2_{\mathrm{ZM}}(\partial\Omega)$ only.

THEOREM 3.15. *Let $\nu_{\mathrm{o}} \in L^2_{\mathrm{ZM}}(\partial\Omega)$ denote the function from Lemma 2.9. The outer potential of $f \in L^2(\partial\Omega; \mathbb{R}^d)$ can be expressed as*

$$\begin{aligned}\mathcal{P}_c f\,(x) = -\mathscr{K}\left(\frac{1}{2} - K\right)^{-1} (B_c f - \langle B_c f, 1\rangle)\,(x) \\ + \langle B_c f, 1\rangle \frac{\mathscr{S}(1-\nu_{\mathrm{o}})\,(x)}{|S\,(1-\nu_{\mathrm{o}})|} \qquad \left(x \in \bar{\Omega}^c\right).\end{aligned} \tag{54}$$

Moreover, the null-space of $\mathcal{P}_c$ is given by

$$\mathcal{N}(\mathcal{P}_c) = \mathcal{N}(B_c) = \left\{ f \in L^2(\partial\Omega; \mathbb{R}^d) \,:\, \left(\frac{1}{2} - K\right) f_\eta = (\nabla_{\mathrm{T}} S)^{\star} f_{\mathrm{T}} \right\}. \tag{55}$$

Proof. From (17) we know that $S(1-\nu_{\mathrm{o}})$ is a nonzero constant; hence the second term in (54) is well defined and converges to $\langle B_c f, 1\rangle$ a.e. as x tends to the boundary nontangentially by (7). Consequently, the right-hand side of (54) converges to $B_c f$ under the same conditions. Taking nontangential limits on $\partial\Omega$ from Ω^o on both sides of (54) while using (11) and (14), we get that

$$\lim_{x \to q} \mathcal{P}_c f\,(x) = -\left(\frac{1}{2} - K\right) f_\eta\,(q) + (\nabla_{\mathrm{T}} S)^{\star} f_{\mathrm{T}}\,(q) = B_c f\,(q)\,. \tag{56}$$

It follows that both sides of (54) are harmonic functions on $\bar{\Omega}^c$ with L^2-bounded nontangential maximal function, whose nontangential limits coincide a.e. on $\partial\Omega$. So, by uniqueness of a solution to the Dirichlet problem, they are equal. It follows that $\mathcal{N}(\mathcal{P}_c) = \mathcal{N}(B_c)$; the second equality in (55) is immediate from the definition of B_c. ∎

Like in the previous section, the L^2-adjoint of B_c, denoted as $B_c^\star$, helps one to characterize the orthogonal complement of the null-space for $\mathcal{P}_c$.

LEMMA 3.16. *The operator $B_c^\star \colon L^2(\partial\Omega) \to L^2(\partial\Omega;\mathbb{R}^d)$ can be expressed as*

$$B_c^\star g = -\eta\left(\frac{1}{2} - K^\star\right)g + \nabla_{\mathrm{T}} S g. \tag{57}$$

The range of $B_c^\star$ satisfies $\mathcal{R}(B_c^\star) = H_+^2(\partial\Omega) = \mathcal{N}(\mathcal{P}_c)^\perp$.

Proof. Let f be in $L^2(\partial\Omega;\mathbb{R}^d)$ with normal component f_η and tangential part f_{T}. Then, for every $g \in L^2(\partial\Omega)$ we get that

$$\langle B_c f, g\rangle = \left\langle -\left(\frac{1}{2} - K\right) f_\eta + (\nabla_{\mathrm{T}} S)^\star f_{\mathrm{T}}, g\right\rangle \tag{58}$$

$$= \left\langle f_\eta, -\left(\frac{1}{2} - K^\star\right) g\right\rangle + \left\langle f_{\mathrm{T}}, \nabla_{\mathrm{T}} S g\right\rangle = \langle f, B_c^\star g\rangle\,. \tag{59}$$

Hence, using (15) we obtain

$$B_c^\star g\,(p) = -\eta\left(\frac{1}{2} - K^\star\right) g\,(p) + \nabla_{\mathrm{T}} S g\,(p) = \lim_{x\to p} \nabla \mathscr{S} g\,(x)\,, \tag{60}$$

where the limit is taken from inside Ω. The rest of the proof follows the same steps as in Lemma 3.5; the only difference is that $\left(\frac{1}{2} - K^\star\right)$ is merely invertible on functions with zero mean and thus we can only define $\varphi = \mathscr{S}\left(\frac{1}{2} - K^\star\right) f_\eta$ if f_η is in $L^2_{\mathrm{ZM}}(\partial\Omega)$. This last condition, however, is guaranteed by the Gauss theorem when f_η is the normal component of a harmonic gradient in Ω. ∎

COROLLARY 3.17. *Setting $O(\partial\Omega) = \mathcal{N}(\mathcal{P}_c) \cap D(\partial\Omega)^\perp$, then the space $L^2(\partial\Omega;\mathbb{R}^d)$ splits into an orthogonal sum as*

$$L^2(\partial\Omega;\mathbb{R}^d) = H_+^2(\partial\Omega) \oplus O(\partial\Omega) \oplus D(\partial\Omega). \tag{61}$$

3.4. Orthogonal projections for the outer decomposition. Let ν_{o} denote the function from Lemma 2.9. Define two complementary projections,

$$P_{\mathrm{B}}^\perp f = \left(\mathbb{1} - P_{\mathrm{B}}\right) f, \qquad P_{\mathrm{B}} f = \begin{cases} \langle f, B_c^\star 1\rangle \frac{B_c^\star 1}{\|B_c^\star 1\|^2} & \text{if } \nu_{\mathrm{o}} \neq 0,\\ 0 & \text{if } \nu_{\mathrm{o}} = 0,\end{cases} \tag{62}$$

and recall that R_{ZM} denotes the projection of $L^2(\partial\Omega)$ to $L^2_{\mathrm{ZM}}(\partial\Omega)$. In this section we prove the following theorem.

THEOREM 3.18. *The operator*

$$P_+ = P_{\mathrm{B}} + P_{\mathrm{B}}^\perp B_c^\star \left(R_{\mathrm{ZM}} B_c B_c^\star\right)^{-1} B_c P_{\mathrm{B}}^\perp \tag{63}$$

defines an orthogonal projection from $L^2(\partial\Omega;\mathbb{R}^d)$ onto $H_+^2(\partial\Omega)$.

For P_{D} as in Definition 3.7, the operator

$$P_c \doteq \mathbb{1} - P_{\mathrm{D}} - P_+ \tag{64}$$

is the orthogonal projection from $L^2(\partial\Omega;\mathbb{R}^d)$ onto $O(\partial\Omega) = \mathcal{N}(\mathcal{P}_c) \cap D(\partial\Omega)^\perp$.

Compared with section 3.2, the only additional ingredient needed for the proof of this theorem is the fact that the operator $R_{\text{ZM}} B_c B_c^\star$ is invertible. We show this in the following lemma.

LEMMA 3.19. $R_{\text{ZM}} B_c B_c^\star \colon L^2_{\text{ZM}}(\partial\Omega) \to L^2_{\text{ZM}}(\partial\Omega)$ *is self-adjoint and invertible.*

Proof. The operator $R_{\text{ZM}} B_c B_c^\star$ is self-adjoint for if $f, g \in L^2_{\text{ZM}}(\partial\Omega)$, we have

$$\langle R_{\text{ZM}} B_c B_c^\star f, g\rangle = \langle B_c B_c^\star f, g\rangle = \langle f, B_c B_c^\star g\rangle = \langle f, R_{\text{ZM}} B_c B_c^\star g\rangle . \tag{65}$$

Hence, it defines a symmetric bounded functional, M, such that for $f, g \in L^2_{\text{ZM}}(\partial\Omega)$,

$$M\,(f, g) = \langle R_{\text{ZM}} B_c B_c^\star f, g\rangle . \tag{66}$$

Since $\frac{1}{2} - K^\star$ is invertible on $L^2_{\text{ZM}}(\partial\Omega)$ by Lemma 2.8, there exists a constant C such that

$$M\,(f, f) = \|B_c^\star f\|^2 \geq \left\| \left(\frac{1}{2} - K^\star\right) f \right\|^2 \geq C\|f\|^2. \tag{67}$$

By the Lax–Milgram theorem, $R_{\text{ZM}} B_c B_c^\star$ has a bounded inverse. ◻

Proof of Theorem 3.18. With P_{D} as above, write the operator P_+ from (63) as

$$P_+ = P_{\text{B}} + P_{\text{B}}^\perp C P_{\text{B}}^\perp \qquad \text{with} \quad C = B_c^\star \left(R_{\text{ZM}} B_c B_c^\star\right)^{-1} B_c,$$

and define P_c through (64). We only show that P_+ is the orthogonal projection from $L^2(\partial\Omega;\mathbb{R}^d)$ onto $H^2_+(\partial\Omega)$. The rest of the proof follows the same steps as the proof of Theorem 3.8.

The operator P_+ is self-adjoint, since P_{B} and $P_{\text{B}}^\perp$ clearly are, and $(R_{\text{ZM}} B_c B_c^\star)^{-1}$ is self-adjoint by Lemma 3.19. Further,

$$P_+ P_+ = P_{\text{B}} + P_{\text{B}}^\perp C P_{\text{B}}^\perp C P_{\text{B}}^\perp. \tag{68}$$

The idempotence will follow once we establish that $C P_{\text{B}}^\perp C P_{\text{B}}^\perp = C P_{\text{B}}^\perp$.

First, we show that $P_{\text{B}}^\perp C P_{\text{B}}^\perp = C P_{\text{B}}^\perp$. To see this, observe from Lemma 2.9 and the definition of $B_c^\star$ that $B_c^\star 1 = B_c^\star \nu_{\text{o}}$. For $f \in L^2(\partial\Omega;\mathbb{R}^d)$, we then get from the definition of C that

$$\left\langle C P_{\text{B}}^\perp f, B_c^\star 1\right\rangle = \left\langle C P_{\text{B}}^\perp f, B_c^\star \nu_{\text{o}}\right\rangle = \left\langle P_{\text{B}}^\perp f, B_c^\star \nu_{\text{o}}\right\rangle = \left\langle P_{\text{B}}^\perp f, B_c^\star 1\right\rangle = 0. \tag{69}$$

Thus, $P_{\text{B}} C P_{\text{B}}^\perp f = 0$ and $P_{\text{B}}^\perp C P_{\text{B}}^\perp f = \left(\mathbb{1} - P_{\text{B}}\right) C P_{\text{B}}^\perp f = C P_{\text{B}}^\perp f$.

Next, we prove that $C C P_{\text{B}}^\perp = C P_{\text{B}}^\perp$. For this, observe that (69) yields $\left\langle B_c C P_{\text{B}}^\perp f, 1\right\rangle = 0$, so that $B_c C P_{\text{B}}^\perp f$ has zero mean. Hence,

$$C C P_{\text{B}}^\perp = B_c^\star \left(R_{\text{ZM}} B_c B_c^\star\right)^{-1} \left(R_{\text{ZM}} B_c B_c^\star\right) \left(R_{\text{ZM}} B_c B_c^\star\right)^{-1} B_c P_{\text{B}}^\perp = C P_{\text{B}}^\perp. \tag{70}$$

Consequently, $C P_{\text{B}}^\perp C P_{\text{B}}^\perp = C P_{\text{B}}^\perp$, and (68) says that $P_+ P_+ = P_+$, guaranteeing that P_+ is an orthogonal projection.

As to the range of P_+, it is easy to see that $\mathcal{N}(P_+) = \mathcal{N}(B_c)$, and from Theorem 3.15 we get $\mathcal{N}(B_c) = \mathcal{N}(\mathcal{P}_c)$. Thus, by Lemma 3.16, we arrive at $\mathcal{R}(P_+) = \mathcal{N}(\mathcal{P}_c)^\perp = H^2_+(\partial\Omega)$. ◻

4. Skew-orthogonal decompositions of fields. In this section we prove two skew-orthogonal decompositions of $L^2(\partial\Omega;\mathbb{R}^d)$. The first is the Hardy–Hodge decomposition of $L^2(\partial\Omega;\mathbb{R}^d)$. The second is the splitting of $L^2(\partial\Omega;\mathbb{R}^d)$ into the spaces $I(\partial\Omega)$, $O(\partial\Omega)$, and $D(\partial\Omega)$.

We begin with the Hardy–Hodge decomposition, which is a simple consequence of the above analysis.

THEOREM 4.1. *The space $L^2(\partial\Omega;\mathbb{R}^d)$ splits into a not necessarily orthogonal direct sum as*

$$L^2(\partial\Omega;\mathbb{R}^d) = H^2_+(\partial\Omega) + H^2_-(\partial\Omega) + D_f(\partial\Omega). \tag{71}$$

Proof. Note from the definition of $B_i^\star$ and $B_c^\star$ that for every $\varphi \in L^2(\partial\Omega)$,

$$(B_i^\star - B_c^\star)\,\varphi = \eta\varphi, \qquad \frac{1}{2}\,(B_i^\star + B_c^\star)\,\varphi = \eta K^\star\varphi + \nabla_{\mathrm{T}} S\varphi. \tag{72}$$

Now, let f be in $L^2(\partial\Omega;\mathbb{R}^d)$. We use the Hodge decomposition from Lemma 3.11 and write $f = P_{\mathrm{D}}^\perp f + P_{\mathrm{D}} f$, with $P_{\mathrm{D}}^\perp f = \eta f_\eta + P_{\mathrm{G}} f_{\mathrm{T}}$. Since $P_{\mathrm{G}} f_{\mathrm{T}}$ is a gradient field, there exists a function $\varphi \in L^2(\partial\Omega)$ such that $P_{\mathrm{D}}^\perp f = \eta f_\eta + \nabla_{\mathrm{T}} S\varphi$. Choosing

$$h_+ = -B_c^\star\left(f_\eta - \left(\frac{1}{2} + K^\star\right)\varphi\right) \in H^2_+(\partial\Omega), \tag{73}$$

$$h_- = \quad B_i^\star\left(f_\eta + \left(\frac{1}{2} - K^\star\right)\varphi\right) \in H^2_-(\partial\Omega), \tag{74}$$

and using (72), we obtain

$$h_+ + h_- = (B_i^\star - B_c^\star)\,(f_\eta - K^\star\varphi) + \frac{1}{2}\,(B_i^\star + B_c^\star)\,\varphi = \eta f_\eta + \nabla_{\mathrm{T}} S\varphi = P_{\mathrm{D}}^\perp f.$$

The desired decomposition is then given by $f = h_+ + h_- + P_{\mathrm{D}} f$.

For uniqueness, we first show that $H^2_-(\partial\Omega) \cap H^2_+(\partial\Omega) = \{0\}$. To see this, assume that g is in $H^2_-(\partial\Omega) \cap H^2_+(\partial\Omega)$. Then, there exist two functions $\phi, \chi \in L^2(\partial\Omega)$ such that

$$B_i^\star \phi = B_c^\star \chi = g. \tag{75}$$

The normal and the tangent components of that equation read

$$\left(\frac{1}{2} + K^\star\right)\phi = -\left(\frac{1}{2} - K^\star\right)\chi \qquad \text{and} \qquad \nabla_{\mathrm{T}} S\phi = \nabla_{\mathrm{T}} S\chi. \tag{76}$$

The equation for the tangent part says that ϕ differs from χ by the null-space of $\nabla_{\mathrm{T}} S$. By Remark 2.10, the null-spaces of $\left(\frac{1}{2} - K^\star\right)$ and $\nabla_{\mathrm{T}} S$ are the same and thus the equation for the normal component reads

$$\left(\frac{1}{2} + K^\star\right)\phi = -\left(\frac{1}{2} - K^\star\right)\phi, \tag{77}$$

leading to $\phi = 0$. It follows that $g = B_i\,\phi = 0$.

Now, assume that $P_{\mathrm{D}}^\perp f = g_- + g_+$ for some $g_- \in H^2_-(\partial\Omega)$ and $g_+ \in H^2_+(\partial\Omega)$; then we have that

$$P_{\mathrm{D}}^\perp f = h_+ + h_- = g_+ + g_-.$$

Hence, the two functions $g_- - h_-$ and $g_+ - h_+$ are in $H^2_-(\partial\Omega) \cap H^2_+(\partial\Omega)$, and thus, $h_- = g_-$ and $h_+ = g_+$. ∎

Next, we prove the skew-orthogonal decomposition involving $I(\partial\Omega)$ and $O(\partial\Omega)$.

LEMMA 4.2. *It holds that $D(\partial\Omega) = D_f(\partial\Omega)$.*

Proof. Recall that by definition $D(\partial\Omega) = \mathcal{N}(\mathcal{P}_i) \cap \mathcal{N}(\mathcal{P}_c)$. Let h be in $\mathcal{N}(\mathcal{P}_i) \cap \mathcal{N}(\mathcal{P}_c)$. Then, by (25) and (55) we have that $B_c h = 0 = B_i h$. Therefore,

$$0 = (B_i - B_c)\, h = h_\eta. \tag{78}$$

Consequently, $\nabla_{\mathrm{T}} S h = 0$ and thus h_{T} is divergence-free. Conversely, if h is in $D_f(\partial\Omega)$, then its normal component is zero by definition and it holds that $B_i h = 0 = B_c h$. From (25) and (55) it follows that h is in $\mathcal{N}(\mathcal{P}_i) \cap \mathcal{N}(\mathcal{P}_c)$. ∎

THEOREM 4.3. *The space $L^2(\partial\Omega; \mathbb{R}^d)$ splits into a not-necessarily orthogonal direct sum as*

$$L^2(\partial\Omega; \mathbb{R}^d) = I(\partial\Omega) + O(\partial\Omega) + D(\partial\Omega). \tag{79}$$

Proof. We will show that for every $f \in L^2(\partial\Omega; \mathbb{R}^d)$ there exists a function $h \in I(\partial\Omega)$ such that $f - h \in \mathcal{N}(\mathcal{P}_c)$; then since $\mathcal{N}(\mathcal{P}_c) = O(\partial\Omega) \oplus D_f(\partial\Omega)$ and since $D(\partial\Omega) = D_f(\partial\Omega)$ the decomposition will follow as $f = h + P_{\mathrm{D}}^{\perp}(f - h) + P_{\mathrm{D}}(f - h)$.

Recall that a function g is in $\mathcal{N}(\mathcal{P}_c)$ if and only if $B_c(g) = 0$. By Theorem 3.3, every $h \in I(\partial\Omega)$ satisfies

$$\left(\frac{1}{2} + K\right) h_\eta = -\left(\nabla_{\mathrm{T}} S\right)^\star h_{\mathrm{T}}. \tag{80}$$

Thus, we require in view of (53) that

$$B_c\,(f - h) = -\left(\frac{1}{2} - K\right) f_\eta + \left(\nabla_{\mathrm{T}} S\right)^\star f_{\mathrm{T}} + h_\eta = 0. \tag{81}$$

Choose $h_\eta = \left(\frac{1}{2} - K\right) f_\eta - \left(\nabla_{\mathrm{T}} S\right)^\star f_{\mathrm{T}}$. By the proof of Corollary 3.13, we can find h_{T} such that $h = \eta h_\eta + h_{\mathrm{T}}$ is in $I(\partial\Omega)$ and $f - h$ is in $\mathcal{N}(\mathcal{P}_c)$ by construction.

As in the previous theorem, uniqueness follows since $I(\partial\Omega) \cap O(\partial\Omega) = \{0\}$. This concludes the proof. ∎

5. Special case—the sphere. In this section we show that if $\partial\Omega$ is a sphere, then the decompositions in (34), (61), (71), and (79) become equal. To begin, we prove the following theorem.

THEOREM 5.1. *The Hardy spaces are orthogonal if and only if $\partial\Omega$ is a sphere.*

It is important to keep in mind that Ω is bounded; otherwise the statement is not true and a half-space whose boundary is a hyperplane is a counterexample (although a half-space is arguably a very large ball). The theorem is known for Clifford-analytic Hardy spaces [35, Thm. 1.1], which comprise Clifford algebra-valued functions. In contrast to this, the present Hardy spaces are smaller as they can be seen as vector-valued Clifford-analytic functions only. Therefore, the above statement requires proof.

Proof. It is well known that Hardy spaces on a sphere are orthogonal (for example, [37]). Thus, we only have to prove the opposite.

Assume that the Hardy spaces are orthogonal. Then, by Lemmas 3.5 and 3.16 we have for every $\varphi, \chi \in L^2(\partial\Omega)$,

$$0 = \left\langle B_i^\star \varphi, B_c^\star \chi \right\rangle = \left\langle B_c B_i^\star \varphi, \chi \right\rangle, \tag{82}$$

which implies the operator identity $B_cB_i^\star = 0$. Then, $B_iB_c^\star = (B_cB_i^\star)^\star = 0$ and $B_cB_i^\star - B_iB_c^\star = 0$. Using the definition of operators B_i, B_c, and their adjoints, the latter identity yields

$$B_cB_i^\star - B_iB_c^\star = K - K^\star = 0. \tag{83}$$

Consequently, the double layer potential must be self-adjoint. This, however, can only happen when $\partial\Omega$ is a sphere by [35, Thm. 4.23]. This concludes the proof. ∎

COROLLARY 5.2. *$I(\partial\Omega)$ and $O(\partial\Omega)$ are orthogonal if and only if $\partial\Omega$ is a sphere.*

Proof. The spaces $I(\partial\Omega)$ and $H^2_-(\partial\Omega)$ are orthogonal by Theorem 3.8, and the spaces $O(\partial\Omega)$ and $H^2_+(\partial\Omega)$ are orthogonal by Theorem 3.18. Moreover, from (71) and (79), we also have $I(\partial\Omega) + O(\partial\Omega) = P_{\mathrm{D}}^\perp L^2(\partial\Omega;\mathbb{R}^d) = H^2_-(\partial\Omega) + H^2_+(\partial\Omega)$. Thus, $I(\partial\Omega)$ is orthogonal to $O(\partial\Omega)$ if and only if $H^2_-(\partial\Omega)$ is orthogonal to $H^2_+(\partial\Omega)$, and the assertion follows from Theorem 5.1. ∎

COROLLARY 5.3. *If $\partial\Omega$ is a sphere, then $I(\partial\Omega) = H^2_+(\partial\Omega)$ and $O(\partial\Omega) = H^2_-(\partial\Omega)$.*

Proof. On a sphere $I(\partial\Omega)$ and $H^2_+(\partial\Omega)$ are both orthogonal to $H^2_-(\partial\Omega)$. Moreover, $I(\partial\Omega) \oplus H^2_-(\partial\Omega) = H^2_+(\partial\Omega) \oplus H^2_-(\partial\Omega)$, by (34) and (71). Hence, we get that $I(\partial\Omega) = H^2_+(\partial\Omega)$. An analogous argument holds for $O(\partial\Omega)$ and $H^2_-(\partial\Omega)$. ∎

By the above corollary it is immediate that (34), (61), (71), and (79) are identical when $\partial\Omega$ is a sphere.

Appendix A. The definition of strongly Lipschitz domains is standard and can be found in many textbooks, including [33, 38]. Nevertheless, the basic differential-geometric notions on Lipschitz manifolds are not easy to ferret out in the literature. For the reader's convenience, we therefore recall the concepts here.

In the following $\mathbb{B}$ will be an open ball in $\mathbb{R}^{d-1}$, and U will denote a doubly truncated cylinder whose cross-section is $\mathbb{B}$.

Strongly Lipschitz domain. We call a bounded region Ω in $\mathbb{R}^d$ a strongly Lipschitz domain if for each $x \in \partial\Omega$ there is a cylinder U, a rigid motion L, and a Lipschitz function $\psi\colon \mathbb{B} \to \mathbb{R}$ such that

$$U \cap \Omega = \{L(y,t) \,: y \in \mathbb{B},\ 0 \le t < \psi(y)\} \quad \text{and} \quad U \cap \partial\Omega = \{L(y,\psi(y)) \,: y \in \mathbb{B}\}.$$

Since $\partial\Omega$ is compact, we can cover it with finitely many such cylinders U_j associated with $\mathbb{B}_j$, L_j, and ψ_j, for $j \in \{1,\dots,N\}$. If, in addition, we introduce the projection onto the first $(d-1)$ components as $P_{d-1}\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$, then the maps $\phi_j := P_{d-1} \circ L_j^{-1}\colon U_j \cap \partial\Omega \to \mathbb{B}_j$ define a system of charts on $\partial\Omega$ with Lipschitz inverse $\phi_j^{-1}\colon \mathbb{B}_j \to \partial\Omega \subset \mathbb{R}^d$ given by $\phi_j^{-1}(y) = (y, \psi_j(y))$; this provides us with the bi-Lipschitz change of charts and makes $\partial\Omega$ a Lipschitz manifold.

Singular and regular points. A point $x \in \partial\Omega$ is called singular if there is a $j \in \{1,\dots,N\}$ such that $x \in U_j$ and ϕ_j^{-1} is not differentiable at $\phi_j(x)$. A point which is not singular is called regular. We denote the set of regular points by $\mathrm{Reg}(\partial\Omega)$ and put $\mathrm{Reg}(\mathbb{B}_j) = \phi_j(\mathrm{Reg}(\partial\Omega) \cap U_j)$. Since ϕ_j is bi-Lipschitz at regular points, its derivative, $D\phi_j^{-1}(y)$, is injective there.

The set of singular points has σ-measure zero on $\partial\Omega$. One can see this as follows: by Rademacher's theorem [39, Thm. 3.2] the set of singular points $\mathbb{B}_j \setminus \mathrm{Reg}(\mathbb{B}_j)$ has Lebesgue measure zero for each j. But the Lebesgue measure on $\mathbb{R}^{d-1}$ coincides with the $(d-1)$-dimensional Hausdorff measure, $\mathcal{H}^{d-1}$, and thus the set of singular points has $\mathcal{H}^{d-1}$-measure zero, which is preserved by Lipschitz functions ϕ_j^{-1} [40].

With the above definition, the set of regular points depends on the atlas. Nevertheless, there is also an intrinsic atlas-independent definition (see, for example, [27]) as the set of those points for which the measure theoretic normal to Ω exists.

Tangent space. At $x \in U_j \cap \mathrm{Reg}(\partial\Omega)$, we define the tangent space $\mathrm{T}_x(\partial\Omega) \subset \mathbb{R}^d$ to be $\mathcal{R}(D\phi_j^{-1}(\phi_j(x)))$. Then, each $X \in \mathrm{T}_x(\partial\Omega)$ has a local representative in the chart (U_j, ϕ_j), which is the unique vector $v \in \mathbb{R}^{d-1}$ such that $X = D\phi_j^{-1}(\phi_j(x))v$. At each regular point the tangent space has dimension $d-1$, and thus, we can define the outer unit normal $\eta(x)$, oriented such that for small $t > 0$ the vector $x + t\eta(x)$ is in $\bar{\Omega}^c$.

Differentiability on $\partial\Omega$. A map $g\colon \partial\Omega \to \mathbb{R}^k$ is said to be differentiable at $x \in \mathrm{Reg}(\partial\Omega)$ if $g \circ \phi_j^{-1}$ is differentiable at $\phi_j(x)$. If v is the local representative of X, then the derivative $Dg(x)\colon \mathrm{T}_x(\partial\Omega) \to \mathbb{R}^k$ is defined by $Dg(x)(X) = D(g\circ\phi_j^{-1})(\phi_j(x))v$. By the chain rule, the derivative is chart independent.

When $g : \partial\Omega \to \mathbb{R}$ is differentiable at $x \in \mathrm{Reg}(\partial\Omega)$, the map $Dg(x)$ is a linear form on $\mathrm{T}_x(\partial\Omega)$, and thus it can be represented as $X \mapsto \langle X, Y\rangle_{\mathbb{R}^d}$ for some unique vector $Y \in \mathrm{T}_x(\partial\Omega)$ that we call the tangential gradient of g at x and denoted by $\nabla_{\mathrm{T}} g(x)$.

If $f\colon \partial\Omega \to \mathbb{R}$ is a Lipschitz function on $\partial\Omega$, then $f \circ \phi_j^{-1} : \mathbb{B}_j \to \mathbb{R}^d$ is also Lipschitz for each j. By Rademacher's theorem, $f\circ\phi_j^{-1}$ is differentiable a.e. on $\mathbb{B}_j$, and consequently, f is differentiable a.e. on $\partial\Omega$. Moreover, the derivatives of a Lipschitz function are uniformly bounded by the Lipschitz constant.

In fact, we can extend f to a function F defined on a small neighborhood around $\partial\Omega$, such that $\nabla_{\mathrm{T}} f$ reads as the Euclidean gradient of F. To verify this, extend f as follows: for each j, define $\tilde{f}_j\colon U_j \to \mathbb{R}$ by $\tilde{f}_j(L_j(y,t)) = f(L_j(y, \psi_j(y)))$. As the differential of $\tilde{f}_j$ is independent of t, it is easily seen that $\tilde{f}_j$ is differentiable a.e. on $\partial\Omega \cap U_j$. Using a smooth partition of unity relative to the U_j's, Ω, and $\bar{\Omega}^c$, we can glue the $\tilde{f}_j$'s together into a single Lipschitz map $F : \cup_j^N U_j \to \mathbb{R}^d$. This map has the following properties: its restriction to $\partial\Omega$ is f; it is differentiable a.e. on $\partial\Omega$; and there its Euclidean gradient, ∇F, coincides with $\nabla_{\mathrm{T}} f$.

Sobolev spaces. The Sobolev space $W^{1,2}(\partial\Omega)$ is the Hilbert space obtained as the completion of Lipschitz functions with respect to the norm

$$\|\psi\|_{W^{1,2}(\partial\Omega)} = \left(\|\psi\|^2_{L^2(\partial\Omega)} + \|\nabla_{\mathrm{T}}\psi\|^2_{L^2(\partial\Omega;\mathbb{R}^d)}\right)^{1/2}. \tag{84}$$

Equivalently, the function ψ is in $W^{1,2}(\partial\Omega)$ if and only if for each chart (U_j, ϕ_j) the function $\psi \circ \phi_j^{-1}$ is in the Euclidean Sobolev space $W^{1,2}(\mathbb{B}_j)$; thus, $W^{1,2}(\partial\Omega)$ agrees with the space $L^2_1(\partial\Omega)$ used in [25, Def. 1.7]. Each $\psi \in W^{1,2}(\partial\Omega)$ has a well defined tangential gradient $\nabla_{\mathrm{T}}\psi \in T(\partial\Omega) \subset L^2(\partial\Omega;\mathbb{R}^d)$; see [33, Thm. 3.17].

LEMMA A.1. *For $f \in L^2(\partial\Omega)$, it holds that $\nabla_T Sf$ defined in Lemma* 2.5 *is the tangential gradient of $Sf \in W^{1,2}(\partial\Omega)$.*

Proof. Let $F \in L^2(\partial\Omega, \mathbb{R}^d)$ be the nontangential limit of $\nabla\mathscr{S}f$ on $\partial\Omega$ from inside, which is known to exist a.e. We will show that the tangential component of F is equal to $\nabla_T Sf$. The limit of $\nabla\mathscr{S}f$ from outside can be handled similarly, and in view of the limiting relation in Lemma 2.5 this will achieve the proof. The statement being local, it is enough to proceed in a coordinate cylinder U_j, and we may assume for simplicity that it has vertical axis so that $\phi_j : U_j \cap \partial\Omega \to \mathbb{B}_j$ is the projection onto the first $d-1$ components and $\phi_j^{-1} : \mathbb{B}_j \to U_j \cap \partial\Omega$ is given by $\phi_j^{-1}(y) = (y, \Psi_j(y))$ for some Lipschitz function $\Psi_j : \mathbb{B}_j \to \mathbb{R}$. Set $e_d := (0, \ldots, 0, 1)^t$, and let $C_{\theta_1,e_n}(\xi)$ be a natural cone of aperture $2\theta_1$ at $\xi \in U_j \cap \partial\Omega$ in the chart (U_j, ϕ_j), contained in the cone $C_{\theta,Z(\xi)}(\xi)$ from the regular family of cones we have fixed. Recall that θ_1 can be taken independent of ξ. For $\varepsilon > 0$ small enough that $(y, \Psi_j(y) - \varepsilon) \in \Omega^+ \cap U_j$ when $y \in \mathbb{B}_j$, the smoothness of $\mathscr{S}f$ in Ω^+ implies that $h_\varepsilon(y) := \mathscr{S}f(y, \Psi_j(y) - \varepsilon)$ is Lipschitz in $\mathbb{B}_j$ with gradient $(\nabla h_\varepsilon(y))^t = (\nabla \mathscr{S}(y, \Psi_j(y) - \varepsilon))^t D\phi_j^{-1}(y)$. Let $\varepsilon_k \to 0$ and observe that $(y, \Psi_j(y) - \varepsilon_k)$ converges to $\xi = (y, \Psi_j(y))$ from within $C_{\theta, e_d}(\xi)$; hence $\nabla \mathscr{S}f(y, \Psi_j(y) - \varepsilon_k)$ converges for m_{d-1}-a.e. $y \in \mathbb{B}_j$ to $F(y, \Psi_j(y))$, while being dominated pointwise in norm by $(\nabla \mathscr{S}f)^M(y, \Psi_j(y))$; here, m_{d-1} indicates $(d-1)$-dimensional Lebesgue measure. As

$$\int_{\mathbb{B}_j} \left((\nabla \mathscr{S}f)^M(y, \Psi_j(y))\right)^2 \sqrt{1 + |\nabla \Psi_j|^2} dm_{d-1} = \int_{U_j \cap \partial\Omega} \left((\nabla \mathscr{S}f)^M(\xi)\right)^2 d\sigma(\xi)$$

because the image of $d\sigma$ in local coordinates is $\sqrt{1 + |\nabla \Psi_j|^2} dm_{d-1}$, and since $|\nabla \Psi_j|$ is uniformly bounded, we get that $(\nabla \mathscr{S}f)^M$ composed with ϕ_j^{-1} lies in $L^2(\mathbb{B}_j)$. Therefore, by dominated convergence, ∇h_{ε_k} converges in $L^2(\mathbb{B}_j)$ to $(D\phi_j^{-1})^t F \circ \phi_j^{-1}$. Likewise, h_{ε_k} converges pointwise a.e. to $Sf(y, \Psi_j(y))$ while being dominated pointwise by $(\mathscr{S}f)^M(y, \Psi_j(y))$ that lies in $L^2(\mathbb{B}_j)$. Therefore, $h_{\varepsilon_k} \to Sf \circ \phi_j^{-1}$ in $L^2(\mathbb{B}_j)$ by dominated convergence. Hence, if we pick a smooth function φ with compact support in $\mathbb{B}_j$ and pass to the limit in the relations $\int_{\mathbb{B}_j} \partial_{y_i} h_{\varepsilon_k} \varphi = -\int_{\mathbb{B}_j} h_{\varepsilon_k} \partial_{y_i} \varphi$, we find that $Sf \circ {\phi_j^{-1}}_{|B(y_0,\delta)}$ lies in $W^{1,2}(\mathbb{B}_j)$ with

$$\nabla(Sf \circ \phi_j^{-1}) = (D\phi_j^{-1})^t F \circ \phi_j^{-1}. \tag{85}$$

Since the normal $\eta(\xi)$ is orthogonal to the columns of $D\phi_j^{-1}(\phi_j(\xi))$ that span the tangent pace $\mathrm{T}_\xi(\partial\Omega)$, a short computation shows that (85) is equivalent to saying that the tangential component of the field F is the gradient of Sf. ☐

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