Analytic Perturbation Theory and its Applications

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Contents

Summary

The properties of matrices depending analytically on a small parameter are investigated in the thesis. There are several motivations to study this problem. Firstly, it is often necessary to deal with the group of problems which have a common structure but differ from each other in one or more parameters. Secondly, admitting that we never have precise data, it is of great importance to analyse the influence of data perturbations. Sometimes even small perturbations of the data cause the dramatic changes in the problem properties. The latter case is called the singular perturbation. Despite a large number of monographs and papers written on perturbation theory, we can still obtain new significant results. Among the main contributions of the present thesis are: the construction of effective algorithms to compute the Laurent series for inverses of singularly perturbed matrices; the perturbation analysis of null spaces with application to the general eigenvalue problem; the calculation of Laurent series for group inverses and Moore-Penrose generalized inverses; the perturbation analysis of the Sylvester matrix equations. In our research we use the complex analytic function technique as well as the traditional apparatus of linear algebra. The obtained theoretical results are applied to Markov chains, Markov decision processes, and to linear and nonlinear mathematical programming.

Declaration

I declare that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university; and that to the best of my knowledge it does not contain any materials previously published or written by another person except where due reference is made in the text of the thesis.

Konstantin Avrachenkov

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Dedication

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Chapter 1

Introduction

Hardly anybody except a PhD student and his supervisors can understand straight away what problems have been solved in the thesis by only looking at the thesis title. Often even a summary of several hundred words is of little help. The author of the present work hopes that this introductory chapter will remove the veil of mystery from the fancy words "Analytic Perturbation Theory".

The main motivation of this thesis can be stated in the following "philosophical" way: as long as we have to work with data, we have to deal with their perturbations. The term *perturbation* signifies that our data deviate from nominal values. As a rule, we do not know the value of this deviation. However, we do expect that this deviation is small and sometimes we can even point out to the most likely direction of this deviation.

Usually one stores and works with data in the matrix form. Therefore, it is natural to analyse the properties of matrices with perturbed elements. This general idea can be written mathematically in the form of the following *additive* perturbation

$$\tilde{A} = A + D, \tag{1.1}$$

where A is a matrix of nominal data, $\tilde{A}$ is a matrix of perturbed data and D is the perturbation itself. There are numerous publications devoted to this subject. An interested reader is refered to the comprehensive book of Sun and Stewart [140] and references therein. This general approach (as any general method) has a drawback that many its theoretical results are too imprecise to be applied in practice. Furthermore, we distinguish between regular and singular perturbations. If the rank of the perturbed data matrix $\tilde{A}$ is different from the rank of the original matrix A, then the perturbation is said to be *singular*, otherwise if the perturbed and original matrices have the same rank, the perturbation is said to be *regular*. Note that there are practically no mathematical tools available to analyse, in detail, the singular perturbation for the general additive scheme.

These latter remarks encourage us to consider a more particular type of perturbation

$$A(\varepsilon) = A + \varepsilon B, \tag{1.2}$$

where the perturbation depends on a small "scale" factor ε. From now on we will refer to ε as a *perturbation parameter*. Note that the perturbation parameter reflects the value of deviation of the perturbed data from the nominal set, and matrix B represents the likely direction of the data deviation. The perturbation model (1.2) is called *linear perturbation*. In linear algebra this object is also known as matrix pencil [58, 83]. Indeed, many results from [58, 83] can be applied in the perturbation analysis of (1.2). The next natural generalization of (1.2) is an *analytic perturbation*

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots\ , \tag{1.3}$$

where the above power series is assumed to be convergent in some neighbourhood of $\varepsilon = 0$.

So far we have motivated our research by the importance of studying the data perturbations. However, there is another huge area of applications for our theoretical results. Practically all mathematical models incorporate parameters in their structure, and, of course, a researcher is interested in the behaviour of the model in respect to the change of these parameters. This is the area of so-called *sensitivity analysis*. Sometimes even a slight change in one parameter produces an enormous difference in the model's output. Fortunately, the latter can be efficiently analysed by the techniques developed for the singular analytic perturbations. The first comprehensive study of the analytic perturbation theory was given in the book of Kato [87]. This book later became a classic in perturbation theory. Some authors [60, 108, 124] have also studied a particular case of (1.3), that is *polynomial perturbation*

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \dots + \varepsilon^p A_p\ . \tag{1.4}$$

Now a sceptical reader has a right to ask the following question: "If the perturbation theory has been greatly developed in several monographs, is it possible to find enough new material for a PhD thesis?" Of course, the author of the present thesis hopes for the positive answer. And this hope is built up on the following observation: Majority of researchers have concentrated their effort on the perturbation analysis of the eigenvalue problem

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon). \tag{1.5}$$

Namely, the researchers were interested in the behaviour of eigenvalues and eigenvectors of the perturbed operator in respect to the change of perturbation parameter. Of course, this is a very important problem for many practical applications. However, there are several other fundamental mathematical problems, where the perturbation analysis can be carried out, which have consequent applications to practical problems. In particular, in this thesis we analyse the system of perturbed linear equations

$$A(\varepsilon)x(\varepsilon) = b(\varepsilon), \tag{1.6}$$

where $A(\varepsilon)$ is not necessarily square or regular. It is, perhaps, surprising that in comparison with the huge amount of the literature available on the topic of the perturbed eigenvalue problem, the research on the perturbed linear system (1.6) is quite limited. For instance, in his encyclopedic book of 427 pages on the perturbation theory Baumgartel devoted only 2 pages to the operator $A^{-1}(\varepsilon)$. And on these pages one can only find some general, nonconstructive, existence results. The primal goal of this thesis is to show that the problem (1.6) perhaps deserves as much atention as the eigenvaue problem (1.5). In fact, in Section 2.4.4 we will demonstrate how the results obtained for the problem (1.6) can be applied for the analysis of (1.5).

Now let us outline the contents and structure of the thesis. The results of the research have been divided in three chapters. The first chapter is devoted to the perturbation analysis of the linear system (1.6), or even more generally, to the perturbation analysis of operator equation

$$L(\varepsilon)x(\varepsilon) = c(\varepsilon),$$

where linear operator $L(\varepsilon)$ depends analytically on the parameter ε. In particular, if $L(\varepsilon)$ has an inverse for $\varepsilon \neq 0$, but sufficiently small, then we investigate the properties of the perturbed inverse operator $L^{-1}(\varepsilon)$ (or matrix-valued function $A^{-1}(\varepsilon)$ in finite-dimensional case). For example, we rely on the fact that $A^{-1}(\varepsilon)$ can always be expanded as a Laurent series

$$A^{-1}(\varepsilon) = \frac{1}{\varepsilon^s}B_{-s} + ... + \frac{1}{\varepsilon}B_{-1} + B_0 + \varepsilon B_1 + ... \ .$$

Then, a most efficient, to date, recursive procedure for the calculating of coefficients $B_k, k = -s, -s+1, ...$ is provided. In the case when matrix $A(\varepsilon)$ is not invertible for any value of ε, we provide the perturbation analysis of its generalized inverse $A^{\dagger}(\varepsilon)$. If in the linear system (1.6) the right hand side $b(\varepsilon) \equiv 0$, then we carry out the perturbation analysis of the null space. Interestingly, the latter can be applied to the general eigenvalue problem (1.5). Finally, in the last section of the first chapter we analyse the perturbation of the Sylvester matrix equation. The latter can again be viewed as general linear operator equation with a particular choice of linear operator $L(\varepsilon) : X \mapsto A(\varepsilon)X + XB(\varepsilon)$. This first chapter serves as a theoretical basis for the next two chapters which are quite independent of each other and can be read separately.

In the second chapter we consider the application of the analytic perturbation theory to Markov chains and Markov decision processes (MDPs, for short). In particular, we study Markov chains and MDPs with analytically perturbed transition matrices. The novelty of our approach is that we do not make any assumptions (e.g., ergodicity) about the structure of the perturbed and original Markov chains. We analyse not only stationary distributions of the perturbed Markov chains but also their fundamental matrices. The latter allows us to perform the perturbation analysis of the mean first passage times. In addition, our methods can be applied for the determination of Blackwell optimal policies in MDPs. The latter is possible since the resolvent of the transition matrix for small values of the interest rate can be expanded as a Laurent series with a nice structure.

In the third chapter we apply the analytic perturbation theory to optimization problems. Firstly, we consider linear programs with coefficients depending on a small parameter. We construct an asymptotic simplex method which iterates over the field of Laurent series. Then, we consider a mathematical program with nonlinear convex objective function and linear constraints with parametric perturbations. We show that a modification of gradient projection method can be applied to the auxiliary mathematical program which is well defined.

Finally, we would like to note that the literature review is also divided in three parts according to the chapters and placed at the beginning of each chapter. We hope that this approach will permit each chapter to be read more or less independently of the other parts of the thesis.

Chapter 2

Analytic Perturbation of Singular Linear Systems

2.1 Background and literature review

This chapter provides a theoretical basis for the numerous applications developed in the next two parts of the thesis. As was mentioned in the introductory chapter, all problems of the present chapter can be viewed as particular cases of the analytically perturbed operator equation

$$L(\varepsilon)x(\varepsilon) = c(\varepsilon). \tag{2.1}$$

In Sections 2.2 and 2.3 we study the inverse of a nearly singular linear operator. Namely, we assume that the perturbed operator $L(\varepsilon)$ is invertible in some small neighbourhood of $\varepsilon = 0$, but the unperturbed operator $L(0)$ has no inverse. We provide efficient recursive methods for the calculation of coefficients of the Laurent series expansion for the inverse operator $L^{-1}(\varepsilon)$. In Section 2.2 we discuss the case of finite dimensional operators, whereas in Section 2.3 we address this problem in the context of general Hilbert spaces.

The problem of inversion of a nearly singular analytic matrix valued function was probably originated in the paper by Keldysh [88]. In that paper he studied the case of a polynomial perturbation

$$A(\varepsilon) = I - A_0 - \varepsilon A_1 - ... - \varepsilon^m A_m,$$

where A_i are compact operators on Hilbert space. In particular, he showed that the principal part of the Laurent series expansion for the inverse operator $A^{-1}(\varepsilon)$ can be given in terms of generalized Jordan chains. The generalized Jordan chains were initially developed in the context of matrix and operator polynomials (see [60, 108, 124] and numerous references therein). However, this concept can be easily generalized for the case of analytic perturbation

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + ...$$

Following [61] and [62], we say that the vectors $\varphi_0, ..., \varphi_{r-1}$ form a *generalized Jordan chain* of the perturbed matrix $A(\varepsilon)$ at $\varepsilon = 0$, if $\varphi_0 \neq 0$ and if

$$\sum_{i=0}^{k} A_i \varphi_{k-i} = 0,$$

for each $k = 0, ..., r-1$. Note that φ_0 is an eigenvector of the unperturbed matrix A_0 corresponding to the zero eigenvalue. The number r is called the length of the Jordan chain and φ_0 is the initial vector. Let $\{\varphi_0^{(j)}\}_{j=1}^{p}$ be a system of linearly independent eigenvectors, which span the null space of A_0. Then one can construct Jordan chains initializing at each of the eigenvectors $\varphi_0^{(j)}$. This generalized Jordan set plays a crucial role in the perturbation analysis of analytic matrix valued function $A(\varepsilon)$. We would like also to mention that the Jordan set can also be expressed in terms of root functions [59, 62].

Firstly we need to discuss the analytic properties of the inverse $A^{-1}(\varepsilon)$ of an analytic matrix valued function. Namely, suppose $A(\varepsilon)$ is invertible in some punctured disc: $0 < |\varepsilon| < \varepsilon_{max}$. Does there exist a Laurent series expansion

$$A^{-1}(\varepsilon) = \frac{1}{\varepsilon^s} B_{-s} + ... + \frac{1}{\varepsilon} B_{-1} + B_0 + \varepsilon B_1 + ... \tag{2.2}$$

which is convergent in some neighbourhood of $\varepsilon = 0$? This question was addressed in a number of papers. Vishik and Lyusternik in [149] studied the case of a linear perturbation $A(\varepsilon) = A_0 + \varepsilon A_1$ and showed that one can always express $A^{-1}(\varepsilon)$ as a Laurent series as long as $A(\varepsilon)$ is a matrix valued function and $A(\varepsilon)$ is invertible in some (maybe punctured) neighbourhood of the origin. In addition, the idea of the undetermined coefficient method for the calculation of Laurent series terms was given in [149]. Later many authors have obtained existence results for the operator valued analytic and meromorphic functions. Ribaric and Vidav [123] have proved the existence of the Laurent series (2.2) for an analytic operator valued function if the zero eigenvalue of the unperturbed operator A_0 is isolated and has finite algebraic multiplicity. They used a decomposition $A(\varepsilon) = P + (A(\varepsilon) - P)$, where P is an eigenprojection corresponding to the zero eigenvalue [87]. Gohberg and Sigal [62] have proved the existence of the Laurent series for the Fredholm meromorphic operator valued function. Also in [62] the structure of the principal part of the Laurent series (2.2) is given in terms of generalized Jordan chains. The results of [62] are based on the local Smith form. This form was first introduced in [58] for matrix polynomials and then it was generalized for analytic operator functions by the authors of [62]. Bart in [10] generalized the notions of ascent and descent of bounded linear operators [102, 143] to holomorphic operator-valued functions. Using the generalized ascent and descent, Bart gave a necessary and sufficient condition for the existence of the Laurent series (2.2) for the inverse of a Fredholm holomorphic operator-valued function. Note that the generalized Jordan eigenvectors described above constitute the bases for ascent and descent spaces defined in [10]. Early Langenhop in [98] provided a similar development for a linear perturbation $A(\varepsilon) = A_0 + \varepsilon A_1$ in the finite dimensional case. In particular, he has derived very useful recurrent formulae for the coefficients of the Laurent series (2.2)

$$B_{-(k+1)} = (-B_{-1}A_0)^k B_{-1} \quad \text{and} \quad B_k = (-B_0A_1)^k B_0, \quad k = 1, 2, \dots . \tag{2.3}$$

Therefore, if the coefficients B_{-1} and B_0 are known, then the other coefficients can be computed easily in a recursive manner. The proof of the above recurrent formulae was simplified further by Schweitzer [133] (see also [135]) and was generalized to the case of analytic perturbation by Korolyuk and Turbin [91]. Howlett [73] showed that the coefficients of the Laurent series (2.2) satisfy a simple recursive relation in the case of polynomial perturbation. Note that in order to use the recurrent formulae (2.3) or their analog for the analytic case [91], one needs first to compute the singular part and the first regular term of (2.2). The approach of Korolyuk and Turbin [91] for the computation of the singular part of (2.2) is restricted to the case of the semisimple zero eigenvalue. Also Campbell and Rose [22, 126] studied the case of linear perturbation. First they provided formulae for the calculation of (2.2), if $A_0A_1 = A_1A_0$, based on Drazin generalized inverses and then they showed how a general linear perturbation can be transformed to the commutative case. In addition, they applied their results to singular differential equations [22, 23, 25]. Other applications of the Laurent series (2.2) can be found in [73, 96, 116, 145] and in the next two chapters of the present thesis. In [114] Pattabhiraman and Lancaster, using the concept of a pseudo-projection which is defined in the spirit of Kato's book [87], have analysed the case of a simple pole. Also we would like to mention the book of Vainberg and Trenogin [145], where they have studied the case of holomorphic Fredholm operator valued function and used the Lyapunov-Schmidt regularization scheme and the techniques of branching theory.

Despite a relatively large number of papers and books on the present topic, few papers have addressed the question of efficient computation of the coefficients in (2.2) and this research direction is far from being completed. Therefore one of the main goals of the present thesis is to provide efficient computational procedures for the Laurent series (2.2).

In Section 2.2 we employ an algebraic approach based on the use of Moore-Penrose generalized inverses and a so-called *reduction process*. We would like to emphasize that our approach differs significantly from the original version of the *reduction process* proposed by Kato [87] for the eigenvalue problem. In Section 2.3 we generalize the computationally efficient approach of Schweitzer and Stewart [135] to the case of Hilbert spaces and analytically perturbed operators. In comparison with existing computational schemes for the inversion of operator valued functions [14, 91, 145], our generalization of the Schweitzer and Stewart approach allows us to treat more complex cases. For instance, in the majority of previous publications the authors consider Fredholm operator valued functions, which are known to have a null space with a finite dimension. However, the approach presented in Section 2.3 allows us to deal with operators that have infinite dimensional null spaces.

In Section 2.4 we study the behaviour of the null space of an analytically perturbed matrix (1.3). We assume that the perturbed matrix $A(\varepsilon)$ has a non-trivial null space for ε different from zero and sufficiently small. It turns out that the dimension of the perturbed null space $\bar{m}$ is less than or equal to the dimension m of the unperturbed matrix [12, 61]. Our primary interest is in the singular case when $\bar{m} < m$. In particular, we are looking for an orthogonal family $V(\varepsilon) = [v_1(\varepsilon), ..., v_{\bar{m}}(\varepsilon)]$ of analytic vector-valued functions which constitute a basis for the perturbed null space. The matrix $V(\varepsilon)$ has to satisfy the equation

$$A(\varepsilon)V(\varepsilon) = 0 \tag{2.4}$$

in some neighbourhood of $\varepsilon = 0$. Note that the above equation can be viewed as a particular case of the operator equation (2.1) with $c(\varepsilon) \equiv 0$. Since the analytic family $V(\varepsilon)$ is known to exist [60, 61], we can construct a solution to our problem in the form of a power series

$$V(\varepsilon) = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \ldots .$$

Upon substituting (1.3) and the above power series into (2.4) and equating coefficients with the same powers of ε, we obtain an infinite system of linear equations. We propose to solve this system by methods similar to those given in Section 2.2. Certainly, the perturbation of null spaces can be considered as a particular case of the general perturbed eigenvalue problem [14, 27, 28, 87, 104, 113, 121, 145, 149]

$$A(\varepsilon)v(\varepsilon) = \lambda(\varepsilon)v(\varepsilon). \tag{2.5}$$

However, it is important to note that our results on the perturbation of null spaces together with a Newton diagram technique [14, 113, 145] can be applied to the solution of the general problem (2.5). This confirms the claim that the study of the singularly perturbed linear equations is no less important than the more traditional perturbation analysis of eigenvalue problems.

All known methods for the perturbation analysis of the eigenvalue problem can be devided into two principal categories. The first category [14, 87] is based on analytic function theory and on spectral theory. Instead of eigenvectors the authors study the behaviour of the spectral eigenprojection which can be given in terms of the resolvent by

$$P(\varepsilon) = \frac{1}{2\pi i} \int_{\Gamma} (A(\varepsilon) - \zeta I)^{-1} d\zeta, \tag{2.6}$$

where Γ is a positively-oriented piecewise smooth closed curve in the complex plane enclosing the eigenvalue under consideration. The second category of methods [27, 28, 104, 113, 121, 145, 149] are developed to obtain the asymptotic expansions for the perturbed eigenvectors and are based on different techniques of linear algebra. The approach described in Section 2.2 belongs to the latter category. We would like to emphasize that most of the existing methods for the general eigenvalue problem have more restrictions than our approach and they seem to be more complicated as well. For example, Rellich [121] studied only symmetric matrices, Kato [87] assumed the semi-simplicity of eigenvalues and Lidskii [104] introduced a special non-degeneracy assumption. None of these assumptions is needed for our approach.

The construction of the Laurent series expansion for the group inverse of an analytically perturbed matrix is yet another application of the perturbation analysis for null spaces. We discuss this issue in Subsection 2.4.4. There we have only outlined the computational procedure. This procedure is realized in detail in Chapter 3 in the context of fundamental matrices of Markov chains.

Note that the group inverse and other generalized inverses are needed when the operator equation (2.1) has many solutions if $\varepsilon = 0$. Therefore in Section 2.5 we continue to carry out the perturbation analysis of different generalized inverses.

Perturbation methods have been thoroughly investigated for the eigenvalue problem (see e.g. [14, 87]), but as pointed out in Sections 2.2 and 2.3 a smaller number of results are available for the inversion of perturbed linear operators, and only a few results are known for the perturbation of generalized inverses.

The primary goal of Section 2.5 is to consider the *reduced resolvent* of a finite-dimensional perturbed operator. This is also known as the *Drazin generalized inverse* in linear algebra [24, 28]. However, as we use spectral theory and the theory of analytic functions we will refer to this object as the reduced resolvent rather than the Drazin inverse.

As before we distinguish between regular and singular perturbations. The perturbation is said to be *regular* if it does not change the dimension of the null space. Otherwise, the perturbation is said to be *singular*. One of the main advantages of the present approach is that it allows analysis of both the regular and singular perturbations in a unified framework.

It turns out that the perturbation analysis of the reduced resolvent can be applied to the perturbation analysis of the group inverse and the *Moore-Penrose generalized inverse*.

Bart [10] and later Bart, Kaashoek and Lay [13] (see also related references therein) have studied the analytic properties of the relative inverse [24] of a meromorphic operator-valued function. In particular, they have given necessary and sufficient conditions for the relative inverse to be a finite-meromorphic function. Note that the Moore-Penrose generalized inverse is a particular member of the family of relative inverses [24]. However, no computational procedure was given in [10, 13] for the calculation of the Laurent series for the relative inverses of operator-valued functions. Stewart [139] provided a comprehensive analysis of the regular perturbations of the Moore-Penrose generalized inverse and pointed out that the error bounds grow infinitely as the perturbation vanishes in the singular case. He has given error bounds and a first order approximation for the general additive perturbation of the generalized inverse. The analytic perturbation is more restrictive than the general additive perturbation. However, it allows a deeper analysis of the singular case. The case of singular perturbations is interesting for it provides information on the lack of continuity of solutions to perturbed linear systems. The work on singular perturbation of the generalized inverse is very limited. Deif [36] proposed an algorithm for computing the Laurent series for the generalized inverse of an analytically perturbed matrix. However, to implement his algorithm one has to obtain power series expansions for *all* the quantities in the singular value decomposition, namely for all the eigenvectors and eigenvalues. This can be very costly, especially for matrices of high dimension. In contrast, we propose an algorithm, which works in appropriately constructed *reduced* subspaces of small dimension.

In Section 2.5 we prove the existence and obtain the Laurent series expansion of the reduced resolvent of a perturbed (finite-dimensional) linear operator. As a by-product, we also obtain the Laurent series of the Moore-Penrose generalized inverse of an analytically perturbed operator.

We provide recursive formulae for the (matrix) coefficients of the regular part of the Laurent series expansion for the reduced resolvent of the perturbed operator. These formulae require knowledge of the coefficients for the singular part and the latter are obtained via a new and simple reduction technique. In particular, this reduction technique uses a novel *group reduced resolvent* different from the one considered in Chatelin [28]. This group reduced resolvent relative to the group of eigenvalues $\Lambda = \{\lambda_i\}_{i=1}^k$ can be given in terms of a Cauchy integral

$$A^{\#\Lambda} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}(A - \zeta I)^{-1} d\zeta,$$

where Γ is a contour in the complex plane which encloses zero and the set of eigenvalues $\{\lambda_i\}_{i=1}^k$, but *none* of the other eigenvalues $\{\lambda_i\}_{i=k+1}^p$. This shows that the approach of Section 2.5 is closely related to the analytic approach of Kato [87], which uses the group eigenprojection in the form (2.6). Furthermore, it seems that the present approach complements the developments of Kato.

Note that if an operator is invertible, then its reduced resolvent is just the ordinary inverse. Therefore, our results are applicable to the problem of the inversion of nearly singular operators (see Sections 2.2, 2.3 and [10, 61, 62, 73, 91, 96, 98, 126, 133, 135, 145, 149]). Hence we have obtained as a by-product another new method for the inversion of singularly perturbed operators, which is based on the techniques of spectral theory.

Last but not least, the limit matrix in the Taylor series expansion of the eigenprojection matrix corresponding to the eigenvalue zero, has a simple expression in terms of (a) the original unperturbed 0-eigenprojection, (b) the perturbation matrices and (c), the singular part of the reduced resolvent. This provides some insight on how the perturbed 0-eigenvectors relate to the original 0-eigenvectors.

The results of Section 2.5 can be immediately applied to the theory of perturbed Markov chains. For example, the deviation matrix of a Markov chain is the negative of the reduced resolvent of the Markov chain generator. To demonstrate this application and verify the validity of our theoretical results, an illustrative example is provided at the end of the section. Another interesting potential application of our results is the perturbation analysis of the least squares problem [36, 139] and the boundary layer phenomenon arising in chemical networks [20].

In the last Section 2.6 of this chapter we provide a perturbation analysis for the Sylvester matrix equation. The Sylvester equation has many important applications (see [18, 28, 69, 84, 116, 150] and references therein). Again it can be viewed as a particular case of the general operator equation (2.1) with operator $L(\varepsilon) : X \mapsto A(\varepsilon)X + XB(\varepsilon)$ acting on the Hilbert space of complex matrices $C^{m\times n}$. Of course, one can use a Kronecker tensor product to write the Sylvester equation in the form $\mathcal{A}(\varepsilon)\mathcal{X} = \mathcal{B}(\varepsilon)$, where $\mathcal{A}(\varepsilon)$ and $\mathcal{B}(\varepsilon)$ are some augmented matrices [97], and then immediately apply the results of Sections 2.2 and 2.3. However, this approach leads to large sparse matrices. In contrast, we propose to take into account the structure of the Sylvester operator. The key element of our method is the novel use of the feasibility conditions of the Sylvester equations [92] expressed in terms of eigenvectors of the original coefficient matrices $A(0)$ and $B(0)$. The latter allows us to obtain a system of reduced equations with coefficient matrices of much smaller dimensions. In Section 2.6 we perform the perturbation analysis under the non-degeneracy assumption, which was first introduced by Lidskii [104] in the context of the perturbed eigenvalue problem. Since this is a practically important case, we explain in Subsection 2.6.3 how the algebraic reduction process of Section 2.2 can be implemented under this assumption. In particular, in the non-degenerate case, the Laurent series for the perturbed solution can have at most a simple pole, and if the unperturbed equation has multiple solutions, then the perturbed solution converges to one of these solutions as the perturbation parameter goes to zero. Finally, we disscuss an important particular case, when only two semisimple eigenvalues of matrices $A(\varepsilon)$ and $B(\varepsilon)$ merge as the perturbation parameter ε goes to zero. This particular case has several applications in control theory [116] and the reduced system of equations for calculating of Laurent series coefficients can be written in a concise matrix form.

Since the methods of Chapter 2 are essentially based on the application of generalized inverse matrices, we review briefly the main definitions and facts from the theory of generalized inverses. An interested reader can find a more detailed discussion in [24, 141].

There are several types of generalized inverses. The *Moore-Penrose generalized inverse* (or *Moore-Penrose pseudo-inverse*) is by far the most commonly used generalized inverse. It can be defined in either geometrical or algebraical terms. First we give a "geometrical" definition.

Let $A \in \mathbb{R}^{m\times n}$ be the matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. And let $N(A) \subseteq \mathbb{R}^n$ and $R(A) \subseteq \mathbb{R}^m$ denote the null space and the range space of this transformation, respectively. The space $\mathbb{R}^n$ can be represented as the direct sum $N(A) \oplus N(A)^\perp$ and the space $\mathbb{R}^m$ can be represented as the direct sum $R(A) \oplus R(A)^\perp$.

Definition 1 *The Moore-Penrose generalized inverse of the linear transformation* $A : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation* $A^\dagger : \mathbb{R}^m \to \mathbb{R}^n$ *defined in the following way. Let* $y \in \mathbb{R}^m$ *and write* $y = y_0 + y_0^\perp$, *where* $y_0 \in R(A)$ *and* $y_0^\perp \in R(A)^\perp$. *Choose* $x \in \mathbb{R}^n$ *such that* $Ax = y_0$ *and write* $x = x_0 + x_0^\perp$, *where* $x_0 \in N(A)$ *and* $x_0^\perp \in N(A)^\perp$. *Then* $A^\dagger y = x_0^\perp$.

Of course, the generalized inverse matrix is just the matrix representation of the corresponding generalized inverse transformation. Next we give an equivalent algebraic defenition.

Definition 2 *If* $A \in \mathbb{R}^{m\times n}$, *then the Moore-Penrose generalized inverse (or pseudo-inverse)* $A^\dagger \in \mathbb{R}^{m\times n}$ *is defined by the equations*

$$AA^\dagger A = A, \tag{2.7}$$

$$A^\dagger AA^\dagger = A^\dagger, \tag{2.8}$$

$$(AA^\dagger)^T = AA^\dagger, \tag{2.9}$$

$$(A^\dagger A)^T = A^\dagger A. \tag{2.10}$$

There are several methods for the computation of Moore-Penrose generalized inverses. The best known and the most computationally stable method is based on the singular value decomposition (SVD). Let $r = r(A)$ be the rank of $A \in \mathbb{R}^{m\times n}$. There exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ and an invertible diagonal matrix $D = diag\{\sigma_1, ..., \sigma_r\}$, whose diagonal elements are the positive square roots of the non-zero eigenvalues of $A^T A$ repeated according to multiplicity, such that

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V.$$

Then the generalized inverse $A^\dagger \in \mathbb{R}^{n\times m}$ can be written in the form

$$A^\dagger = V^T \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T.$$

The numbers $\sigma_1, ..., \sigma_r$ are usually referred to as the *singular values* of A. More details on the singular value decomposition and the computation of the orthogonal matrices U and V can be found in [24, 141]. Another method for the computation of $A^\dagger$ based on the more elementary LU decomposition is discussed in [141].

The following properties of the Moore-Penrose generalized inverse will be used in the sequel (for the proof see, e.g., [24]).

$$(A^T)^\dagger = (A^\dagger)^T \tag{2.11}$$

$$A^T = A^T A A^\dagger = A^\dagger A A^T \tag{2.12}$$

$$(A^T A)^\dagger = A^\dagger A^{T\dagger} \tag{2.13}$$

$$A^\dagger = (A^T A)^\dagger A^T = A^T (A A^T)^\dagger \tag{2.14}$$

One can immediately conclude from Definition 1 that the generalized inverse is an equation solver, that is

Lemma 1 *Consider the assumed feasible system of linear equations*

$$Ax = b, \tag{2.15}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then x is a solution of this system if and only if

$$x = A^\dagger b + v \quad ,$$

where $v \in \mathbb{R}^{n\times 1}$ belongs to the right null space [1] of A.

[1] We say that $v \in \mathbb{R}^{n\times 1}$ belongs to the right null space or simply the null space of $A \in \mathbb{R}^{m\times n}$ if $Av = 0$.

The next lemma provides a simple condition for the feasibility of linear systems [24, 141].

Lemma 2 *The system of linear equations* (2.15) *is feasible if and only if* $wb = 0$ *for all vectors* $w \in \mathbb{R}^{1\times m}$ *in the left null space* [2] *of* A.

An important particular case of the Moore-Penrose generalized inverse is the so-called *group inverse*, defined as follows:

Definition 3 *Suppose that* A *is a square matrix. The group inverse* A^g*, if it exists, is characterized as the unique matrix satisfying the three equations*

$$AA^gA = A, \tag{2.16}$$

$$A^gAA^g = A^g, \tag{2.17}$$

$$AA^g = A^gA. \tag{2.18}$$

Existence of the group inverse of $A \in \mathbb{R}^{n\times n}$ is equivalent to the possibility to decompose the space $\mathbb{R}^n$ into a *direct* sum of the null space and the range of A (see e.g. [24]).

We now show that computing the Moore-Penrose generalized inverse reduces to computing the group inverse of a square symmetric matrix. This result seems to be new.

Lemma 3 *The Moore-Penrose generalized inverse of* A *can be calculated by the formulae*

$$A^\dagger = (A^TA)^gA^T = A^T(AA^T)^g. \tag{2.19}$$

PROOF: By (2.14) to prove the above formulae, we need only to verify that the Moore-Penrose generalized inverse $(A^TA)^\dagger$ is also the group inverse of A^TA. Indeed, $(A^TA)^\dagger$ satisfies (2.16)–(2.18). It is obvious that (2.16) and (2.17) hold, since by definition, the generalized inverse satisfies (2.7), (2.8). The last identity (2.18) is obtained via

$$(A^TA)(A^TA)^\dagger = A^TAA^\dagger A^{T\dagger} = A^TAA^\dagger A^{\dagger T} = A^TA^{\dagger T} = (A^\dagger A)^T = A^\dagger A = (A^TA)^\dagger(A^TA),$$

using (2.13), (2.11), (2.12), (2.10) and (2.14) respectively. Thus, the matrix $(A^TA)^\dagger$ satisfies (2.18), and, therefore, $(A^TA)^g = (A^TA)^\dagger$, which immediately yields (2.19).

□

Now let us discuss another type of generalized inverses, *Drazin inverses*. The Drazin inverse can be defined and calculated in the following way. If $A \in \mathbb{R}^{n\times n}$, then it can be represented by the following decomposition [24]

$$A = W \begin{bmatrix} S & 0 \\ 0 & N \end{bmatrix} W^{-1}, \tag{2.20}$$

[2] We say that $w \in \mathbb{R}^{1\times m}$ belongs to the left null space of $A \in \mathbb{R}^{m\times n}$ if $wA = 0$.

where N is nilpotent and S is invertible. Then, the Drazin inverse is defined by

$$A^{\#} = W \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}.$$

Note that the Drazin inverse is not an equation solver. However, based on algebraic properties, Drazin inverses have more in common with usual inverse matrices than Moore-Penrose generalized inverses do [24]. In spectral theory the Drazin inverse is known as *reduced resolvent* [87].

The group inverse is also a particular case of the Drazin inverse. Namely, whenever the group inverse exists for matrix A, A can be decomposed into (2.20) with $N = 0$. Actually, the group inverse represents a situation when the Moore-Penrose generalized inverse and the Drazin inverse coincide.

2.2 The inversion of singularly perturbed matrices

2.2.1 Preliminaries

Let $A(\varepsilon)$ be an analytic perturbation of matrix $A_0 \in \mathbb{R}^{n\times n}$. That is, the power series

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots \quad , \tag{2.21}$$

is well defined for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \varepsilon_{max}$. We adopt a standard assumption that $A(\varepsilon)$ is invertible in a punctured disc: $0 < |\varepsilon| < \varepsilon_{max}$. This immediately implies (see references mentioned in the "Background and literature review" section) that the inverse matrix valued function $A^{-1}(\varepsilon)$ has the Laurent series (2.2). To simplify notations, we write the Laurent series in the following form

$$A^{-1}(\varepsilon) = \frac{1}{\varepsilon^s}(X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots). \tag{2.22}$$

Upon substituting the above series into the identity $A(\varepsilon)A^{-1}(\varepsilon) = I$ and collecting coefficients with the same power of ε, we obtain the system of *fundamental equations*.

$$\begin{aligned}
A_0X_0 &= 0 && (2.23.0)\\
A_0X_1 + A_1X_0 &= 0 && (2.23.1)\\
&\vdots && \vdots\\
A_0X_s + \dots + A_sX_0 &= I && (2.23.s)\\
A_0X_{s+1} + \dots + A_{s+1}X_0 &= 0 && (2.23.s+1)\\
&\vdots && \vdots
\end{aligned}$$

Tha above infinite system of linear equations uniquely determine the coefficients of the Laurent series (2.22). This fact has been noted in [14, 73, 91, 149, 145].

Our aim is to provide an efficient computational scheme to determine the sequence $\{X_i\}_{i=0}^{\infty}$ from a knowledge of the sequence $\{A_i\}_{i=0}^{\infty}$. Of course, in order to make the procedure finite, one has to specify some truncation parameters. For example, one could specify the number of coefficients required in the regular part of the Laurent expansion. This would then imply a (possibly posterior) bound on how many matrices of the series $\{A_i\}_{i=0}^{\infty}$ are needed. We also observe that the order s of the pole depends only on some finite subsequence $\{A_i\}_{i=0}^{p}$ of the sequence $\{A_i\}_{i=0}^{\infty}$, where the number p is not known in advance. The order s of the pole, and hence p, can be determined either by using the method of Sain and Massey (see [73, 130] and the introductory part to Section 2.2.3) or by applying the complete reduction process, which will be described in the last part of Section 2.2.3.

In this section we suggest three related methods for computing the coefficients of the Laurent series (2.22). All three methods are based on the recursive solution of the infinite set of fundamental equations (2.23). The first method uses the Moore-Penrose generalized inverse matrix to solve a set of linear equations and extends the work in [66]. The second method has one reduction step and in the third method we perform a complete reduction process with s number of steps. Of course, an intermediate process with the number of reductions between 1 and s could also be developed.

We emphasize that we are mainly interested in the case where A_0 is singular, as otherwise the analysis and the resulting procedure are quite straightforward.

2.2.2 Recursive solution of the fundamental equations

In this subsection we show that the fundamental equations (2.23) can be solved in a recursive manner and that we need to consider only $s+1$ fundamental equations at a time to determine subsequent coefficients of the Laurent series (2.22).

Before proceeding further, let us introduce some more notation. For any integer $t \geq 0$ define the matrix $\mathcal{A}^{(t)} \in R^{n(t+1)\times n(t+1)}$ [61, 66, 73] by setting

$$\mathcal{A}^{(t)} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_t & A_{t-1} & \cdots & A_1 & A_0 \end{bmatrix}$$

Let $\mathcal{G}^{(t)} \stackrel{def}{=} [\mathcal{A}^{(t)}]^{\dagger}$ be the Moore-Penrose generalized inverse of $\mathcal{A}^{(t)}$ and define the matrices

$G_{ij}^{(t)} \in \mathbb{R}^{n\times n}$ for $0 \leq i,j \leq t$ by

$$\mathcal{G}^{(t)} = \begin{bmatrix} G_{00}^{(t)} & \cdots & G_{0t}^{(t)} \\ \vdots & \ddots & \vdots \\ G_{t0}^{(t)} & \cdots & G_{tt}^{(t)} \end{bmatrix}$$

where $G_{ij}^{(t)} \in \mathbb{R}^{n\times n}$ is the (i,j)-th block of $\mathcal{G}^{(t)}$.

Let us prove the following lemma.

Lemma 4 *Let s be the order of the singularity of $A^{-1}(\varepsilon)$. Any eigenvector of $\mathcal{A}^{(s)}$ corresponding to the zero eigenvalue possesses the property that its first n elements are zero.*

PROOF: Suppose on the contrary that there exists an eigenvector $\Phi \in \mathbb{R}^{n(s+1)}$ such that

$$\mathcal{A}^{(s)}\Phi = 0 \tag{2.24}$$

and not all of its first n entries are zeros. Then, we partition vector Φ into $s+1$ blocks and rewrite (2.24) in the form

$$A_0\phi_0 = 0$$

$$A_0\phi_1 + A_1\phi_0 = 0$$

$$\vdots$$

$$A_0\phi_s + ... + A_s\phi_0 = 0$$

with $\phi_0 \neq 0$. The latter means that we have found a generalized Jordan chain of length $s+1$. However, from the results of Gohberg and Sigal [62], we conclude that the maximal length of the generalized Jordan chain of $A(\varepsilon)$ at $\varepsilon = 0$ is s. Thus, we come to a contradiction and hence $\phi_0 = 0$.

□

The next theorem provides the theoretical basis for the recursive solution of the infinite system of fundamental equations (2.23).

Theorem 1 *Each coefficient $X_k, k = 0, 1, ...$ is uniquely determined by the previous coefficients $X_0, ..., X_{k-1}$ and the set of s fundamental equations $(2.23.k) - (2.23.k+s)$.*

PROOF: It is obvious that the sequence of Laurent series coefficients $\{X_i\}_{i=0}^{\infty}$ is itself a solution to the fundamental equations (2.23). Let the coefficients $X_i, i = 0, ..., k-1$ be determined. Now we show that the set of fundamental equations (2.23.k)-(2.23.k+s) uniquely determines the next coefficient X_k. Suppose that there exists another solution $\tilde{X}_k$. Since X_k and $\tilde{X}_k$ are both solutions, we can write

$$\mathcal{A}^{(s)} \begin{bmatrix} \tilde{X}_k \\ \vdots \\ \tilde{X}_{k+s} \end{bmatrix} = \begin{bmatrix} J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i} \end{bmatrix}$$

and

$$\mathcal{A}^{(s)} \begin{bmatrix} X_k \\ \vdots \\ X_{k+s} \end{bmatrix} = \begin{bmatrix} J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i} \end{bmatrix},$$

where the matrix J_i is defined as follows:

$$J_i = \begin{cases} I, & i = s \\ 0, & \text{otherwise} \end{cases}.$$

and $\tilde{X}_{k+1}, ..., \tilde{X}_{k+s}$ are any particular solutions of the nonhomogenous linear system (2.23.k)-(2.23.k+s). Note that the above systems have identical right hand sides. By subtracting the second system from the first, we get

$$\mathcal{A}^{(s)} \begin{bmatrix} \tilde{X}_k - X_k \\ \vdots \\ \tilde{X}_{k+s} - X_{k+s} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This means that $[\tilde{X}_k - X_k \cdots \tilde{X}_{k+s} - X_{k+s}]^T$ is a matrix of vectors in the null space of $\mathcal{A}^{(s)}$. However, by Lemma 4, the first n rows of this matrix are zero. In other words, $\tilde{X}_k - X_k = 0$ which proves the theorem.

□

Also Lemma 4 provides, as stated next in Proposition 1, an explicit expression for X_0, the leading term of the Laurent series expension (2.22). This will be used later, in the first method of Section 2.2.3, for the initialization of a recursive procedure for computing all other terms.

Proposition 1 *The leading term X_0 of the Laurent series expansion* (2.22) *is given by*

$$X_0 = G_{0s}^{(s)}.$$

PROOF: According to Lemma 1, the general solution of the first $s+1$ fundamental equations is given by

$$\begin{bmatrix} X_0 \\ \vdots \\ \tilde{X}_s \end{bmatrix} = \mathcal{G}^{(s)} \begin{bmatrix} 0 \\ \vdots \\ I \end{bmatrix} + V = \begin{bmatrix} G_{0s}^{(s)} \\ \vdots \\ G_{ss}^{(s)} \end{bmatrix} + \begin{bmatrix} V_0 \\ \vdots \\ V_s \end{bmatrix},$$

where each column of matrix $V \in \mathbb{R}^{sn \times n}$ belongs to the right null space of $\mathcal{A}^{(s)}$. From Lemma 4 we conclude that $V_0 = 0$ and hence $X_0 = G_{0s}^{(s)}$.

□

2.2.3 Solution methods

In this section we discuss three methods for solving the system of fundamental equations (2.23). The first method is based on a direct application of the Moore-Penrose generalized inverse. The second method involves the replacement of the original system of fundamental equations by a system of equations with a reduced order, whose matrix coefficients are of smaller dimensions. In the third method we show that the reduction process can be applied recursively to reduce the problem to a non-singular system. Since all methods depend to some extent on prior knowledge of s we begin by discussing a procedure for the determination of s. However, as will be seen in the last subsection, it is possible to use the third method without prior knowledge of s.

The determination of the order of the pole

The rank test on the matrix $\mathcal{A}^{(t)}$ proposed by Sain and Massey in [130] is likely to be the most effective procedure for determining the value of s. The calculation of rank is essentially equivalent to the reduction of $\mathcal{A}^{(t)}$ to row echelon normal form and it can be argued that we should use row operations to successively calculate the ranks of $\mathcal{A}^{(0)}$,$\mathcal{A}^{(1)}$,$\mathcal{A}^{(2)}$,... and find the minimum value of t for which $\mathrm{rank}\mathcal{A}^{(t)} = \mathrm{rank}\mathcal{A}^{(t-1)} + n$. This minimum value of t is equal to the order s of the pole [73, 130]. Note that previous row operations for reduction of $\mathcal{A}^{(t-1)}$ to echelon form are replicated in the reduction of $\mathcal{A}^{(t)}$ and do not need to be repeated. For example, if a certain combination of row operations reduces A_0 to row echelon form then the same operations are used again as part of the reduction of

$$\begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix}$$

to row echelon form.

Basic generalized inverse method

Assuming s known, Proposition 1 gives an explicit expression for X_0. In this section we define a recursive formula for the coefficients X_k, $k = 1, 2, ...$ The formula will be given in terms of the generalized inverse $\mathcal{G}^{(s)} = [\mathcal{A}^{(s)}]^{\dagger}$. Furthemore, we would like to note that in the solution we use only the first n rows of the generalized inverse, namely, $[G_{00}^{(s)} \ \cdots \ G_{0s}^{(s)}]$.

The procedure initialises with

$$X_0 = G^{(s)}_{0s}.$$

Then, the terms X_k, for $k = 1, 2, ...$ can be obtained recursively from the fundamental equations (2.23.k)–(2.23.$k+s$) which can be rewritten in the form

$$\mathcal{A}^{(s)} \begin{bmatrix} \tilde{X}_k \\ \vdots \\ \tilde{X}_{k+s} \end{bmatrix} = \begin{bmatrix} J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i} \end{bmatrix}$$

Again as in Proposition 1, the general solution to the above system is given in the form.

$$\begin{bmatrix} X_k \\ \tilde{X}_{k+1} \\ \vdots \\ \tilde{X}_{k+s} \end{bmatrix} = \begin{bmatrix} G^{(s)}_{00} & \cdots & G^{(s)}_{0s} \\ G^{(s)}_{10} & \cdots & G^{(s)}_{1s} \\ \vdots & \ddots & \vdots \\ G^{(s)}_{s0} & \cdots & G^{(s)}_{ss} \end{bmatrix} \begin{bmatrix} J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ J_{k+1} - \sum_{i=1}^{k} A_{i+1} X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i} \end{bmatrix} + \begin{bmatrix} 0 \\ V_1 \\ \vdots \\ V_s \end{bmatrix},$$

Thus the Laurent series coefficient X_k is given by the following recursive expression

$$X_k = \sum_{j=0}^{s} G^{(s)}_{0j} (J_{j+k} - \sum_{i=1}^{k} A_{i+j} X_{k-i}) \quad , \quad k \geq 1, \tag{2.25}$$

where the matrix J_i is defined as before by

$$J_i = \begin{cases} I, & i = s \\ 0, & \text{otherwise.} \end{cases}$$

Note that $J_{s+j+k} = 0$ for all $j = 0, 1, 2, ...$ where $k \geq 1$ and hence the matrices J_{j+k} in the expression (2.25) disappear when the coefficients of the regular part of the Laurent series (2.22) are computed.

Remark 1 *The formula* (2.25) *is a generalisation of the recursive formula for the Neumann expansion. Indeed, let A_0 be invertible. Then $s = 0$ and $G^{(0)} = A_0^{-1}$. Consequently,* (2.25) *becomes the classical formula*

$$X_k = -A_0^{-1} \sum_{i=1}^{k} A_i X_{k-i}, \quad k \geq 1$$

with $X_0 = A_0^{-1}$.

Remark 2 *Probably from a computational point of view it is better not to compute the generalized inverse $\mathcal{G}^{(s)}$ beforehand, but rather to find the SVD or LU decomposition of $\mathcal{A}^{(s)}$ and then use these decompositions to solve the linear system* (2.23.k)–(2.23.$k+s$) *of fundamental equations. This is the standard approach for solving linear systems with various right hand sides.*

The formulae of this section rely on a knowledge of the matrix $\mathcal{G}^{(s)}$. However, we note from [24, Ch. 12] that the computation of the Moore-Penrose inverse for non-full rank matrices may be unstable. Therefore we will develop more computationally stable algorithms, to be presented in the ensuing sections.

The one step reduction process

In this subsection we describe an alternative scheme that can be used in the case where it is relatively easy to compute a basis for each of the left and right null spaces of A_0. Specifically, let $m = n - r(A_0)$ be the dimension of the null space of A_0, let $Q \in \mathrm{I\!R}^{\mathrm{n}\times\mathrm{m}}$ be a matrix whose m columns form a basis for the right null space and let $M \in \mathrm{I\!R}^{\mathrm{m}\times\mathrm{n}}$ be a matrix whose m rows form a basis for the left null space of A_0. Of course, although $m = 0$ and hence $s = 0$ is possible, we are interested in the singular case where $m \geq 1$. Recall that $A_0^\dagger$ denotes the Moore-Penrose generalized inverse of the unperturbed matrix A_0.

Let us consider the first $s+1$ fundamental equations as defined in Section 2.2.1 but with arbitrary right hand sides.

$$\begin{aligned} A_0\tilde{X}_0 &= R_0 && (2.26.0)\\ A_0\tilde{X}_1 + A_1\tilde{X}_0 &= R_1 && (2.26.1)\\ &\vdots && \vdots\\ A_0\tilde{X}_s + \ldots + A_s\tilde{X}_0 &= R_s. && (2.26.s) \end{aligned}$$

We are interested only in the case where the above system has a solution. In such a case it follows from Lemma 4 that $\tilde{X}_0$ is uniquely defined. The following theorem provides a key technique for the reduction process.

Theorem 2 *Suppose that the system of equations* (2.26) *is feasible. The solution* $\tilde{X}_0$ *to this system is given by*

$$\tilde{X}_0 = A_0^\dagger R_0 + Q\tilde{Y}_0$$

where $\{\tilde{Y}_i\}_{i=0}^{s-1} \subset \mathrm{I\!R}^{\mathrm{m}\times\mathrm{n}}$ *solves a reduced system of equations in the form*

$$\begin{aligned} B_0\tilde{Y}_0 &= T_0 && (2.27.0)\\ B_0\tilde{Y}_1 + B_1\tilde{Y}_0 &= T_1 && (2.27.1)\\ &\vdots && \vdots\\ B_0\tilde{Y}_{s-1} + \ldots + B_{s-1}\tilde{Y}_0 &= T_{s-1}. && (2.27.s-1) \end{aligned}$$

The matrices $B_k \in \mathbb{R}^{m\times m}$ *and* $T_k \in \mathbb{R}^{m\times n}$, $0 \le k \le s-1$, *are defined by*

$$B_k = M\left(\sum_{p=1}^{k+1}(-1)^{p-1}\sum_{\nu_1+\ldots+\nu_p=k+1} A_{\nu_1}A_0^{\dagger}A_{\nu_2}\cdots A_0^{\dagger}A_{\nu_p}\right)Q, \tag{2.28}$$

$$T_k = M\left(\sum_{i=0}^{k}\left(\sum_{p=1}^{k-i+1}(-1)^{p}\sum_{\nu_1+\ldots+\nu_p=k-i+1} A_{\nu_1}A_0^{\dagger}A_{\nu_2}\cdots A_0^{\dagger}A_{\nu_p}\right)A_0^{\dagger}R_i + R_{k+1}\right), \tag{2.29}$$

where $\nu_1, \ldots, \nu_p$ *are positive natural numbers. Moreover, the matrix* $\tilde{Y}_0$ *is uniquely defined by the reduced system* (2.27).

PROOF: According to Lemma 1 the general solution to the matrix equation (2.26.0) is

$$\tilde{X}_0 = A_0^{\dagger}R_0 + Q\tilde{Y}_0, \tag{2.30}$$

where $\tilde{Y}_0 \in \mathbb{R}^{m\times n}$ is some arbitrary matrix. Lemma 2 shows that the feasibility condition for equation (2.26.1) is

$$M(R_1 - A_1\tilde{X}_0) = 0.$$

Substituting expression (2.30) for the general solution $\tilde{X}_0$ into the above feasibility condition, one finds that $\tilde{Y}_0$ satisfies the equation

$$M(R_1 - A_1(A_0^{\dagger}R_0 + Q\tilde{Y}_0)) = 0$$

which can be rewritten as

$$MA_1Q\tilde{Y}_0 = M(R_1 - A_1A_0^{\dagger}R_0).$$

Thus we have obtained the first reduced fundamental equation (2.27.0) with

$$B_0 \triangleq MA_1Q \quad \text{and} \quad T_0 \triangleq M(R_1 - A_1A_0^{\dagger}R_0).$$

Next we observe that the general solution of equation (2.26.1) is represented by the formula

$$\tilde{X}_1 = A_0^{\dagger}(R_1 - A_1X_0) + Q\tilde{Y}_1 \tag{2.31}$$

with $\tilde{Y}_1 \in \mathbb{R}^{m\times n}$. Moving on and applying the feasibility condition stated in Lemma 2 to equation (2.26.2), we obtain

$$M(R_2 - (A_1\tilde{X}_1 + A_2\tilde{X}_0)) = 0.$$

The substitution of expressions (2.30) and (2.31) into the above condition yields

$$MA_1(A_0^{\dagger}(R_1 - A_1\tilde{X}_0) + Q\tilde{Y}_1) + MA_2(A_0^{\dagger}R_0 + Q\tilde{Y}_0) = MR_2$$

which is rearranged to give

$$MA_1Q\tilde{Y}_1 + M(-A_1A_0^\dagger A_1 + A_2)Q\tilde{Y}_0 = M(R_2 - A_1A_0^\dagger R_1 + (A_1A_0^\dagger A_1 - A_2)A_0^\dagger R_0).$$

The previous equation is the reduced equation (2.27.1) with

$$B_1 \triangleq M(-A_1A_0^\dagger A_1 + A_2)Q \quad \text{and} \quad T_1 \triangleq M(R_2 - A_1A_0^\dagger R_1 + (A_1A_0^\dagger A_1 - A_2)A_0^\dagger R_0).$$

Note that this equation imposes restrictions on $\tilde{Y}_1$ as well as on $\tilde{Y}_0$. By proceeding in the same way, we eventually obtain the complete system of equations (2.27) with coefficients given by formulae (2.28) and (2.29) each of which can be justified by induction in a straightforward way. Finally, notice that the matrix $\tilde{Y}_0$ is uniquely determined by the reduced system of equations (2.27). This follows immediately from the facts that $\tilde{X}_0$ is determined uniquely by the system of equations (2.26) (see arguments of Theorem 1) and that the matrix Q has full rank.

□

Remark 3 *The matrices B_k and T_k given in* (2.28) *and* (2.29) *can be recursively computed in the spirit of [91, 134]. Specifically, define the series of matrices $\{U_k\}_{k=0}^{\infty}$ by setting*

$$U_k \triangleq \sum_{p=1}^{k+1}(-1)^{p-1} \sum_{\nu_1+\ldots+\nu_p=k+1} A_{\nu_1}A_0^\dagger A_{\nu_2}\cdots A_0^\dagger A_{\nu_p}, \quad k = 0, 1, \ldots$$

which is equivalent to the recursive definition

$$U_k = A_{k+1} - \sum_{i=1}^{k} A_iA_0^\dagger U_{k-i} \quad , \quad k = 1, 2, \ldots$$

initialising with $U_0 = A_1$. Then, (2.28) *and* (2.29) *can be simplified to give*

$$B_k = MU_kQ \tag{2.32}$$

and

$$T_k = M\left(-\sum_{i=0}^{k} U_{k-i}A_0^\dagger R_i + R_{k+1}\right) \tag{2.33}$$

for $k = 0, \ldots, s-1$.

Remark 4 *It is important to note that all equations* (2.27.0)–(2.27.$s-1$) *are needed in order to determine Y_0 uniquely. This in turn implies that B_0 is singular if $s > 1$. Actually, it is this observation which leads to the fact that the reduction process can now be applied again to the system* (2.27.0)–(2.27.$s-1$)*. Details of a complete reduction process are given in the next subsection.*

Remark 5 *Note that, since* $\dim B_k = \dim N(A)$*, the matrices* $B_k, k = 0, ..., s-1$ *have significantly smaller dimensions than the coefficients* A_k *of the original fundamental equations.*

Since the reduced system (2.27) has the same structure as the original system of fundamental equations, we can again apply the generalized inverse technique. Towards this end define

$$\mathcal{B}^{(t)} \triangleq \begin{bmatrix} B_0 & 0 & 0 & \cdots & 0 \\ B_1 & B_0 & 0 & \cdots & 0 \\ B_2 & B_1 & B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_t & B_{t-1} & \cdots & B_1 & B_0 \end{bmatrix}$$

and

$$\mathcal{H}^{(t)} = \begin{bmatrix} H_{00}^{(t)} & \cdots & H_{0t}^{(t)} \\ \vdots & \ddots & \vdots \\ H_{t0}^{(t)} & \cdots & H_{tt}^{(t)} \end{bmatrix} \triangleq [\mathcal{B}^{(t)}]^{\dagger}.$$

By carrying out a similar computation to the one presented in the proof of Proposition 1, we obtain

$$\tilde{Y}_0 = \sum_{i=0}^{s-1} H_{0i}^{(s-1)} T_i.$$

Once $\tilde{Y}_0$ is determined it is possible to obtain $\tilde{X}_0$ which is given by

$$\tilde{X}_0 = A_0^{\dagger} R_0 + Q\tilde{Y}_0 = A_0^{\dagger} R_0 + Q \sum_{i=0}^{s-1} H_{0i}^{(s-1)} T_i.$$

Furthermore, substituting for T_i for $0 \le i \le s-1$ from (2.33) and changing the order of summation gives

$$\tilde{X}_0 = \left(A_0^{\dagger} - \sum_{i=0}^{s-1} Q H_{0i}^{(s-1)} M U_i A_0^{\dagger} \right) R_0 + \sum_{j=1}^{s} \left(Q H_{0j-1}^{(s-1)} M - \sum_{i=j}^{s-1} Q H_{0i}^{(s-1)} M U_{i-j} A_0^{\dagger} \right) R_j. \tag{2.34}$$

Note that we are using the convention the sum disappears when the lower limit is greater than the upper limit in the limits of the sum.

Next we apply the above results in order to solve the system of fundamental equations (2.23). As before, consider the set (2.23.k)–(2.23.$k+s$) of fundamental equations. This set uniquely determines the coefficient X_k of the Laurent series expansion (2.22). Indeed, we have

$$\begin{array}{rcl} A_0 X_k & = & J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ A_0 X_{k+1} + A_1 X_k & = & J_{k+1} - \sum_{i=1}^{k} A_{i+1} X_{k-i} \\ & \vdots & \\ A_0 X_{k+s} + ... + A_s X_k & = & J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i}. \end{array}$$

The above system is like the one given in (2.26) with $R_j = J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}$. Hence, from formula (2.34) with the appropriate R_j we obtain

$$\begin{aligned} X_k &= \left(A_0^\dagger - \sum_{i=0}^{s-1} QH_{0i}^{(s-1)} MU_i A_0^\dagger\right)(J_k - \sum_{i=1}^{k} A_i X_{k-i}) \\ &+ \sum_{j=1}^{s}\left(QH_{0j-1}^{(s-1)} M - \sum_{i=j}^{s-1} QH_{0i}^{(s-1)} MU_{i-j} A_0^\dagger\right)(J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}) \end{aligned} \tag{2.35}$$

for all $k \geq 1$. In particular, the coefficient of the first singular term in (2.22) can be given by the formula

$$X_0 = QH_{0s-1}^{(s-1)} M\ . \tag{2.36}$$

Remark 6 *The advantage of* (2.35) *vis a vie* (2.25) *is that only the generalized inverses of smaller matrices* $\mathcal{B}^{(s)}$ *and* A_0 *are needed instead of the generalized inverse of the large augmented matrix* $\mathcal{A}^{(s)}$.

The complete reduction process

The reduced system (2.27) has a similar structure to the original system (2.26) but:

1. the number of fundamental equations is reduced from $s+1$ to s,
2. the dimension of the unkown matrices is reduced from n to $m = \dim N(A_0)$, and
3. the matrices on the right hand sides have changed.

These changes do not affect the possibility of applying the reduction procedure more than once. As can be deduced from Theorems 1 and 2, it is possible to make up to s reduction steps. We now outline how these steps can be executed. Since this should be routine, no details are given. We start by introducing the sequence of reduced systems. The fundamental matrix equations for the l-th reduction step are

$$\begin{aligned} A_0^{(l)} X_0^{(l)} &= R_0^{(l)} && (2.37.0) \\ A_0^{(l)} X_1^{(l)} + A_1^{(l)} X_0^{(l)} &= R_1^{(l)} && (2.37.1) \\ &\vdots && \vdots \\ A_0^{(l)} X_{s-l}^{(l)} + \ldots + A_{s-l}^{(l)} X_0^{(l)} &= R_{s-l}^{(l)} && (2.37.s-l) \end{aligned}$$

With $l = 0$, one gets the original system (2.26) and with $l = 1$ one gets the reduced system (2.27) for the first reduction step. Initialising with $R_i^{(0)} = 0$, $0 \leq i \leq s-1$, and $R_s^{(0)} = I$ and with $A_i^{(0)} = A_i$, $0 \leq i \leq s$, the matrices $A_j^{(l)}$ and $R_j^{(l)}$, $0 \leq j \leq s-l$, for each reduction step $1 \le l \le s$ can computed subsequently by a recursion similar to (2.32) and (2.33):

$$U_0^{(l)} = A_1^{(l-1)} \quad , \quad U_j^{(l)} = A_{j+1}^{(l-1)} - \sum_{i=1}^{j} A_i^{(l-1)} A_0^{(l-1)\dagger} U_{j-i}^{(l)}, \quad j = 1, \dots, s-l$$

$$A_j^{(l)} = M^{(l)} U_j^{(l)} Q^{(l)} \quad j = 0, \dots, s-l,$$

$$R_j^{(l)} = M^{(l)} \left(- \sum_{i=0}^{j} U_{j-i}^{(l)} A_0^{(l-1)\dagger} R_i^{(l-1)} + R_{j+1}^{(l-1)} \right) \quad j = 0, \dots, s-l$$

where $Q^{(l)}$ and $M^{(l)}$ are the basis matrices for the right and left null spaces respectively of the matrix $A_0^{(l-1)}$ and where $A_0^{(l-1)\dagger}$ is the Moore-Penrose generalized inverse of $A_0^{(l-1)}$. After s reduction steps, one gets the final system of reduced equations

$$A_0^{(s)} X_0^{(s)} = R_0^{(s)} \ .$$

Since the original system (2.26) has a unique solution and Theorem 2 states the equivalence of the n-th and $(n+1)$-th systems of reduced equations, the matrix $A_0^{(s)}$ is invertible. Thus,

$$X_0^{(s)} = [A_0^{(s)}]^{-1} R_0^{(s)} \ . \tag{2.38}$$

The original solution $X_0 = X_0^{(0)}$ can be readily retrieved by the backwards recursive relationship

$$X_0^{(l-1)} = A_0^{(l-1)\dagger} R_0^{(l-1)} + Q^{(l)} X_0^{(l)} \quad l = s, ..., 1. \tag{2.39}$$

By taking $R_j^{(0)} = J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}$, $0 \le j \le s$, one gets the algorithm for computing the Laurent series coefficients X_k. Of course, recursive formulae similar to (2.25) and (2.35) can be obtained, but it is quite complicated and probably unnecessary to write them down for the general case.

Remark 7 *The order s of the pole can also be obtained from the reduction process. We continue the reduction process until $A_0^{(l)}$ becomes non-singular. The number of reduction steps exactly equals the order of the pole.*

Note also that the sequence of matrices $A_0^{(l)}$, $l \ge 0$, can be computed independently of the right hand sides. Once s is determined, one can compute the right hand sides $R_j^{(l)}$, $1 \le l \le s$, $0 \le j \le s-l$.

2.2.4 Example

Let us consider the following perturbed matrix

$$A(\varepsilon) = A_0 + \varepsilon A_1 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

where $\mathrm{rank}(A_0) = 2$. Construct the augmented matrices

$$\mathcal{A}^{(0)} = A_0 \quad \text{and} \quad \mathcal{A}^{(1)} = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix}.$$

and note that $\mathrm{rank}(\mathcal{A}^{(1)}) - \mathrm{rank}(\mathcal{A}^{(0)}) = 5 - 2 = 3$, which is the dimension of the original coefficients A_0 and A_1. Therefore, according to the test of Sain and Massey [130], the Laurent expansion for $A^{-1}(\varepsilon)$ has a simple pole. Alternatively, we can compute a basis for $N(\mathcal{A}^{(1)})$, which in this particular example consists of only one vector

$$q^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}^T.$$

The first three zero elements in $q^{(1)}$ confirm that the Laurent series has a simple pole. Next we compute the generalized inverse of $\mathcal{A}^{(1)}$ given by

$$\mathcal{A}^{(1)\dagger} = \mathcal{G}^{(1)} = \begin{bmatrix} G_{00}^{(1)} & G_{01}^{(1)} \\ G_{10}^{(1)} & G_{11}^{(1)} \end{bmatrix} = \begin{bmatrix} 1/3 & -5/12 & -1/12 & 1/8 & 1/8 & -1/8 \\ 0 & 1/4 & 1/4 & 1/8 & 1/8 & -1/8 \\ 1/3 & -5/12 & -1/12 & -3/8 & -3/8 & 3/8 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

Consequently,

$$X_0 = G_{01}^{(1)} = \frac{1}{8} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -3 & -3 & 3 \end{bmatrix}. \tag{2.40}$$

Alternatively, we know that X_0 is uniquely determined by the fundamental equations

$$\begin{aligned} A_0 X_0 &= 0, \\ A_0 X_1 + A_1 X_0 &= I. \end{aligned}$$

After one reduction step these equations reduce to

$$M A_1 Q W_0 = M,$$

where

$$M = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 & -3 \end{bmatrix}^T.$$

Hence,

$$W_0 = (MA_1Q)^{-1}M = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

and

$$X_0 = QW_0 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -3 & -3 & 3 \end{bmatrix}.$$

The latter expression is identical with (2.40) and coincides with the one computed by expanding $A^{-1}(\varepsilon)$ with the help of the MATLAB symbolic toolbox. Note that even for this three dimensional example the direct symbolic calculation of the Laurent series takes a relatively long time.

2.2.5 Alternative proof of the results of Subsection 2.2.2

Here we show an alternative way to prove the results of Subsection 2.2.2. The following approach is based on the elegant technique introduced by Haviv and Ritov in [66, 67].

Theorem 3 *The number τ of the first fundamental equations (2.6) which uniquely determine the leading Laurent series coefficient X_0 is equal to $s+1$, where s is the order of the pole of $A^{-1}(\varepsilon)$ at zero.* [3]

PROOF: We start by proving that $\tau \geq s+1$. Aiming for a contradiction assume that $\tau < s+1$ and consider the first τ fundamental equations

$$\begin{aligned} A_0\tilde{X}_0 &= 0 \\ A_0\tilde{X}_1 + A_1\tilde{X}_0 &= 0 \\ &\vdots \\ A_0\tilde{X}_{\tau-1} + \ldots + A_{\tau-1}\tilde{X}_0 &= 0. \end{aligned}$$

It is obvious that $\tilde{X}_0 = \tilde{X}_1 = \ldots = \tilde{X}_{\tau-1} = 0$ is a solution. On the other hand the first τ coefficients of the Laurent series (2.22) is also a solution. Since the definition of τ implies that the solution for $\tilde{X}_0$ is unique it follows that the first coefficient of the Laurent series (2.22) is equal to zero. This contradicts the fact that s is the order of the pole.

We next show that τ equals $s+1$. Namely, we prove that the first $s+1$ fundamental equations uniquely determine X_0. Suppose on the contrary that this is not true and that there exists another solution $Y_0, Y_1, \ldots, Y_s \in \mathbb{R}^{n\times n}$ to the first $s+1$ fundamental equations

[3] Note that *a priori* the option that $\tau = \infty$ is not ruled out.

with $Y_0 \neq X_0$. In this case consider the sequence $\{Z_i\}_{i=0}^{\infty}$ of matrices in $\mathbb{R}^{n\times n}$ defined by $Z_i = X_i - Y_i$ for $0 \le i \le s$, and $Z_i = X_i$ for $i > s$. Clearly, for $0 < |\varepsilon| < \bar{\varepsilon}$,

$$\frac{1}{\varepsilon^s}A(\varepsilon)[Z(\varepsilon) + Y(\varepsilon)] = I,$$

where $Z(\varepsilon) \triangleq \sum_{i=0}^{\infty} Z_i\varepsilon^i$ and $Y(\varepsilon) \triangleq \sum_{i=0}^{s} Y_i\varepsilon^i$. Equivalently we may write

$$\frac{1}{\varepsilon^s}A(\varepsilon)Z(\varepsilon) = I - \frac{1}{\varepsilon^s}A(\varepsilon)Y(\varepsilon) \quad .$$

But since the series $\{Y_i\}_{i=0}^{s}$ solves the set of the first $s+1$ fundamental equations, the above right hand side has order of magnitude $O(\varepsilon)$ and hence

$$\frac{1}{\varepsilon^s}A(\varepsilon)Z(\varepsilon) = O(\varepsilon) \quad .$$

Multiplication of the last identity by $A^{-1}(\varepsilon)$ on the left leads to

$$\frac{1}{\varepsilon^s}Z(\varepsilon) = O(\frac{1}{\varepsilon^{s-1}}) \quad .$$

This is a contradiction, since $Z_0 \neq 0$ and the left hand side has order of magnitude $O(1/\varepsilon^{s-1})$.

□

Then the important Lemma 4 becomes a corollary of the above theorem.

2.3 The inversion of singularly perturbed operators on Hilbert space

2.3.1 Preliminaries

In [135] Schweitzer and Stewart proposed a computationally efficient approach for the inversion of a matrix pencil $(A + zB)$. Here we generalize the approach of [135] to the case when A and B are linear bounded operators acting from one complex Hilbert space H to another complex Hilbert space K. Let $\mathcal{L}(H, K)$ denote the space of linear bounded operators. Furthermore, we show how the results of [135] and their generalization for complex Hilbert spaces can be applied to polynomial perturbations

$$A(z) = A_0 + zA_1 + ... + z^m A_m, \tag{2.41}$$

and analytic perturbations

$$A(z) = A_0 + zA_1 + z^2A_2 + ..., \tag{2.42}$$

where $A_k \in \mathcal{L}(H, K)$.

2.3.2 Regular perturbation

Firstly, in order to make the present Section 2.3 self-contained, let us consider the application of the classical Neumann series expansion to the case of regular perturbations. The results can be summarized as follows:

Theorem 4 *Let $A, B \in \mathcal{L}(H, K)$ and let A be a one-to-one mapping of H onto K. Then the perturbation operator can be expanded as Maclaurin series*

$$(A + zB)^{-1} = \sum_{n=0}^{\infty} z^n (-A^{-1}B)^n A^{-1} \tag{2.43}$$

for all $|z| < 1/\|A^{-1}B\|$.

PROOF: Note that

$$(A + zB)^{-1} = (A(I + zA^{-1}B))^{-1} = (I + zA^{-1}B)^{-1}A^{-1}$$

Then, by applying the Neumann series to $(I + zA^{-1}B)^{-1}$, we obtain the expansion (2.43) with the radius of convergence $1/\|A^{-1}B\|$.

□

Note that the case of regular polynomial or analytical perturbations can be treated in a similar way.

2.3.3 Singular perturbation

Now let us consider the case of a singular perturbation where the operator A does not have an inverse but the perturbed operator $A+zB$ does for sufficiently small z but distinct from zero. It turns out that, under certain conditions, the inverse of the perturbed operator can be expanded as a Laurent series. Here we develop one set of conditions for the existence of Laurent series with a finite order pole and provide an algorithm for the calculation of its coefficients. Before proceeding further, let us recall several important facts about Hilbert spaces:

- If $A \in \mathcal{L}(H, K)$ is a bounded linear map and if $M = \mathcal{N}(A)$ is the null space of A then M is a closed subspace of H.
- The orthogonal compliment $X^{\perp}$ of any subspace $X \subset H$ is always closed.

Then, one can define a natural projection in the Hilbert space. The following theorem provides a basis for this definition [106].

Theorem 5 *Let H be a Hilbert space and let $M \subseteq H$ be a closed subspace of H. For each $x \in H$ there is a unique element $x_M \in M$, which is called the projection of x into M, such that*

$$\|x - x_M\| \leq \|x - y\|$$

for all $y \in M$. Furthermore

$$\langle x - x_M, y \rangle = 0$$

for all $y \in M$.

Definition 4 *Let H be a Hilbert space and let $M \subseteq H$ be a closed subspace of H. For each $x \in H$ let $x_M \in M$ be the unique projection of x into M. Let $P_M : H \mapsto M \subseteq H$ be defined by*

$$P_M x = x_M$$

for all $x \in H$. The operator P_M is called the orthogonal projection operator onto the closed subspace M and we write

$$x = P_M x + (I - P_M)x = x_M + x_M^{\perp}.$$

It is easy to show that the defined above operator P_M indeed possesses all properties of the orthogonal projection operator.

To provide conditions for the existence of the Laurent series, we need to prove the next lemma.

Lemma 5 *Let $A, B \in \mathcal{L}(H, K)$ be bounded linear maps. Let $M = \mathcal{N}(A)$ and define $N = B(M) \subseteq K$. If there exists a fixed value $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ for which $(A + z_0 B)^{-1}$ exists, then B is bounded below on M and N is a closed subspace of K.*

Proof[4] By the Banach Inverse Mapping Theorem [106] the map $(A + z_0 B)$ is bounded below on H. Therefore we can find $\epsilon > 0$ such that

$$\|(A + z_0 B)x\| \geq \epsilon \|x\|$$

for all $x \in H$. Since $Am = 0$ for all $m \in M$ it follows that

$$\|Bm\| \geq \frac{\epsilon}{|z_0|} \|m\|$$

for all $m \in M$. Now if $\{n_r\}$ is a Cauchy sequence in $N = B(M)$ then $n_r = Bm_r$ where $\{m_r\}$ is a corresponding sequence in M. Because B is bounded below on M the sequence

[4] The proof of this lemma was given by P.G. Howlett.

$\{m_r\}$ must also be a Cauchy sequence. If $m_r \to m$ and $n_r \to n$ then it is easy to see that $Bm = n$. Thus $n \in B(M) = N$.

$\square$

Since $M = \mathcal{N}(A)$ is a closed subspace and the orthogonal complement $M^\perp$ is also closed it follows that $H_1 = M$ and $H_2 = M^\perp$ are each Hilbert spaces. Let $P_M \in \mathcal{L}(H, H)$ denote the natural projection onto the subspace $M \subseteq H$ and define associated mappings $P_1 \in \mathcal{L}(H, H_1)$ and $P_2 \in \mathcal{L}(H, H_2)$ by setting

$$P_1 x = P_M x, \quad \text{and} \quad P_2 x = (I - P_M)x$$

for each $x \in H$. The mappings $P_i^* \in \mathcal{L}(H_i, H)$ are defined for $i = 1, 2$ by the identities

$$\langle P_i^* \xi_i, x \rangle = \langle \xi_i, P_i x \rangle = \langle \xi_i, x \rangle$$

for each $\xi_i \in H_i$ and $x \in H$. Therefore $P_i^* \xi_i = \xi_i$ and we write $P_i^* = I_i$. Define $R \in \mathcal{L}(H, H_1 \times H_2)$ by the formula

$$Rx = \begin{pmatrix} P_1 x \\ P_2 x \end{pmatrix}$$

for each $x \in H$. The mapping $R^* \in \mathcal{L}(H_1 \times H_2, H)$ is given by

$$R^* \xi = R^* \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = P_1^* \xi_1 + P_2^* \xi_2 = \xi_1 + \xi_2$$

for each

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in H_1 \times H_2.$$

Since

$$\begin{aligned} \langle Rx_1, Rx_2 \rangle &= \langle P_1 x_1, P_1 x_2 \rangle + \langle P_2 x_1, P_2 x_2 \rangle \\ &= \langle P_1 x_1 + P_2 x_1, P_1 x_2 + P_2 x_2 \rangle \\ &= \langle x_1, x_2 \rangle \end{aligned}$$

for each $x_1, x_2 \in H$ the mapping R defines a unitary equivalence between H and $H_1 \times H_2$. We write $H \cong H_1 \times H_2$.

In the same way we note that $N = B(M)$ is a closed subspace and since the orthogonal complement $N^\perp$ is also closed it follows that $K_1 = N$ and $K_2 = N^\perp$ are each Hilbert spaces. Let $Q_N \in \mathcal{L}(K, K)$ denote the natural projection onto the subspace $N \subseteq K$ and define associated mappings $Q_1 \in \mathcal{L}(K, K_1)$ and $Q_2 \in \mathcal{L}(K, K_2)$ by setting

$$Q_1 y = Q_N y, \quad \text{and} \quad Q_2 y = (I - Q_N)y$$

for each $y \in K$. The mappings $Q_j^* \in \mathcal{L}(K_j, K)$ are defined for $j = 1, 2$ by the identities

$$\langle Q_j^* \eta_j, y \rangle = \langle \eta_j, Q_j y \rangle = \langle \eta_j, y \rangle$$

for each $\eta_j \in K_j$ and $x \in H$. Therefore $Q_j^* \eta_j = \eta_j$ and we write $Q_j^* = I_j$. Define $S \in \mathcal{L}(K, K_1 \times K_2)$ by the formula

$$Sy = \begin{pmatrix} Q_1 y \\ Q_2 y \end{pmatrix}$$

for each $y \in K$ and note that $S^* \in \mathcal{L}(K_1 \times K_2, K)$ is given by

$$S^* \eta = S^* \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = Q_1^* \eta_1 + Q_2^* \eta_2 = \eta_1 + \eta_2$$

for each

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in K_1 \times K_2.$$

The mapping S defines a unitary equivalence between K and $K_1 \times K_2$. We write $K \cong K_1 \times K_2$.

We partition the operators A and B in the form

$$SAR^* = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad SBR^* = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

where $A_{ij}, B_{ij} \in \mathcal{L}(H_i, K_j)$ and where $A_{11} = Q_1 A P_1^* = 0$, $A_{12} = Q_1 A P_2^*$, $A_{21} = Q_2 A P_1^* = 0$, and $A_{22} = Q_2 A P_2^*$; and $B_{11} = Q_1 B P_1^*$, $B_{12} = Q_1 B P_2^*$, $B_{21} = Q_2 B P_1^* = 0$, and $B_{22} = Q_2 B P_2^*$. The block-triangular structure follows from the next equalities:

$$A P_1^* \xi_1 = A \xi_1 = 0, \quad \forall \xi_1 \in H_1,$$

and

$$Q_2 B P_1^* \xi_1 = Q_2 B \xi_1 = Q_2 \eta_1 = 0, \quad \forall \xi_1 \in H_1.$$

Now we are able to formulate the main result of this section. This is a generalization of the approach given in Schweitzer and Stewart [135].

Theorem 6 *Suppose that $A \in \mathcal{L}(H, K)$ is singular and that $B_{11} \in \mathcal{L}(H_1, K_1)$ is a $1-1$ mapping of H_1 onto K_1. The mapping $(A + zB) \in \mathcal{L}(H, K)$ is a $1-1$ mapping of H onto K if and only if $z \neq 0$ and $(A_{22} + zB_{22}) \in \mathcal{L}(H_2, K_2)$ is a $1-1$ mapping of H_2 onto K_2. In this case*

$$\begin{aligned} (A + zB)^{-1} &= P_1^* B_{11}^{-1} Q_1 / z \\ &\quad + [P_2^* - P_1^* B_{11}^{-1} (A_{12} + zB_{12})/z](A_{22} + zB_{22})^{-1} Q_2. \end{aligned} \tag{2.44}$$

If $A_{22} \in \mathcal{L}(H_2, K_2)$ is a $1-1$ mapping of H_2 onto K_2, then for some real number $\epsilon > 0$ the operator $(A + zB)^{-1}$ can be represented in the region $0 < |z| < \epsilon$ by a convergent Laurent series with a pole of order 1 at $z = 0$.

PROOF: Since

$$(A + zB) = S^* \begin{pmatrix} zB_{11} & A_{12} + zB_{12} \\ 0 & A_{22} + zB_{22} \end{pmatrix} R$$

where R and S are unitary operators it follows that $(A + zB)^{-1}$ exists if and only if

$$\begin{pmatrix} zB_{11} & A_{12} + zB_{12} \\ 0 & A_{22} + zB_{22} \end{pmatrix}^{-1}$$

exists. If we set $x = R\xi$ and $y = S\eta$ then we may say that the system of equations

$$(A + zB)x = y$$

has a unique solution $x \in H$ for each $y \in K$ if and only if the system of equations

$$\begin{aligned} (zB_{11})\xi_1 + (A_{12} + zB_{12})\xi_2 &= \eta_1 \\ (A_{22} + zB_{22})\xi_2 &= \eta_2 \end{aligned}$$

has a unique solution $\xi \in H_1 \times H_2$ for each $\eta \in K_1 \times K_2$. If the latter system is rewritten in the form

$$\begin{aligned} (A_{22} + zB_{22})\xi_2 &= \eta_2 \\ (zB_{11})\xi_1 &= \eta_1 - (A_{12} + zB_{12})\xi_2 \end{aligned}$$

then it is clear there will be a unique solution if and only if $z \neq 0$ and both B_{11} is a $1 - 1$ mapping of H_1 onto K_1 and $(A_{22} + zB_{22})$ is a $1 - 1$ mapping of H_2 onto K_2. In this case

$$\begin{aligned} \xi_2 &= (A_{22} + zB_{22})^{-1}\eta_2 \\ \xi_1 &= B_{11}^{-1}\left[\eta_1 - (A_{12} + zB_{12})\xi_2\right]/z \end{aligned}$$

and hence, by back substitution,

$$\begin{aligned} x &= P_1^*\xi_1 + P_2^*\xi_2 \\ &= \{P_1^* B_{11}^{-1} Q_1/z + \left[P_2^* - P_1^* B_{11}^{-1}(A_{12} + zB_{12})/z\right](A_{22} + zB_{22})^{-1} Q_2\}y. \end{aligned}$$

Thus we obtain the desired formula for $(A + zB)^{-1}$.

□

Remark 8 *Recall that if A is singular and $(A + z_0 B)^{-1}$ exists for some $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ then B is bounded below on H_1. Equivalently we can say that $B_{11} \in \mathcal{L}(H_1, K_1)$ is bounded below. It follows that B_{11} is a $1-1$ mapping of H_1 onto K_1.*

Remark 9 *It is possible to reduce polynomial perturbations to linear perturbations using an augmentation procedure and to then invert as above. For more detail see the next subsection.*

Now let us note that Theorem 6 points to an iterative procedure for the computation of the Laurent series for $(A+zB)^{-1}$. Indeed, if the operator $(A+zB)$ is invertible in some domain around $z=0$, then the *reduced* operator $(A_{22}+zB_{22})$ is invertible as well. Note that the operator $(A_{22}+zB_{22})$ satisfies all conditions of Theorem 6 and hence formula (2.44) can be applied again. If the new unperturbed reduced operator is invertible, then the orginal perturbed operator $(A+zB)$ can be expanded in a Laurent series with the pole of order two and we can terminate this *reduction* process. Otherwise, we can perform the next iteration. Note that the present reduction process is different from the one proposed in Section 2.2. However, these two reduction processes have the folowing common feature: the reduced operators act on "smaller" underlying spaces. In the case of finite dimensional Hilbert spaces this implies that the reduction process has a finite number of steps and the inverse of the linearly perturbed operator can always be expanded as a Laurent series with a finite order pole. However, in the case of infinite dimensional Hilbert spaces the inverse operator $(A+zB)^{-1}$ can have an essential singularity at $z=0$ and therefore the described above iterative process can, in principle, go on indefenitely. Of course, one can impose further restrictions on operators A and B to guarantee that the Laurent series has a finite order pole. For instance, one possibility is to assume that the operator A is Fredholm. However, we would like to emphasize again that the methods of [10, 91, 145] constructed for Fredholm operators can handle less general cases than the method presented above. In particular, our approach can treat the problem with the operator A having an infinite dimensional null space.

2.3.4 Extension to polynomial and analytic perturbation

Now let us show that the method described in the previous subsection can be easily applied to polynomial or even analytic perturbations. First we shall show that the polynomial perturbation (2.41) can be transformed to an equivalent linear perturbation problem.

Note that the problem of inversion of the perturbation operator is equivalent to the solution of the infinite system of *fundamental equations* (see Section 2.2). Suppose that the Laurent expansion has a finite order pole, as before in (2.22). Then, in the case of polynomial perturbation (1.4) the coefficients of the Laurent series $\{X_k\}_{k=0}^{\infty}$ have to satisfy the fundamental equations

$$\sum_{i=0}^{m} A_i X_{l-i} = \begin{cases} I, & l = s, \\ 0, & \text{otherwise} \end{cases} \tag{2.45}$$

The approach of Schweitzer and Stewart can be applied to the polynomial case by transforming the original perturbation problem to a suitable *linear* perturbation problem in *augmented* space. There are a number of methods to implement this idea [10, 14, 11, 60, 108, 124]. Here we discuss two schemes which seem to be most appropriate for the present approach.

Theorem 7 *Suppose the inverse of the polynomial perturbation operator* (2.41) *can be expanded as a Laurent series* (2.22) *in some punctured neighbourhood of the origin. Define augmented operators from H^m to K^m by setting*

$$\mathcal{A} := \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m-1} & A_{m-2} & \cdots & A_0 \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} A_m & A_{m-1} & \cdots & A_1 \\ 0 & A_m & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad \mathcal{I} := \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then the solution of the linear perturbation problem

$$(\mathcal{A} + z\mathcal{B})\mathcal{X}(z) = z^{s_*}\mathcal{I} \tag{2.46}$$

can be expanded as a Laurent series

$$\mathcal{X}(z) = \frac{1}{z^{s_*}} \left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_0 \\ \vdots \\ X_{mod(s,m)} \end{bmatrix} + \begin{bmatrix} X_{mod(s,m)+1} \\ \vdots \\ \\ \\ \vdots \\ X_{mod(s,m)+m} \end{bmatrix} + ... \right) \tag{2.47}$$

where $s_* = \lfloor s/m \rfloor + 1$ and $\{X_k\}_{k=0}^{\infty}$ are coefficients of the Laurent series (2.22)*, and where $\lfloor s/m \rfloor$ is the integer part and $mod(s,m)$ is the remainder of the division of s by m.*

PROOF: The proof is based on the fact that the system of fundamental equations for (2.46) precisely coincides with the system of fundamental equations (2.45).

□

Thus, we can apply the reduction process of Section 2.3.3 first to the augmented problem and then we can easily retrieve the coefficients of the original Laurent series (2.2).

The next scheme was analysed in [14, 108]. Note that the method was used for the perturbation analysis of the generalized eigenvalue problem but not for the inversion of perturbation operators.

Theorem 8 *Let the augmented operators be defined as follows:*

$$\mathcal{A} := \begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \qquad \mathcal{B} := \begin{bmatrix} 0 & 0 & \cdots & A_m \\ -I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I & 0 \end{bmatrix}$$

Then the uppermost block of the solution $\mathcal{X}(z)$ for equation (2.46) with $s_* = s$ coincides with the solution of the original equation $A(z)X(z) = z^s I$.

Each of the presented augmentation schemes has its own merits. Using the second method to obtain the first k coefficients of the Laurent series (2.22) one needs to calculate k augmented coefficients $\mathcal{X}_i, i = 0, ..., k-1$, whereas if one utilizes the first augmentation scheme, one needs to compute about m times less augmented coefficients. However, in the second method both augmented matrices $\mathcal{A}$ and $\mathcal{B}$ are close to upper triangular form and have a lot of zero elements. Since the Schweitzer and Stewart method is based on simultaneous reduction of matrices $\mathcal{A}$ and $\mathcal{B}$ to upper triangular form, each iteration of the second method could be more computationally efficient.

Now we show that in turn the results on polynomial perturbation can be applied to the general case of analytical perturbation (2.42). Suppose again that the Laurent series for the inverse $A^{-1}(z)$ has a pole of finite order s. Then the number of terms which uniquely determine the inversion procedure is finite (see Section 2.2). Namely, there exists m such that the inverse $(A_0 + ... + z^m A_m)^{-1}$ exists for sufficiently small z. Moreover, any $m \geq s$ can be taken. Therefore we may write

$$\begin{aligned} A^{-1}(z) &= [(A_0 + ... + z^s A_s) + z^{s+1} A_{s+1} + ...]^{-1} \\ &= [(A_0 + ... + z^s A_s)(I + (A_0 + ... + z^s A_s)^{-1}(z^{s+1} A_{s+1} + ...))]^{-1} \\ &= [I + (A_0 + ... + z^s A_s)^{-1}(z^{s+1} A_{s+1} + ...)]^{-1}(A_0 + ... + z^s A_s)^{-1} \qquad (2.48) \end{aligned}$$

By using the method of augmented operators, we obtain the following Laurent series

$$(A_0 + ... + z^s A_s)^{-1} = \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k.$$

Now substitute the above expansion into (2.48).

$$\begin{aligned} A^{-1}(z) &= \left[I + \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k (z^{s+1} A_{s+1} + z^{s+2} A_{s+2} + ...) \right]^{-1} \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k \\ &= \left[I + \sum_{k=1}^{\infty} z^k \sum_{i=0}^{k-1} \tilde{X}_i A_{s+k-i} \right]^{-1} \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k \end{aligned}$$

By expanding the expression

$$\left[I + \sum_{k=1}^{\infty} z^k \sum_{i=0}^{k-1} \tilde{X}_i A_{s+k-i} \right]^{-1}$$

as a Neumann power series, we obtain a simple recursive formulae for coefficients $X_k, k = 0, 1, ...$ of the Laurent series for $A^{-1}(z)$. For instance,

$$X_{-s} = \tilde{X}_{-s},$$

$$X_{-s+1} = \tilde{X}_{-s+1} - \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s},$$

$$X_{-s+2} = \tilde{X}_{-s+2} - \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s+1} - \tilde{X}_{-s+1} A_{s+1} \tilde{X}_{-s} - \tilde{X}_{-s} A_{s+2} \tilde{X}_{-s} + \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s}.$$

2.4 Perturbation of null spaces with application to the eigenvalue problem and generalized inverses

2.4.1 Preliminaries

The primary goal of this section is to analyse the null space of an analytically perturbed matrix

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{2.49}$$

with $A_k \in R^{n \times n}, k = 0, 1, ...,$ and the above series converging in the region $0 < |\varepsilon| \leq \varepsilon_{\max}$ for some positive $\varepsilon_{\max}$. We assume that the unperturbed matrix A_0 has eigenvalue zero with multiplicity $m \geq 1$ and that the perturbed matrices $A(\varepsilon)$ also have eigenvalue zero with multiplicity $\bar{m}$ for ε sufficiently small but different from zero. We emphasize that the dimension of the perturbed null space does not depend on ε in some small punctured neighbourhood around $\varepsilon = 0$ [60, 61] (see also Subsection 2.4.7). When the perturbation parameter ε deviates from zero, the zero eigenvalues of the unperturbed matrix can split into zero and non-zero eigenvalues [14, 87]. This fact implies that $\bar{m} \leq m$. More detailed discussion on the stability properties of null spaces can be found in [12] and references therein. We assume that $\bar{m} \geq 1$ and (for computational purposes) that the value of $\bar{m}$

should be known in advance. The case where $\bar{m} = 0$ and hence $A(\varepsilon)$ is invertible for $\varepsilon \neq 0$ and sufficiently small, was dealt with in Sections 2.2 and 2.3. A perturbation is said to be *regular* if $\bar{m} = m$, and it is said to be *singular* if $\bar{m} < m$.

We denote by $v_i, i = 1, ..., m$, m orthonormal eigenvectors of A_0 corresponding to the eigenvalue zero and form the matrix $V := [v_1, ..., v_m]$. This matrix satisfies the following equations

$$A_0 V = 0, \tag{2.50}$$

$$V^T V = I_m. \tag{2.51}$$

Similarly, let $v_i(\varepsilon), i = 1, ..., \bar{m}$ be linearly independent eigenvectors of the perturbed matrix $A(\varepsilon)$ corresponding to the eigenvalue zero. Again one can form the matrix $V(\varepsilon) := [v_1(\varepsilon), ..., v_{\bar{m}}(\varepsilon)]$, which satisfies the equation

$$A(\varepsilon)V(\varepsilon) = 0. \tag{2.52}$$

From Chapter S6 of [60],the results of [61], and Section 2.4.7 of this thesis, we know that there exists a holomorphic family of vector-valued functions $v_i(\varepsilon)$ which constitute a basis for the null space of $A(\varepsilon)$ for $\varepsilon \neq 0$. Therefore, $V(\varepsilon)$ can be expressed as a power series in some neighbourhood of zero

$$V(\varepsilon) = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \cdots. \tag{2.53}$$

Of course, there exists an orthonormal family of perturbed eigenvectors $v_i(\varepsilon), i = 1, ..., \bar{m}$. However, it is more convenient to construct a "quasi-orthonormal" family of eigenvectors described by the following condition

$$V_0^T V(\varepsilon) = I_{\bar{m}}, \tag{2.54}$$

where V_0 is the first coefficient of the power series expansion (2.53). The above normalized conditions were used in [27, 28] for the case of regular perturbations. Note that even though this family of eigenvectors is not orthonormal for $\varepsilon \neq 0$, it is linearly independent when ε is sufficiently small [27, 28]. Also note that (2.54) was introduced in order to make $V(\varepsilon)$ unique once the leading term V_0 is determined (and some degrees of freedom exist here). Furthermore, one can always obtain an orthonormal basis from a "quasi-orthonormal" basis by applying a Gram-Schmidt-like procedure over the vectors with elements as power series expansions. This procedure will be discussed in more detail in Section 2.4.6. As mentioned above, we distinguish between two cases, the regular case where $\bar{m} = m$ and the singular case where $1 \leq \bar{m} < m$. To justify this terminology, note that only in the regular case it is possible to take $V_0 = V$. Moreover, as will be demonstrated in Section 2.4.6 in the case of singular perturbation only, the group inverse of the perturbed matrix (when it exists) has a singularity, that is, a pole at $\varepsilon = 0$. Later on we also show that the group inverse of $A(\varepsilon)$, denoted next by $A^g(\varepsilon)$, can be expanded as a Laurent series:

$$A^g(\varepsilon) = \frac{1}{\varepsilon^s} B_{-s} + \cdots + \frac{1}{\varepsilon} B_{-1} + B_0 + \varepsilon B_1 + \cdots \tag{2.55}$$

where s is the order of the pole at the singularity.

One of the main purposes of this section is to obtain an efficient recursive algorithm for the computation of coefficients $V_k, k = 0, 1, \ldots$ as well as for the computation of $B_l, l = -s, -s+1, \cdots$, determining the Laurent expansion (2.55). We would like to point out that, at least with our procedure, the latter task cannot be accomplished without prior completion of the former task. The algorithm for computing $V_k, k = 0, 1, \ldots$ is based on recursively solving a system of *fundamental equations*. The fundamental equations named $F0, F1, \ldots$, are obtained by substituting (2.49) and (2.53) into (2.52) to give

$$\begin{aligned}
A_0 V_0 &= 0 && (F0)\\
A_0 V_1 + A_1 V_0 &= 0 && (F1)\\
A_0 V_2 + A_1 V_1 + A_2 V_0 &= 0 && (F2)\\
&\vdots && \vdots\\
A_0 V_k + A_1 V_{k-1} + \cdots + A_{k-1} V_1 + A_k V_0 &= 0 && (Fk)\\
&\vdots && \vdots
\end{aligned}$$

The normalization condition (2.54) gives rise to another system of equations named $N0, N1, \ldots$ and given by

$$\begin{aligned}
V_0^T V_0 &= I && (N0)\\
V_0^T V_1 &= 0 && (N1)\\
&\vdots && \vdots\\
V_0^T V_k &= 0 && (Nk)\\
&\vdots && \vdots
\end{aligned}$$

We will refer to the latter system as the system of *normalization equations*. We treat the cases of regular and singular perturbations separately. In Section 2.4.3 we provide an algorithm for computing the coefficients V_k, $k \geq 0$, in the regular case. This algorithm is based on a straightforward recursive procedure. The singular case is treated in Section 2.4.4, where we suggest three algorithms for computing $\{V_i\}_{i=0}^{\infty}$. The first is based on defining an augmented matrix which leads to the solution. The second algorithm is based on reducing the dimension of the equations to a set of equations whose type coincides with the regular case. The third algorithm is a combination of the previous two algorithms and is based on repeating the reduction process a few times before solving the resulting system with the help of a generalized inverse. The number of reductions needed is determined by the series $\{A_i\}_{i=0}^{\infty}$ itself. Finally, in Section 2.4.5 we show how our results can be applied to a perturbation analysis of the general eigenvalue problem and in Section 2.4.6 we present a second application of our results to the computation of the Laurent series expansion for the perturbed group inverse (2.55). Before proceeding further, we would like to mention that the results of the present section are also based on the fundamental Lemmas 1 and 2 from Section 2.1.

2.4.2 Examples

Before proceeding to the formal analysis let us illustrate the problem by a couple of examples. First we consider an example of the regular perturbation. Namely, let the perturbed matrix be given by

$$A(\varepsilon) = A_0 + \varepsilon A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The null spaces of A_0 and $A(\varepsilon)$ are both one-dimensional and they are spanned by the following vectors.

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v(\varepsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \varepsilon^2 \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} + \ldots$$

We can see that $v(\varepsilon)$ is holomorphic and converges to v as ε goes to zero. Next we consider an example of the singular perturbation.

$$A(\varepsilon) = A_0 + \varepsilon A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The null space of A_0 is two-dimensional and it is spanned by

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

and the null space of $A(\varepsilon)$ is one-dimensional and it is spanned by the holomorphic vector-valued function

$$v(\varepsilon) = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + \varepsilon^2 \begin{bmatrix} -1/(4\sqrt{2}) \\ 0 \\ -1/(4\sqrt{2}) \end{bmatrix} + \ldots \tag{2.56}$$

Thus, we can see that $v(\varepsilon)$ converges to the vector which belongs to the unperturbed null space as $\varepsilon \to 0$ but there is a gap in the dimensions between the perturbed and unperturbed null spaces. The latter phenomenon justifies the name *singular* perturbation.

2.4.3 Regular perturbations

Let us obtain the feasibility conditions for the systems (F) and (N) in the case of a regular perturbation.

Lemma 6 *If the perturbation is regular then the sequence of matrices $\{A_k\}_{k=0}^{\infty}$ satisfies the following conditions:*

$$U^T \left(\sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1+...+\nu_p=k+1} A_{\nu_1} A_0^{\dagger} A_{\nu_2} ... A_0^{\dagger} A_{\nu_p} \right) V = 0, \quad k = 0, 1, ... , \tag{2.57}$$

where V and U are bases for the right and left null spaces of matrix A_0, respectively.

PROOF: From equation (F0) we conclude that

$$V_0 = V C_0 \tag{2.58}$$

where C_0 is some coefficient matrix. Since we consider the case of a regular perturbation, the rank of V_0 equals m. This in turn implies that $C_0 \in R^{m \times m}$ and that it is a full rank matrix.

Since $U^T A_0 = 0$, we obtain by Lemma 2, the following feasibility condition for equation (F1):

$$U^T A_1 V_0 = 0$$

Upon substituting (2.58) into the above expression, we get that

$$U^T A_1 V C_0 = 0.$$

Moreover, since C_0 is a full rank matrix, we conclude that

$$U^T A_1 V = 0, \tag{2.59}$$

which is the first feasibility condition of (2.57).

Since the perturbation is regular, there exists a holomorphic basis $V(\varepsilon)$ for the perturbed null space such that $V = V(0)$. The coefficients $V_k, k = 0, 1, ...$ of the power series (2.53) satisfy the fundamental equations (F0),(F1),... Hence the feasibility condition for equation (F1) is satisfied and we can write its general solution in the form

$$V_1 = V C_1 - A_0^{\dagger} A_1 V C_0. \tag{2.60}$$

where $C_1 \in R^{m\times m}$ is some matrix. According to Lemma 2, the feasibility condition for the next fundamental equation (F2) is

$$U^T(A_1V_1 + A_2V_0) = 0 \quad .$$

By substituting (2.58) and (2.60) into the above expression we obtain

$$U^T A_1 V C_1 + U^T(A_2 - A_1A_0^{\dagger}A_1)VC_0 = 0.$$

Now using the first feasibility condition (2.59) and the fact that C_0 is a full rank matrix, we obtain that

$$U^T(A_2 - A_1A_0^{\dagger}A_1)V = 0,$$

which is condition (2.57) for the case that $k = 1$. By induction all feasibility conditions (2.57) can be proved to hold.

□

The next theorem provides a recursive formula for the computation of the coefficients $V_k, k = 0, 1, \dots$.

Theorem 9 *Let the matrix $A(\varepsilon)$ be regularly perturbed. Then there exists a holomorphic family of eigenvectors $V(\varepsilon)$ corresponding to the zero eigenvalue and satisfying the normalization condition (2.54). Moreover, the coefficients of the power series for $V(\varepsilon)$ can be calculated recursively by the following formula:*

$$V_k = -A_0^{\dagger}\sum_{j=1}^{k} A_jV_{k-j} \ , \quad k = 1, 2, \dots, \tag{2.61}$$

where $V_0 = V$.

PROOF: It follows from the proof of Lemma 6 that the general solution of the fundamental equations is

$$V_k = VC_k - A_0^{\dagger}\sum_{j=1}^{k} A_jV_{k-j} \ , \quad k = 1, 2, \dots$$

with $V_0 = VC_0$. By choosing $C_0 = I_m$, we get $V_0 = V$, which satisfies the normalization condition (N0).

Now the coefficients $C_k, k = 1, 2, \dots$, are uniquely determined by the normalization conditions (Nk). Namely, we have

$$V_0^TV_k = V_0^T(VC_k - A_0^{\dagger}\sum_{j=1}^{k} A_jV_{k-j}) = 0,$$

or, equivalently,

$$V^T V C_k - V^T A_0^\dagger \sum_{j=1}^{k} A_j V_{k-j} = 0,$$

since $V_0 = V$ when $C_0 = I_m$. Recalling that $V^T V = I$ and $V^T A_0^\dagger = 0$, since $N(A) = R(A^\dagger)^\perp$ [24], we obtain that $C_k = 0, k = 1, 2, ...$ as required.

□

Next, we would like to address the issue of the radius of convergence. First, above we have implicitly assumed that the series $\{A_i\}_{i=0}^\infty$ has a positive radius of convergence. This is, for example, the case when $\| A_i \| \leq ar^i$ for some constants a and r and where $\| \dot{\|}$ is some matrix norm. In this case the radius of convergence is (at least) r^{-1}. The next theorem follows the similar technique of [73] and gives a bound on the radius of convergence of the series given in (2.53) with coefficients as in (2.61) for this special case.

Theorem 10 *Suppose $||A_i|| \leq ar^i$ for some positive constants a and r, then the radius of convergence of the series $V(\varepsilon) = V_0 + \varepsilon V_1 + \cdots$, where V_k is computed by (2.61), is at least* $(1 + a||A_0^\dagger||)^{-1} r^{-1}$.

Proof: First, we prove by induction the following inequality

$$||V_k|| \leq ||V_0||(1 + a||A_0^\dagger||)^k r^k \quad . \tag{2.62}$$

which trivially holds when $k = 0$. Now suppose inequality (2.62) holds for the coefficients $V_0, .., V_{k-1}$. From (2.61), we obtain

$$||V_k|| \leq ||A_0^\dagger|| \sum_{j=1}^{k} ||A_j|| ||V_{k-j}|| \leq a||A_0^\dagger|| \sum_{j=1}^{k} r^j ||V_{k-j}||.$$

Now using inequality (2.62) for $j = 0, 1, \ldots, k-1$, (which is the inductive hypothesis) we get

$$\begin{aligned} ||V_k|| &\leq a||A_0^\dagger|| \sum_{j=1}^{k} r^j ||V_0||(1 + a||A_0^\dagger||)^{k-j} r^{k-j} \\ &\leq a||A_0^\dagger|| ||V_0|| r^k \sum_{j=1}^{k} (1 + a||A_0^\dagger||)^{k-j}. \end{aligned}$$

Note that

$$\sum_{j=1}^{k} (1 + a||A_0^\dagger||)^{k-j} = \frac{(1 + a||A_0^\dagger||)^k - 1}{1 + a||A_0^\dagger|| - 1} = \frac{(1 + a||A_0^\dagger||)^k - 1}{a||A_0^\dagger||}.$$

Thus,

$$\begin{aligned} ||V_k|| &\leq ||V_0||r^k[(1+a||A_0^\dagger||)^k - 1] \\ &\leq ||V_0||r^k(1+a||A_0^\dagger||)^k \end{aligned}$$

as required. Consequently, the radius of convergence for the power series $V(\varepsilon) = V_0 + \varepsilon V_1 + \cdots$ is at least $(1+a||A_0^\dagger||)^{-1}r^{-1}$.

□

2.4.4 Singular perturbations

In this subsection we deal with the case of singular perturbations, namely when the dimension of the perturbed null space $N(A(\varepsilon)), 0 < \varepsilon < \bar{\varepsilon}$ is strictly less than the dimension of $N(A_0)$. Next we propose two algorithms. The first is based on generalized inverses applied to some augmented matrices, whereas the second is based on a reduction technique. Both methods have their own merits. Finally, we also suggest a way to combine these two approaches.

An approach based on augmented matrices

Some definitions are required prior to introducing our analysis for the singlar case, that is, the case where $\bar{m} < m$. First, as in Section 2.2, for any integer t, $t \geq 0$, we define the augmented matrix $\mathcal{A}^{(t)} \in R^{n(t+1)\times n(t+1)}$ [61, 67, 73]:

$$\mathcal{A}^{(t)} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_t & A_{t-1} & \cdots & A_1 & A_0 \end{bmatrix}$$

Second, we partition the generalized inverse $\mathcal{G}^{(t)} := [\mathcal{A}^{(t)}]^\dagger$ into a block structure which corresponds to the structure of the augmented matrix $\mathcal{A}^{(t)}$. Namely,

$$\mathcal{G}^{(t)} = \begin{bmatrix} G_{00}^{(t)} & \cdots & G_{0t}^{(t)} \\ \vdots & \ddots & \vdots \\ G_{t0}^{(t)} & \cdots & G_{tt}^{(t)} \end{bmatrix},$$

where $G_{ij}^{(t)} \in R^{n\times n}$ for $0 \leq i, j \leq t$.

Third, let $B_t \subseteq R^n$ be the linear subspace of vectors w such that for some vector $v \in N(\mathcal{A}^{(t)}) \subseteq R^{n(t+1)}$, the first n entries in v coincide with w. Since $\bar{v} \in N(\mathcal{A}^{(t+1)})$

implies that the first $n(t+1)$ entries of $\bar{v}$ form a vector $v \in N(\mathcal{A}^{(t)})$, $B_{t+1} \subseteq B_t$ for any $t \geq 0$ and hence $dim(B_t)$ is non-increasing with t. Finally, let $\tau = \arg\min_t\{dim(B_t)\}$. In other words, τ is the smallest value of t where the minimum of $dim(B_t)$ is attained.

Theorem 11 *For any $V_0 \in B_\tau$, there exists a sequence $\{V_i\}_{i=0}^{\infty}$ which solves*

$$\mathcal{A}^{(t)} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.63}$$

for any $t \geq 0$. In particular, $\bar{m} = dim(B_\tau)$.

PROOF: A necessary (but not sufficient) condition for V_0 to be a leading term in such a series is that $A_0V_0 = 0$, i.e., that $V_0 \in B_0$. But what is further required is that for this V_0, there exists a V_1 such that $A_0V_1 + A_1V_0 = 0$, i.e., that $V_0 \in B_1$. Conversely, any $V_0 \in B_1$ (coupled with an appropriate V_1), solves (2.63) for $t = 1$. Similarly, one can see that $V_0 \in B_2$ (coupled with the corresponding V_1 and V_2 which exist by the definition of B_2) if and only if (2.63) holds for $t = 2$. By induction, we conclude that V_0 leads to a solution for (2.63) for any $t \geq 0$, if any only if $V_0 \in B_t$ for any $t \geq 0$, i.e., if and only if $V_0 \in B_\tau$. The explanation of $\bar{m} = dim(B_\tau)$ follows from the fact, that for each $V_0 \in B_\tau$ one can construct an analytic perturbed eigenvector $V(\varepsilon) = V_0 + \varepsilon V_1 + ...$ Thus, the dimension of B_τ coincides with the dimension of the perturbed null space.

□

The equations (F0)–(Fτ) are

$$\mathcal{A}^{(\tau)} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_\tau \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.64}$$

Above we argued that any vector in B_τ will lead to a solution of (2.64). Adding the normalization condition (N0) is now equivalent to requiring that V_0 be an orthonormal basis. Finally, any such orthonormal basis will do for our purposes.

Once V_0 is determined, the next goal is the determination of the corresponding V_1. Using (F1)–(F($\tau+1$)) we have,

$$\mathcal{A}^{(\tau)} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{\tau+1} \end{bmatrix} = \begin{bmatrix} -A_1V_0 \\ -A_2V_0 \\ \vdots \\ -A_{\tau+1}V_0 \end{bmatrix} \tag{2.65}$$

which is as (2.63) with $t = \tau$ but with a different righthand side. Note that by definition of τ and by the fact that $V_0 \in B_\tau$, (2.65) is solvable. Hence, by Lemma 1,

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{\tau+1} \end{bmatrix} = [\mathcal{A}^{(\tau)}]^\dagger \begin{bmatrix} -A_1V_0 \\ -A_2V_0 \\ \vdots \\ -A_{\tau+1}V_0 \end{bmatrix} + y$$

for *some* $y \in Null(\mathcal{A}^{(\tau)})$. However, note that not any $y \in Null(\mathcal{A}^{(\tau)})$ will do since in (2.65) we have not considered all fundamental equations. However, for any $w \in B_\tau$ there exists such a y with w being its first n entries. Moreover, any such w leads to a vector V_1 such that coupled with V_0, they are the leading two terms in a series expansion for $V(\varepsilon)$. The reason is that whatever was true for V_0 is now true for V_1 since in the latter case one gets the same set of equations but with a different righthand side. The normalization condition (N1), coupled with the fact that V_0 is chosen, imply a unique value for the matrix V_1.

Above we have shown how the value of V_0 leads to the value of V_1. Next, we show that this is the case in general. Specifically, once $V_0, ..., V_k$ are determined, one can compute V_{k+1} by the recursive formula provided in the next theorem.

Theorem 12 *The solution of the system of fundamental equations (F) coupled with the normalization conditions (N) is given by the following recursive formula*

$$V_{k+1} = -(I - V_0V_0^T) \sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+1} A_{i+j}V_{k+1-i}, \tag{2.66}$$

where V_0 is any orthogonal basis of the linear subspace B_τ.

PROOF: Consider the set of fundamental equations (F.k+1)-(F.k+1+τ). Since they are feasible, the general solution is of the form

$$\begin{bmatrix} V_{k+1} \\ \vdots \\ V_{k+1+\tau} \end{bmatrix} = \mathcal{G}^{(\tau)} \begin{bmatrix} -\sum_{i=1}^{k+1} A_iV_{k+1-i} \\ \vdots \\ -\sum_{i=1}^{k+1} A_{i+\tau}V_{k+1-i} \end{bmatrix} + y$$

where $y \in Null(\mathcal{A}^{(\tau)})$. Since the first n entries of y constitute a vector w in B_τ and V_0 is an orthogonal basis of B_τ, the general solution for V_{k+1} can be written as follows

$$V_{k+1} = -\sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+\tau} A_{i+\tau}V_{k+1-i} + V_0C_{k+1}, \tag{2.67}$$

where C_{k+1} is some matrix coefficient that can be determined from the normalization condition (N.k+1). Specifically,

$$-V_0^T \sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+\tau} A_{i+\tau} V_{k+1-i} + C_{k+1} = 0,$$

and hence

$$C_{k+1} = V_0^T \sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+\tau} A_{i+\tau} V_{k+1-i}.$$

Substituting the above expression for the coefficient C_{k+1} into the formula (2.67) results in the recursive formula (2.66). This completes the proof.

□

Remark 10 *We would like to point out that although above we call for $[A^{(\tau)}]^\dagger$, only its first $\bar{m}$ rows are required in order to carry out the desired computations.*

An algorithm based on a reduction process

Next, we show that by using a reduction process one can transform the system of fundamental equations (F) to another system with coefficient matrices of reduced dimensions. The latter system can be solved by the algorithm proposed in Section 2.4.2 for the regular case. Thus, we reduce the singular problem to the regular one. The next theorem is a key to the reduction process.

Theorem 13 *A solution of the fundamental equations (F) together with the normalization conditions (N) is given by the following recursive formula*

$$V_k = VW_k - A_0^\dagger \sum_{j=1}^{k} A_j V_{k-j}, \quad k = 1, 2, \ldots \tag{2.68}$$

with $V_0 = VW_0$, and where the sequence of auxiliary matrices W_k, $k \geq 0$, is a solution to the next system of reduced fundamental equations (RF):

$$\begin{aligned}
B_0 W_0 &= 0 && (RF0)\\
B_0 W_1 + B_1 W_0 &= 0 && (RF1)\\
B_0 W_2 + B_1 W_1 + B_2 W_0 &= 0 && (RF2)\\
&\vdots && \vdots\\
B_0 W_k + B_1 W_{k-1} + \cdots + B_{k-1} W_1 + B_k W_0 &= 0 && (RFk)\\
&\vdots && \vdots
\end{aligned}$$

and reduced normalization conditions (RN):

$$W_0^T W_0 = I_{\bar{m}} \qquad (RN0)$$
$$W_0^T W_k = 0 \quad k = 1, 2, \ldots \; . \qquad (RNk)$$

with the coefficient matrices B_k, $k \geq 0$, are given by the formula

$$B_k = U^T \left(\sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1 + \cdots + \nu_p = k+1} A_{\nu_1} A_0^{\dagger} A_{\nu_2} A_0^{\dagger} \ldots A_0^{\dagger} A_{\nu_p} \right) V, \qquad (2.69)$$

where ν_k are positive integers.

Proof: From the fundamental equation (F0) we conclude that V_0 belongs to the null space of A_0, that is

$$V_0 = V W_0, \qquad (2.70)$$

where $W_0 \in R^{m \times m_1}$ is some coefficient matrix, and where m_1 is a number to be determined with $\bar{m} \leq m_1 \leq m$. By Lemma 2 the equation (F1) is feasible if and only if

$$U^T A_1 V_0 = 0 \; .$$

Substituting the expression (2.70) for V_0, one gets that

$$U^T A_1 V W_0 = 0 \; .$$

This is the first equation of the reduced system (RF) with $B_0 = U^T A_1 V$. Note that m_1 above is the dimension of the null space of B_0. Next we consider the fundamental equations (F1). Their solution in the general form is

$$V_1 = V W_1 - A_0^{\dagger} A_1 V_0, \qquad (2.71)$$

where $W_1 \in R^{m \times m_1}$ is some coefficient matrix, which describes the general solution of the corresponding homogeneous system and where $-A_0^{\dagger} A_1 V_0$ is a particular solution of (F1). The coefficient matrices W_0 and W_1 have to be chosen so that they satisfy the feasibility condition for the next fundamental equation (F2)

$$U^T (A_1 V_1 + A_2 V_0) = 0.$$

Upon substitution of V_0 (see (2.70)) and V_1 (see (2.71)) into the above condition, one obtains

$$U^T A_1 V W_1 + U^T (A_2 - A_1 A_0^{\dagger} A_1) V W_0 = 0$$

which is the reduced fundamental equation (RF1) with $B_1 = U^T (A_2 - A_1 A_0^{\dagger} A_1) V$.

Note that the recursive formula (2.68) is just the general form of the solution of the fundamental equation (Fk). The reduced system of equations (RF) is the set of feasibility conditions for $W_k, k = 0, 1, \ldots$, which are obtained in a way similar to the above considerations. The general formula (2.69) for the coefficients can now be established by induction.

Next, we show that the new normalization conditions (RN) also hold. First, consider the normalization condition for W_0. Substituting the expression (2.70) into (N0), we get

$$(VW_0)^T VW_0 = I_{\bar{m}}$$

or

$$W_0^T V^T V W_0 = I_{\bar{m}} \quad .$$

Recall that we have chosen the basis V for the null space of A_0 such that $V^T V = I_m$. The latter implies that

$$W_0^T W_0 = I_{\bar{m}}.$$

Thus, we have obtained the normalization condition (RN0). Next we show that the normalization conditions (RNk), $k \geq 1$, hold as well. To this end, substitute the recursive expression (2.68) into (Nk) to obtain

$$V_0^T V W_k - V_0^T A_0^\dagger \left(\sum_{j=1}^{k} A_j V_{k-j}\right) = 0.$$

Note that since V_0 belongs to the null space of A_0 and since $N(A) = R(A^\dagger)^\perp$ [24], $V_0^T A_0^\dagger = 0$. Thus,

$$V_0^T V W_k = 0.$$

By substituting V_0 from (2.70) and taking into account that $V^T V = I_m$, we get that

$$W_0^T V^T V W_k = W_0^T W_k = 0,$$

which is the normalization condition (RNk). This completes the proof.

□

Remark 11 *Note that the computation of the coefficient matrices B_k, $k \geq 0$, by (2.69) will be tedious. Therefore, similarly to Section 2.2 we propose to compute these coefficients recursively. Specifically, define the series of matrices $\{S_k\}_{k=0}^{\infty}$ as follows:*

$$S_k = \sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1+\ldots+\nu_p=k+1} A_{\nu_1} A_0^\dagger A_{\nu_2} A_0^\dagger \ldots A_{\nu_p}, \quad k = 0, 1, \ldots$$

These auxiliary matrices can be computed by the following recursion

$$S_k = A_{k+1} - \sum_{i=1}^{k} A_i A_0^{\dagger} S_{k-i}, \quad k = 1, 2, ... \tag{2.72}$$

initialising with $S_0 = A_1$. *Then the coefficient matrices* B_k, $k \geq 0$, *are simply given by*

$$B_k = U^T S_k V.$$

We would like to point out that the reduced system of equations (RF) together with the normalization condition (RN) have exactly the same structure as the initial system of fundamental equations (F) with the normalization conditions (N). Thus, one has two options as how to proceed from here. The first is to solve it using the procedure described in the first subsection of 2.4.3, that is, to use the augmented matrix approach. The second is to apply the reduction technique again; now to the system composed of (RF) and (RN). If the latter option is pursued, then once again one may face the same alternative and so on. At first sight, it might seem that one may need to carry out an infinite number of reduction steps. However, as it turns out, termination is guaranteed after a finite number of steps. The next theorem addresses this issue.

Theorem 14 *Suppose that* $\{B_k^{(l)}\}_{k=0}^{\infty}, l = 1, 2, ...$ *are the coefficients of the reduced system obtained at the* l*-th reduction step* ($B_k^{(1)} = B_k$). *Also, let* m_l *be the dimension of the null space of* $B_0^{(l)}$. *Then, the reduction process terminates after a finite number of steps with* $m_l = \bar{m}$, *where* $\bar{m}$ *is the dimension of the null space of the perturbed matrices* $A(\varepsilon)$, $0 < |\varepsilon| < \varepsilon_{\max}$. *Furthermore, the final system of reduced equations (namely, the system of reduced fundamental equations derived at the last reduction step) can be solved by the iterative procedure which was proposed for the case of regular perturbations described in Section 2.4.2.*

PROOF: Note that after each reduction step the dimension of the null space of $B_0^{(l)}$ does not increase. Since we deal with a finite-dimensional problem and since the sequence m_l, $l \geq 1$, is of integers, we conclude that the sequence of m_l achieves its limit, say m_*, in a finite number of steps. Next we argue that this limit m_* equals $\bar{m}$ and once it is reached there is no need to make any further reductions. Note also that the solution to the final system of reduced equations (the reduction process terminates when the null space of $B_0^{(l)}$ has dimension m_*) can be obtained by the recursive algorithm proposed in Section 2.4.2.

The latter means that a basis for the null space of the perturbed matrix $A(\varepsilon)$ is constructed and this basis is holomorphic with the parameter ε. This basis is formed by m_* linearly independent vectors. However, according to our assumptions the dimension of the null space of $A(\varepsilon)$ is $\bar{m}$. This implies that the limit m_* equals $\bar{m}$.

□

Finally we would like to suggest a practical implementation of the above scheme. If one applies the reduction process as described above to calculate V_k, then one needs to compute $B_i, i = 0, ..., k + \bar{m}$, $B_i^{(2)}, i = 0, ..., k + \bar{m} - 1$ and so on. This could result in a large amount of calculations even when the recursive formula given in Remark 11 is used. Fortunately, it is possible to develop a recursion which uses only a finite number of precomputed $B_i^{(l)}$. Namely, suppose that we have already obtained $V_0, ..., V_k, k > r$, where r denotes the number of reduction steps needed to obtain the final system of reduced equations. Then we can rewrite the fundamental system as follows:

$$\begin{aligned} A_0 V_{k+1} &= -(A_1 V_k + \cdots + A_{k+1} V_0) \\ A_0 V_{k+2} + A_1 V_{k+1} &= -(A_2 V_k + \cdots + A_{k+2} V_0) \\ &\vdots \\ A_0 V_{k+r+1} + \cdots + A_r V_{k+1} &= -(A_{r+1} V_k + \cdots + A_{k+r+1} V_0) \end{aligned}$$

This system of equations can be effectively solved by the same reduction technique. Moreover, note that the auxiliary matrices such as $B_i^{(l)}$ can be stored and used afterwards to compute the next terms $V_{k+2}, V_{k+3}, \cdots$

Remark 12 *It is worth pointing out that in most applications the number of reduction steps r is small and hence we need to compute and to store only a small number of coefficients $B_i^{(l)}$.*

Remark 13 *If it is needed, the estimation of the convergence radius can be also obtained for the singular case. This can be done by applying recursively the arguments of Theorem 9.*

2.4.5 Perturbation of general eigenvalue problem

The results on the perturbation of null spaces can be immediately applied to the general perturbed eigenvalue problem [14, 27, 28, 87, 104, 113, 121, 145, 149]

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon). \tag{2.73}$$

Recall that the perturbed eigenvalue $\lambda(\varepsilon)$ satisfies the secular equation

$$det(A(\varepsilon) - \lambda(\varepsilon)I) = 0.$$

This is a polynomial equation

$$(-1)^n \lambda^n + a_{n-1}(\varepsilon)\lambda^{n-1} + ... + a_1(\varepsilon)\lambda + a_0(\varepsilon) = 0,$$

where the coefficients $a_i(\varepsilon)$ are analytic functions. Using the method of the Newton diagram [14, 113, 145], one can obtain the Puiseux expansion of the perturbed eigenvalue

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon^{1/p}\lambda_1 + \varepsilon^{2/p}\lambda_2 + ...$$

where p is a positive integer. Now let us introduce the new variable $\eta := \varepsilon^{1/p}$. Note that the perturbed eigenvalue depends analytically on this new variable. Consequently, the system of equations for the perturbed eigenvectors can be written in the form

$$[A(\eta^p) - \lambda(\eta)I]x(\eta) = 0.$$

Hence, we have reduced the general perturbed eigenvalue problem to the problem of analytic perturbation of the null space, which can be effectively solved by the methods described in Sections 2.4.2 and 2.4.3.

2.4.6 Perturbations of group inverse

Next we discuss the applications of the above analysis for the designing of several methods for calculating the Laurent series expansion (2.55) of the perturbed group inverse. We assume that the perturbed group inverse $A^g(\varepsilon)$ exists for $\varepsilon \in (0, \bar{\varepsilon})$. In the previous subsections we have developed the power series expansion for the eigenvectors corresponding to the zero eigenvalue of $A(\varepsilon)$. The matrix $V(\varepsilon)$ forms a basis for the null space of $A(\varepsilon)$, though it is not necessarily orthogonal. To proceed further we need to construct the orthogonal projection $P(\varepsilon)$ on the null space of $A(\varepsilon)$. There are (at least) two possible directions to pursue here. The first is based on the direct application of the formula:

$$P(\varepsilon) = V(\varepsilon)(V^T(\varepsilon)V(\varepsilon))^{-1}V^T(\varepsilon), \quad 0 < |\varepsilon| < \varepsilon_{\max}. \tag{2.74}$$

Since $\lim_{\varepsilon\to 0} V^T(\varepsilon)V(\varepsilon) = V_0^T V_0 = I_{\bar{m}}$, the matrix $(V^T(\varepsilon)V(\varepsilon))^{-1}$ is the "well-behaved" inverse of the perturbed identity matrix $I_{\bar{m}}$. In this case we can apply directly the Neumann series [87]. The expansion is of the form

$$(V^T(\varepsilon)V(\varepsilon))^{-1} = I - \varepsilon(V_0^T V_1 + V_1^T V_0) + ...$$

Multiplying the above expression by $V(\varepsilon)$ from the left and by $V^T(\varepsilon)$ from the right, we obtain the power series for the orthogonal projection $P(\varepsilon)$.

The second method is based on a Gram-Schmidt-like orthogonalization process. Firstly, we perform the Gram-Schmidt procedure (without normalization) over the analytic vector-valued functions $v_i(\varepsilon), i = 1, ..., \bar{m}$, which constitute the "quasi-orthogonal" basis $V(\varepsilon)$. Note that summation, multiplication and division operations which are used in the orthogonalization procedure can be carried out on power series as well as on numbers.

This results in an orthogonal basis for the perturbed null space. Each new basis element is a vector-valued function analytic in the punctured disc: $0 < |\varepsilon| < \varepsilon_{max}$. Next we show that the normalization procedure leads to a basis whose elements are analytic vector-valued functions at $\varepsilon = 0$. Indeed, consider a vector-valued function $v(\varepsilon)$ which is analytic in $0 < |\varepsilon| < \varepsilon_{max}$. It can be expanded as a Laurent series. And let $v_i(\varepsilon) = \varepsilon^m v_{im} + \varepsilon^{m+1} v_{im+1} + ...$ be the largest element (in absolute value and for sufficiently small ε) of the vector $v(\varepsilon)$. Then, clearly

$$||v(\varepsilon)|| = \sqrt{v_1^2(\varepsilon) + ... + v_2^2(\varepsilon)} = \varepsilon^m(\nu_0 + \varepsilon\nu_1 + ...), \quad \nu_0 \neq 0.$$

The latter implies that the normalized vector $v(\varepsilon)/||v(\varepsilon)||$ can be expanded as a series with positive powers of ε and with a non-zero leading coefficient. As a result of the above procedure we obtain an orthonormal basis, say $\bar{V}(\varepsilon)$. Then the orthogonal projection is given by

$$P(\varepsilon) = \bar{V}^T(\varepsilon)\bar{V}(\varepsilon).$$

Next we suggest some methods for finding the Laurent series expansion for the group inverse $A^g(\varepsilon)$ which are based on results given in Sections 2.2 and 2.3. In particular, we assume that the Laurent series expansion for the inverse of a singularly perturbed martix is available, for example by any method suggested in Sections 2.2 and 2.3. Moreover, the power series for $P(\varepsilon)$ is assumed to have been developed already.

The first method is based on a straightforward application of the inversion of singularly perturbed matrices. It utilizes the formula [40, 95]

$$A^g(\varepsilon) = [A(\varepsilon) + P(\varepsilon)]^{-1} - P(\varepsilon),$$

where the projection $P(\varepsilon)$ is given by (2.74).

The second method is based on the solution of the following system of equations [87]

$$A(\varepsilon)A^g(\varepsilon) = I - P(\varepsilon), \tag{2.75}$$

$$P(\varepsilon)A^g(\varepsilon) = 0, \tag{2.76}$$

where $P(\varepsilon)$ is the same projection as before. Note that equation (2.76) can be written in the equivalent form,

$$V^T(\varepsilon)A^g(\varepsilon) = 0, \tag{2.77}$$

which might be more useful for practical computations.

Finally, the third method is based on the construction of the matrix $C(\varepsilon) = [\tilde{A}(\varepsilon), V(\varepsilon)]$, where the matrix $\tilde{A}(\varepsilon) \in R^{n\times(n-m)}$ consists of columns of $A(\varepsilon)$ which span its range. The matrix $C(\varepsilon)$ is invertible for ε sufficiently small but distinct from zero, and the perturbed group inverse $A^g(\varepsilon)$ can be immediately retrieved by [44]

$$A^g(\varepsilon) = [I - P(\varepsilon)]C^{-1}(\varepsilon).$$

We would like to discuss in more detail the second method based on equations (2.75) and (2.76). This approach is a generalization of the results given in [8, 66, 67]. By substituting the power series (2.49),(2.55) and the power series for the projection $P(\varepsilon)$ into equations (2.75), (2.76) and equating terms with the same power of ε, we obtain two new systems of fundamental equations

$$\begin{aligned} A_0B_{-s} &= 0 && (G0)\\ A_0B_{-s+1} + A_1B_{-s} &= 0 && (G1)\\ &\vdots && \vdots\\ A_0B_0 + A_1B_{-1} + \cdots + A_{s-1}B_{-s+1} + A_sB_{-s} &= I - P_0 && (Gs)\\ A_0B_1 + A_1B_0 + \cdots + A_sB_{-s+1} + A_{s+1}B_{-s} &= -P_1 && (Gs+1)\\ &\vdots && \vdots \end{aligned}$$

and

$$\begin{aligned} P_0B_{-s} &= 0 && (H0)\\ P_0B_{-s+1} + P_1B_{-s} &= 0 && (H1)\\ &\vdots && \vdots \end{aligned}$$

The next result provides a way to solve recursively the above infinite systems of equations.

Theorem 15 *Equations* $(G0) - (Gs)$ *together with the equation* $(H0)$ *constitute the minimal set of fundamental equations that uniquely determine the first singular coefficient* B_{-s} *of the Laurent series* (2.55).

PROOF: By the same arguments as in Section 2.2.4 we conclude that it is necessary to take into account at least the first $s+1$ fundamental equations from the set (G) and the first equation from the set (H0).

Now we prove that this set of equations is sufficient to determine uniquely the first Laurent series coefficient B_{-s}. (The demonstration below is also similar to the one given in [67] and Section 2.2.4.) Suppose on the contrary that there exists another solution $C_{-s}, C_{-s+1}, ..., C_0 \in R^{n\times n}$ to (G0)-(Gs) and (H0) such that $C_{-s} \neq B_{-s}$. Consider the sequence $\{D_i\}_{i=-s}^{\infty}$ defined by $D_i = B_i - C_i$ for $-s \le i \le 0$ and $D_i = B_i$ for $i > 0$. Clearly, for sufficiently small ε we have by (2.75)

$$A(\varepsilon)[D(\varepsilon) + C(\varepsilon)] = I - P(\varepsilon)$$

and hence

$$A(\varepsilon)D(\varepsilon) = I - P(\varepsilon) - A(\varepsilon)C(\varepsilon).$$

Using fundamental equations (G0)-(Gs), we conclude that

$$A(\varepsilon)D(\varepsilon) = O(\varepsilon)$$

Now we multiply the above equation by $A^g(\varepsilon)$ from the left hand side and get that

$$[I - P(\varepsilon)]D(\varepsilon) = O(\frac{1}{\varepsilon^{s-1}})$$

Since $P_0B_0 - P_0C_0 = 0 - 0 = 0$ and consequently the left hand side of the above equation is of order $O(1/\varepsilon^s)$, we come to a contradiction.

□

The above theoretical result points to a numerical procedure. First, solve the set of equations (G0)-(Gs) and (H0) to find B_{-s}. Note that, since this is a linear system, a number of efficient methods is available for its solution. For example, one can use methods similar to those described in Section 2.2 and in Subsections 2.4.2 and 2.4.3. Then, use equations (G1)-(Gs+1) and (H1) to find B_{-s+1} and so on. We would like to emphasize that the system of equations (G1)-(Gs+1) and (H1) has exactly the same left hand sides as the system of equations (G0)-(Gs) and (H0). Thus, the auxiliary results which were obtained during the solution of the system (G0)-(Gs) and (H0) can be applied again. Moreover, one can use the quantities obtained from the reduction process of the fundamental system (F) in order to solve the system (G).

2.4.7 On the existence of a holomorphic basis for the perturbed null space

To make the present analysis of the perturbed null spaces self-contained, we include a proof that there exists a holomorphic family of eigenvectors which constitute a basis for the perturbed null space. The proof of the next theorem was obtained independently of the analogous results in [60, 61].

Theorem 16 *Let $A(\varepsilon)$ be an analytic perturbation of the matrix A_0 and let $A(\varepsilon)$ have a non-trivial null space for ε belonging to some small neighbourhood of zero. Then there exists a basis $V(\varepsilon)$ of this subspace, whose vectors are analytic functions in a small neighbourhood of the origin.*

Proof: If the perturbed matrix $A(\varepsilon)$ has a non-trivial null space, then it has $r := rank(A(\varepsilon))$ linearly independent rows and the other rows can be expressed as their linear combination. Without loss of generality, we can assume that these are the first r rows of the matrix $A(\varepsilon)$. We now have $n-r$ free variables. To construct $n-r$ linearly independent and analytic solutions, let us assign the value of the determinant of the upper left $r \times r$ block to these free variables. Then, according to the Cramer rule the linearly independent solutions of the system (2.52) can be written in the form

$$x_i^{(j)}(\varepsilon) = \begin{cases} -det(A_i^{(j)}(\varepsilon)), & i = 1, ..., r \\ det(A_{rr}(\varepsilon)), & i = j + r \\ 0, & \text{otherwise} \end{cases}$$

where $A_{rr}(\varepsilon)$ is the upper left $r \times r$ block of the matrix $A(\varepsilon)$ and $A_i^{(j)}(\varepsilon)$ is a matrix formed by inserting the upper r segment of the $(j+r)$-th column in place of the i-th column of the matrix $A_{rr}(\varepsilon)$.

Since $x_i^{(j)}(\varepsilon)$ are polynomials of matrix element $a_{ij}(\varepsilon)$, which are analytic, they are analytic functions as well. Moreover, since in some small neighbourhood of the origin $det(A_{rr}(\varepsilon)) \not\equiv 0$, the solutions $x^{(j)}, j = 1, .., n-r$ are linearly independent for ε sufficiently small but different from zero.

□

To illustrate the obscure notations of the above theorem let us again consider the second example of Subsection 2.4.2. In this case we have $n = 3$, $r = 2$ and

$$x_3^{(1)}(\varepsilon) = det(A_{22}(\varepsilon)) = \begin{vmatrix} \varepsilon & 1 \\ 0 & 1 \end{vmatrix} = \varepsilon,$$

$$x_1^{(1)}(\varepsilon) = -det(A_1^{(1)}(\varepsilon)) = -\begin{vmatrix} 0 & 1 \\ \varepsilon & 1 \end{vmatrix} = \varepsilon, \quad x_2^{(1)}(\varepsilon) = -det(A_2^{(1)}(\varepsilon)) = -\begin{vmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{vmatrix} = -\varepsilon^2.$$

Thus, $x^{(1)}(\varepsilon) = [\varepsilon \ -\varepsilon^2 \ \varepsilon]$. By normalizing this vector and expanding each its component as a power series in ε, we obtain (2.56).

2.5 Analytic perturbation of reduced resolvents and generalized inverses

2.5.1 Preliminaries

In this section we recall several useful facts from spectral theory and from the theory of analytic functions. A more detailed exposition can be found for example in [87] and [111].

Any (finite-dimensional) linear operator $A \in \mathbb{R}^{n\times n}$ possesses the following spectral representation

$$A = \sum_{i=0}^{p} (\lambda_i P_i + D_i), \tag{2.78}$$

where $p+1$ is the number of distinct eigenvalues of A, P_i is the eigenprojection and D_i is the nilpotent operator corresponding to the eigenvalue λ_i. By convention, λ_0 is the zero eigenvalue of A, that is, $\lambda_0 = 0$. In the case, where there is no zero eigenvalue, the eigenvalues are enumerated from $i = 1$. The *resolvent* is another very important operator in spectral theory.

Definition 5 *The following operator-valued function of the complex parameter ζ is called the resolvent of the operator $A \in \mathbb{R}^{n\times n}$.*

$$R(\zeta) = (A - \zeta I)^{-1}.$$

The resolvent satisfies the *resolvent identity*:

$$R(\zeta_1) - R(\zeta_2) = (\zeta_1 - \zeta_2) R(\zeta_1) R(\zeta_2), \tag{2.79}$$

for all $\zeta_1, \zeta_2 \in \mathbb{C}$. The resolvent has singularities at the points $\zeta = \lambda_k$, where λ_k are the eigenvalues of A. In a neighbourhood of each singular point λ_k the resolvent can be expanded as a Laurent series

$$R(\zeta) = -\sum_{n=1}^{m_k-1} \frac{1}{(\zeta-\lambda_k)^{n+1}} D_k^n - \frac{1}{\zeta - \lambda_k} P_k + \sum_{n=0}^{\infty} (\zeta - \lambda_k)^n S_k^{n+1}, \tag{2.80}$$

where S_k is the *reduced resolvent* corresponding to the eigenvalue λ_k with geometric multiplicity m_k.

In particular, the reduced resolvent S_0 corresponding to the zero eigenvalue is denoted by $A^{\#}$ and we simply refer to it as the *reduced resolvent*. Note that the reduced resolvent is also known in linear algebra as the *Drazin generalized inverse*. The reduced resolvent has the following basic properties [87]

$$AA^{\#} = I - P_0, \tag{2.81}$$

$$P_0 A^{\#} = 0. \tag{2.82}$$

This characterization shows that $A^{\#}$ is the "inverse" of A in the complementary subspace to the generalized null space of A. Here by *generalized* null space we mean a subspace which is spanned by all eigenvectors and generalized (Jordan) eigenvectors corresponding to the zero eigenvalue. Note that P_0 is a projection onto this generalized null space.

Moreover, if the underlying space admits a decomposition into the *direct* sum of the null space and the range of the operator A (recall that this is a necessary and sufficient condition for the existence of the group inverse [24]), then the reduced resolvent and the group inverse coincide, and the following Laurent expansion holds

$$R(\zeta) = -\frac{1}{\zeta} P_0 + \sum_{n=0}^{\infty} \zeta^n (A^{\#})^{n+1}. \tag{2.83}$$

Since the reduced resolvent is the constant term in the Laurent series (2.80) and (2.83), it can be calculated via the following Cauchy integral formula [87, 111]

$$A^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta)\, d\zeta, \tag{2.84}$$

where Γ_0 is a closed positively-oriented contour in the complex plane, enclosing 0 but no other eigenvalue of A. The above formula will play a crucial role in the sequel.

The reduced resolvent also has a simple expression in terms of eigenprojections, eigenvalues and nilpotent operators of the original operator A. Namely,

$$A^{\#} = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i} P_i + \sum_{j=1}^{m_i - 1} (-1)^j \frac{1}{\lambda_i^{j+1}} D_i^j \right\}. \tag{2.85}$$

The above expression again demonstrates that the reduced resolvent is the inverse operator in the complementary subspace to the generalized null space. Moreover, this expression exactly represents the inverse operator A^{-1} whenever A has no zero eigenvalue.

2.5.2 Existence of a Laurent series expansion

Consider a finite-dimensional linear operator $A_0 \in \mathbb{R}^{m \times n}$ and the following analytic perturbation $A(z)$ of A_0, given by:

$$A(z) = A_0 + zA_1 + z^2 A_2 + \dots\ , \tag{2.86}$$

where $|z| < z_{max}$ is a small complex perturbation parameter. If we analyse the reduced resolvent or the group inverse of the perturbed operator, we always assume that $m = n$. In this section we prove the existence of a Laurent series expansion for the reduced resolvent $A^{\#}(z)$ of the analytically perturbed operator $A(z)$. The latter will imply the existence of a Laurent series expansion for both the group inverse and the Moore-Penrose generalized inverse.

First consider the resolvent $R(\zeta, z) := (A(z) - \zeta I)^{-1}$ of the perturbed operator $A(z)$. One can expand $R(\zeta, z)$ in a power series with respect to the complex variable z near $z = z_0$, as follows (see e.g. [87])

$$R(\zeta, z) \;=\; R(\zeta, z_0) + \sum_{n=1}^{\infty} (z - z_0)^n R^{(n)}(\zeta, z_0), \tag{2.87}$$

where

$$R^{(n)}(\zeta, z_0) := \sum_{\nu_1+\ldots+\nu_p=n} (-1)^p R(\zeta, z_0) A_{\nu_1} R(\zeta, z_0) A_{\nu_2} \cdots R(\zeta, z_0) A_{\nu_p} R(\zeta, z_0),$$

where A_{ν_k} are the coefficients of $A(z)$ and $\nu_k \geq 1$. The above expansion is called the second Neumann series for the resolvent. It is *uniformly* convergent for z sufficiently close to z_0 and $\zeta \in D$, where D is a compact subset of the complex plane which does not contain the eigenvalues of $A(z_0)$ [87].

Theorem 17 *Let $A(z)$ be the analytic perturbation of the matrix A_0 given by (2.86). Then, the reduced resolvent $A^{\#}(z)$ of the perturbed operator $A(z)$ can be expanded as a Laurent series*

$$A^{\#}(z) = \sum_{n=-s}^{+\infty} z^n H_n \tag{2.88}$$

for $0 < |z| < z_{max}$.

PROOF: We first show that there exists a domain $0 < |z| < z_{max}$ such that $A^{\#}(z)$ can be expanded in Taylor series at any point z_0 in this domain. For a fixed, arbitrary $z > 0$, (2.84) becomes

$$A^{\#}(z) = \frac{1}{2\pi i} \int_{\Gamma_0(z)} \frac{1}{\zeta} R(\zeta, z) d\zeta, \tag{2.89}$$

where $\Gamma_0(z)$ is a closed positively-oriented curve enclosing the origin but no other eigenvalue of $A(z)$.

With z_{max} less than the modulus of any non-zero eigenvalue of A_0, expand the perturbed resolvent in the power series (2.87), around the point z_0 (with $0 < |z_0| < z_{max}$). Then, substituting that series in the integral formula (2.89), yields

$$A^{\#}(z) = \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} \left[R(\zeta, z_0) + \sum_{n=1}^{\infty} (z - z_0)^n R^{(n)}(\zeta, z_0) \right] d\zeta.$$

Since the power series for $R(\zeta, z)$ is uniformly convergent for z sufficiently close to z_0, we can integrate the above series term by term,

$$\begin{aligned} A^{\#}(z) &= \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} R(\zeta, z_0)\, d\zeta + \sum_{n=1}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} [R^{(n)}(\zeta, z_0)]\, d\zeta \\ &= A^{\#}(z_0) + \sum_{n=1}^{\infty} (z - z_0)^n H_n(z_0), \end{aligned} \tag{2.90}$$

where the coefficients are defined by

$$H_n(z_0) := \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} [R^{(n)}(\zeta, z_0)]\, d\zeta.$$

The convergence of power series (2.90) in some non-empty domain $0 < |z| < z_{max}$ can be shown by using the bounds for the contour integrals (for very similar development see for example [87, Ch.2, Sec.3]). From the power series (2.90), we can see that $A^{\#}(z)$ is holomorphic in the domain $0 < |z| < z_{max}$. The formal convergence arguments Consequently, by Laurent's theorem [111, v. II, p. 7], we conclude that $A^{\#}(z)$ possesses a Laurent series expansion at $z = 0$ (with radius of convergence z_{max}), i.e.,

$$A^{\#}(z) = \sum_{n=-\infty}^{+\infty} z^n H_n. \tag{2.91}$$

We next show that the pole at $z = 0$ can be at most of finite order. Consider the spectral representation (2.85) for the reduced resolvent of the perturbed operator $A(z)$

$$A^{\#}(z) = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i(z)} P_i(z) + \sum_{j=1}^{m_i-1} (-1)^j \frac{1}{\lambda_i^{j+1}(z)} D_i^j(z) \right\}.$$

From Kato [87], we know that the perturbed eigenvalues $\lambda_i(z)$ are bounded in $|z| \le z_{max}$ and they have at most algebraic singularities. Furthermore, the eigenprojections $P_i(z)$ and nilpotents $D_i(z)$ can also have only algebraic singularities and poles of finite order. Therefore, none of the functions $\lambda_i(z)$, $P_i(z)$ and $D_i(z)$ can have an essential singularity. This latter fact implies that their finite sums, products or divisions as in $A^{\#}(z)$, do not have an essential singularity as well and, consequently, the order of pole in (2.91) is finite. This completes the proof.

□

Note that the Laurent series for the perturbed group inverse does not always exist. Indeed, the existence of the group inverse of the unperturbed operator does not imply the existence of the group inverse of the perturbed operator. However, existence results for the Laurent series for the group inverse $A^g(z)$ and the Moore-Penrose generalized inverse $A^{\dagger}(z)$ are direct consequences of the above theorem. The following is a general sufficient condition for the existence of the Laurent series for the perturbed group inverse.

Corollary 1 *Let the group inverse $A^g(z)$ of the analytically perturbed operator $A(z)$ exist in some non-empty (possibly punctured) neighbourhood of $z = 0$. Then the group inverse $A^g(z)$ can be expanded as a Laurent series around $z = 0$ with a non-zero radius of convergence.*

PROOF: Whenever the group inverse exists, it coincides with the reduced resolvent. Therefore, the existence of a Laurent series follows from Theorem 17.

□

By contrast, the Laurent series for the Moore-Penrose generalized inverse $A^{\dagger}(z)$ always exists.

Corollary 2 *Let $A^{\dagger}(z)$ be the Moore-Penrose generalized inverse of the analytically perturbed matrix $A(z)$. Then, $A^{\dagger}(z)$ possesses a Laurent series expansion*

$$A^{\dagger}(z) = \frac{1}{z^s}B_{-s} + ... + \frac{1}{z}B_{-1} + B_0 + zB_1 + ... \tag{2.92}$$

in some non-empty region around the origin.

PROOF: Rewriting (2.19) for the perturbed operator $A(z)$ yields

$$A^{\dagger}(z) = (A^T(z)A(z))^g A^T(z).$$

Note that the group inverse of a symmetric matrix always exists. Since the group inverse is a particular case of the reduced resolvent and the product or sum of two analytic functions is an analytic function again, we can write

$$A^{\dagger}(z) = \mathcal{A}^{\#}(z)A^T(z), \tag{2.93}$$

where $\mathcal{A}(z) = \sum_{k=0}^{\infty} z^k \mathcal{A}_k$ with $\mathcal{A}_k = \sum_{i=0}^{k} A_i^T A_{k-i}$. Note that $\mathcal{A}(z)$ is an analytically perturbed operator. According to Theorem 17, $\mathcal{A}^{\#}(z)$ has a Laurent series expansion, and from (2.93), we may conclude that the Laurent series for the perturbed pseudo-inverse $A^{\dagger}(z)$ exists as well.

□

Remark 14 *We would like to note that the statement of Corollary 2 can be considered as a particular case of more general existence results in [10, 13]. However, the reduced resolvent does not belong to the family of relative inverses and hence one cannot apply (at least, not immediately) the results of [10, 13] to the proof of Theorem 17.*

Remark 15 *We would like to emphasize that according to* (2.93), *computing the Laurent series of the perturbed pseudo-inverse $A^{\dagger}(z)$ reduces to computing the Laurent series of a reduced resolvent. Moreover, $\mathcal{A}(z)$ is a symmetric perturbation, that is, $\mathcal{A}_k^T = \mathcal{A}_k, k = 0, 1, ...$ This guarantees that the reduction process defined in Section 2.5.4 is indeed applicable in this case.*

2.5.3 Recursive formulae for the regular part coefficients

In this subsection, we derive recursive formulae for the coefficients of the regular part of the Laurent series (2.88). We use the same analytic technique as in [87, 91], based on Cauchy contour integrals and resolvent-like identities.

First, observe that the structure of the reduced resolvent $(A_0 + zA_1 + z^2A_2 + ...)^{\#}$ is similar to the structure of the classical resolvent $(A_0 - zI)^{-1}$. Moreover, $A^{\#}(z)$ becomes precisely the resolvent if $A_1 = -I$ and $A_k = 0$ for $k \geq 2$. Therefore, one can expect that these two mathematical objects have some similar features. It turns out that the reduced resolvent of an analytically perturbed operator $A(z)$ satisfies an identity similar to the resolvent identity (2.79).

Lemma 7 *The reduced resolvent $A^{\#}(z)$ of the analytically perturbed operator $A(z) = \sum_{k=0}^{\infty} z^k A_k$ satisfies the resolvent-like identity:*

$$A^{\#}(z_1) - A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k - z_1^k)A^{\#}(z_1)A_k A^{\#}(z_2) + A^{\#}(z_1)P_0(z_2) - P_0(z_1)A^{\#}(z_2), \quad (2.94)$$

where $P_0(z)$ is the eigenprojection matrix corresponding to the zero eigenvalue.

PROOF: Consider the following expression

$$A(z_2) - A(z_1) = \sum_{k=1}^{\infty}(z_2^k - z_1^k)A_k.$$

Premultiplying by $A^{\#}(z_1)$ and postmultiplying by $A^{\#}(z_2)$, yields

$$A^{\#}(z_1)A(z_2)A^{\#}(z_2) - A^{\#}(z_1)A(z_1)A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k - z_1^k)A^{\#}(z_1)A_k A^{\#}(z_2).$$

Then, using (2.81), we get

$$A^{\#}(z_1)[I - P_0(z_2)] - [I - P_0(z_1)]A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k - z_1^k)A^{\#}(z_1)A_k A^{\#}(z_2).$$

Equivalently,

$$A^{\#}(z_1) - A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k - z_1^k)A^{\#}(z_1)A_k A^{\#}(z_2) + A^{\#}(z_1)P_0(z_2) - P_0(z_1)A^{\#}(z_2),$$

which is the desired identity (2.94).

□

In the next theorem, we obtain a general relation between the coefficients of the Laurent series (2.88).

Theorem 18 *Let $H_k, k = -s, -s+1, \ldots$ be the coefficients of the Laurent series* (2.88) *and $P_0(z) = \sum_{k=0}^{\infty} z^k P_{0k}$ be a power series for the eigenprojection corresponding to the zero eigenvalue of the perturbed operator. Then the coefficients $H_k, k = -s, -s+1, \ldots$ satisfy the following relation*

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = -(\eta_n + \eta_m - 1)H_{n+m+1}$$

$$-\begin{cases} 0, & m<0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1}+z_1P_{0m+2}+...]dz_1, & m\ge 0 \end{cases}$$

$$-\begin{cases} 0, & n<0 \\ \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+...]A^{\#}(z_2)dz_2, & n\ge 0, \end{cases} \tag{2.95}$$

where

$$\eta_m := \begin{cases} 1, & m\ge 0, \\ 0, & m<0. \end{cases}$$

For the sake of clarity of presentation, the detailed proof is postponed to the auxiliary Section 2.5.7. Now the recursive formula for the coefficients of the regular part of the Laurent series (2.88) becomes a corollary of the above general result.

Corollary 3 *Suppose that the coefficients $H_k, k=-s,...,-1,0$ and $P_{0k}, k=0,1,...$ are given. Then, the coefficients of the regular part of the Laurent expansion (2.88) can be computed by the following recursive formula:*

$$\begin{aligned} H_{m+1} &= -\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s} H_{-j}A_{i+j+1}\right)H_{m-i} - \sum_{i=1}^{m} P_{0m+1-i}H_i \\ &\quad -(P_{0m+1}H_0+...+P_{0m+1+s}H_{-s}) - (H_{-s}P_{0m+1+s}+...+H_0P_{0m+1}) \end{aligned} \tag{2.96}$$

for $m=0,1,\ldots$.

Proof: Let us take $n=0$, $m>0$ and then simplify the last two terms in (2.95), so that

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1}+z_1P_{0m+2}+...]dz_1 \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{1}{z_1}[\frac{1}{z_1^s}H_{-s}+...+\frac{1}{z_1}H_{-1}+H_0+...][P_{0m+1}+z_1P_{0m+2}+...]dz_1 \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{1}{z_1}\frac{1}{z_1^{m+1}}\frac{1}{z_1^s}[H_{-s}P_{0m+1+s}+...+H_0P_{0m+1}]dz_1 \\ &= H_{-s}P_{0m+1+s}+...+H_0P_{0m+1} \end{aligned} \tag{2.97}$$

The last term can be dealt with in a similar fashion.

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+...]A^{\#}(z_2)dz_2 \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{01}+z_2P_{02}+...]A^{\#}(z_2)dz_2 \end{aligned}$$

$$= \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{z_2} \frac{1}{Z_2^m} [P_{01} + z_2 P_{02} + ...][\frac{1}{z_1^s} H_{-s} + ... + \frac{1}{z_1} H_{-1} + H_0 + ...] dz_2$$

$$= \frac{1}{2\pi i} \int_{\Gamma_2} \frac{1}{z_2} [P_{01} H_m + P_{02} H_{m-1} + ... + P_{0m} H_1 + P_{0m+1} H_0 + ... + P_{0m+1+s} H_{-s}] dz_2$$

$$= \sum_{i=1}^{m} P_{0m+1-i} H_i + (H_{-s} P_{0m+1+s} + ... + H_0 P_{0m+1}) \tag{2.98}$$

Substituting (2.97) and (2.98) into (2.95) with $n = 0$ and $m > 0$, we obtain

$$\sum_{k=1}^{\infty} \sum_{i=0}^{k-1} H_{n-i} A_k H_{m+i-k+1} = -H_{m+1} - (H_{-s} P_{0m+1+s} + ... + H_0 P_{0m+1})$$

$$-(P_{0m+1} H_0 + ... + P_{0m+1+s} H_{-s}) - \sum_{i=1}^{m} P_{0m+1-i} H_i.$$

Rearranging terms in the above expression, we get

$$\begin{aligned} H_{m+1} \quad = \quad & - \sum_{i=0}^{m+s} \left(\sum_{j=0}^{s} H_{-j} A_{i+j+1} \right) H_{m-i} - \sum_{i=1}^{m} P_{0m+1-i} H_i \\ & -(P_{0m+1} H_0 + ... + P_{0m+1+s} H_{-s}) - (H_{-s} P_{0m+1+s} + ... + H_0 P_{0m+1}) \end{aligned}$$

which is the recursive formula (2.96).

□

Remark 16 *If the perturbed operator $A(z)$ is invertible for $0 < |z| < z_{max}$, then the inverse $A^{-1}(z)$ can be expanded as a Laurent series*

$$A^{-1}(z) = \frac{1}{z^{-s}} H_{-s} + ... + \frac{1}{z} H_{-1} + H_0 + z H_1 + ...$$

and the formula (2.96) becomes

$$H_{m+1} = - \sum_{i=0}^{m+s} \left(\sum_{j=0}^{s} H_{-j} A_{i+j+1} \right) H_{m-i}, \quad m = 0, 1, ...$$

which is the formula given by Korolyuk and Turbin [91]. Furthermore, if the perturbed operator is invertible and the perturbation is linear $A(z) = A_0 + z A_1$ we retrieve the recurrent formula obtained by Langenhop [98], and Schweitzer and Stewart [133, 135]

$$H_{m+1} = (-H_0 A_1) H_m, \quad m = 0, 1, ...$$

Note that, using (2.96) together with (2.93), one can easily derive the recursive formulae for the regular part of the Laurent series expansion (2.92) of the Moore-Penrose generalized inverse.

2.5.4 Reduction process

We have seen that the regular terms in the Laurent series expansion (2.88) of $A^{\#}(z)$ can be computed recursively by (2.96). However, to apply (2.96), one first needs to compute the terms $H_{-s}, \ldots, H_{-1}, H_0$, that is, the terms of the singular part. One way to achieve this was discussed in Section 2.4.5. That approach is based on the recursive solution of the fundamental equations, which are obtained by substitution of the Laurent series (2.88) into (2.81-2.82) and the identification of terms with the same powers of z. One may show (see Section 2.4.5) that a *finite* subsytem of the fundamental equations provides a *unique* solution for $H_{-s}, \ldots, H_{-1}, H_0$. This subsystem can be solved by an algebraic reduction process similar to the one defined in Section 2.2. However, here we would like to pursue an alternative approach based on spectral theory and analytic function theory. In particular, we define a *group reduced resolvent* which permits us to treat *singular* and *regular* perturbations in a unified framework.

Regular case

In this subsection we apply analytic function techniques to express the power series for the reduced resolvent of the perturbed operator in the case of *regular* perturbations, that is, when the dimension of the null space of the operator does not change with perturbations. In other words, when there is no splitting of the zero eigenvalue of the perturbed operator at $z = 0$. The latter implies that the expansion (2.90) is valid in some neighbourhood of $z_0 = 0$, and any contour $\Gamma_0 := \Gamma_0(0)$ chosen so that it does not enclose eigenvalues other than zero. Namely, the expansion (2.90) takes the form

$$A^{\#}(z) = A_0^{\#} + \sum_{n=1}^{\infty} z^n A_n^{\#}, \tag{2.99}$$

where

$$A_0^{\#} = A^{\#}(0), \quad A_n^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} [R^{(n)}(\zeta)]\, d\zeta$$

and

$$R^{(n)}(\zeta) = \sum_{\nu_1+\ldots+\nu_p=n} (-1)^p R(\zeta) A_{\nu_1} R(\zeta) A_{\nu_2} \cdots R(\zeta) A_{\nu_p} R(\zeta).$$

It turns out that it is possible to express the coefficients $A_n^{\#}$ in terms of

- the unperturbed reduced resolvent $A^{\#}(0)$,
- the eigenprojection P_0 corresponding to the zero eigenvalue of A_0,
- the perturbation matrices $A_n, n = 1, 2, \ldots$.

The next theorem gives the precise statement.

Theorem 19 *Suppose that the operator A_0 is perturbed analytically as in (2.86) and assume that the zero eigenvalue of A_0 is semi-simple and the perturbation is regular. Then, the matrices $A_n^{\#}, n = 1, 2, \ldots$ in the expansion (2.99) are given by the following formula*

$$A_n^{\#} = \sum_{p=1}^{n} (-1)^p \sum_{\substack{\nu_1 + \ldots + \nu_p = n \\ \mu_1 + \ldots + \mu_{p+1} = p+1 \\ \nu_j \geq 1, \mu_j \geq 0}} S_{\mu_1} A_{\nu_1} S_{\mu_2} \ldots A_{\nu_p} S_{\mu_{p+1}}, \tag{2.100}$$

where $S_0 := -P_0$ and $S_k := (A_0^{\#})^k, k = 1, 2, \ldots.$

Proof: Since Γ_0 encloses only the zero eigenvalue, we have by (2.99)

$$A_n^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R^{(n)}(\zeta) d\zeta = \sum_{\nu_1 + \ldots \nu_p = n} (-1)^p \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta) A_{\nu_1} R(\zeta) \ldots A_{\nu_p} R(\zeta) d\zeta.$$

$$= \sum_{\nu_1 + \ldots \nu_p = n} (-1)^p \mathrm{Res}_{\zeta=0} \{ \frac{1}{\zeta} R(\zeta) A_{\nu_1} R(\zeta) \ldots A_{\nu_p} R(\zeta) \}$$

In order to compute the above residue, we replace $R(\zeta)$ by its Laurent series (2.83) in the expression

$$\frac{1}{\zeta} R(\zeta) A_{\nu_1} R(\zeta) \ldots A_{\nu_p} R(\zeta),$$

and collect the terms with $1/\zeta$, that is, the terms

$$\sum_{\sigma_1 + \ldots + \sigma_{p+1} = 0} S_{\sigma_1 + 1} A_{\nu_1} S_{\sigma_2 + 1} \ldots A_{\nu_p} S_{\sigma_p + 1}.$$

Next, we change indicies $\mu_k := \sigma_k + 1, k = 1, \ldots, p + 1$, and rewrite the above sum as:

$$\sum_{\mu_1 + \ldots + \mu_{p+1} = p+1} S_{\mu_1} A_{\nu_1} S_{\mu_2} \ldots A_{\nu_p} S_{\mu_p},$$

which yields (2.100).

□

Singular case

We now show that by using a *reduction process*, we can transform the original *singular* problem into a *regular* one. We would like to emphasize that the reduction process of this section is different from the algebraic reduction technique proposed in Sections 2.2 and

2.4. This reduction process can be viewed as complimentary to the existing reduction process based on spectral theory which was applied by many authors [87, 14, 40, 29] to the eigenvalue problem. Moreover, to the best of our knowledge, applying the reduction technique to analytical perturbations of generalized inverses is new.

To develop the reduction technique in the context of generalized inverses, we will need to introduce a new notion of *group reduced resolvent*. A definition based on spectral representation is as follows:

Definition 6 *Let $A : R^n \to R^n$ be a linear operator with the spectral representation (2.78). Then, the* group reduced resolvent $A^{\#\Lambda}$ *relative to the group of eigenvalues $\Lambda := \{\lambda_i\}_{i=0}^k$ is defined as follows:*

$$A^{\#\Lambda} \stackrel{def}{=} \sum_{i=k+1}^{p} \left\{ \frac{1}{\lambda_i} P_i + \sum_{j=1}^{m_i-1} (-1)^j \frac{1}{\lambda_i^{j+1}} D_i^j \right\}$$

Remark 17 *Note that the reduced resolvent (see formula (2.85)) is a particular case of the group reduced resolvent. In this case, the group of eigenvalues consists only of the zero eigenvalue. In addition, we would like to mention the book of Chatelin [28], where a different group reduced resolvent is defined algebraically via a Sylvester equation.*

From our definition, the properties of a group reduced resolvent follow easily. In particular, in the next theorem, we will obtain an alternative analytic expression of the group reduced resolvent that will play a crucial role in perturbation analysis.

Theorem 20 *Let A be a linear operator with representation (2.78). Then, the group reduced resolvent relative to the eigenvalues $\Lambda = \{\lambda_i\}_{i=0}^k$ is given by*

$$A^{\#\Lambda} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta} (A - \zeta I)^{-1} d\zeta, \tag{2.101}$$

where Γ is a contour in the complex plane which encloses the set of eigenvalues $\{\lambda_i\}_{i=0}^k$, but none of the other eigenvalues $\{\lambda_i\}_{i=k+1}^p$.

Proof: It is a well known fact [87] that the resolvent can be represented by

$$-\sum_{i=0}^{p} \left[\frac{1}{\zeta - \lambda_i} P_i + \sum_{j=1}^{m_i} \frac{1}{(\zeta - \lambda_i)^{j+1}} D_i^j \right].$$

Substituting the above expression into the integral of (2.101) yields

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta} (A - \zeta I)^{-1} d\zeta \;=\; -\frac{1}{2\pi i} \int_\Gamma \sum_{i=0}^{p} \left[\frac{1}{\zeta(\zeta - \lambda_i)} P_i + \sum_{j=1}^{m_i} \frac{1}{\zeta(\zeta - \lambda_i)^{j+1}} D_i^j \right] d\zeta.$$

Using the fact that for every positive integer l

$$\mathrm{Res}_{\zeta=0}\frac{1}{\zeta(\zeta-\lambda)^l}=\frac{1}{(-\lambda)^l} \quad \text{and} \quad \mathrm{Res}_{\zeta=\lambda}\frac{1}{\zeta(\zeta-\lambda)^l}=-\frac{1}{(-\lambda)^l},$$

we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta &= \sum_{i=0}^{k}\mathrm{Res}_{\zeta=\lambda_i}\left\{-\sum_{i=0}^{p}\left[\frac{1}{\zeta(\zeta-\lambda_i)}P_i+\sum_{j=1}^{m_i}\frac{1}{\zeta(\zeta-\lambda_i)^{j+1}}D_i^j\right]\right\} \\ &= \sum_{i=k+1}^{p}\left\{\frac{1}{\lambda_i}P_i+\sum_{j=1}^{m_i-1}(-1)^j\frac{1}{\lambda_i^{j+1}}D_i^j\right\}.\end{aligned}$$

According to Definition 6, the latter expression is equal to the group reduced resolvent, so that the proof is complete.

□

Lemma 8 *Let* $P=\sum_{i=0}^k P_i$ *be the projection corresponding to the group of eigenvalues* $\Lambda=\{\lambda_i\}_{i=0}^k$, *then*

$$A^{\#\Lambda}=(A[I-P])^{\#}.$$

PROOF: Since $P_iP_j=\delta_{ij}P_i$ and $D_iP_j=\delta_{ij}D_i$, we have $P_i[I-P]=0$, $D_i[I-P]=0$ for $i=0,...,k$ and $P_i[I-P]=P_i$, $D_i[I-P]=D_i$ for $i=k+1,...,p$. Then, by (2.78) and above

$$(A[I-P])^{\#}=\left(\sum_{i=0}^{p}(\lambda_iP_i+D_i)[I-P]\right)^{\#}=\left(\sum_{i=k+1}^{p}(\lambda_iP_i+D_i)\right)^{\#}.$$

Using formula (2.85), we obtain

$$(A[I-P])^{\#}=\sum_{i=k+1}^{p}\left\{\frac{1}{\lambda_i}P_i+\sum_{j=1}^{m_i-1}(-1)^j\frac{1}{\lambda_i^{j+1}}D_i^j\right\}.$$

The latter is equal to the group reduced resolvent $A^{\#\Lambda}$ by Definition 6.

□

Now equipped with this new notion of *group reduced resolvent*, we go back to perturbation analysis. The group of the perturbed eigenvalues $\lambda_i(z)$ such that $\lambda_i(z)\to 0$ as $z\to 0$ is called the 0-group (see e.g. Kato [87]). We denote the 0-group of eigenvalues by Ω. The eigenvalues of the 0-group split from zero when the perturbation parameter differs from zero. Since the eigenvalues of the perturbed operator are analytic functions except at some points with algebraic singularities [87], each eigenvalue of the 0-group (other than 0) can be written as

$$\lambda_i(z)=z^\nu\lambda_{i\nu}+o(z^\nu), \tag{2.102}$$

with $\lambda_{ik} \neq 0$ and ν is a positive rational number. The reduction technique is essentially based on the semi-simplicity assumption of reduced operators [87], which will be introduced below. Under that assumption, the power ν in (2.102) must be an integer. The latter implies that we can partition the 0-group into subsets that we call z^l-groups. Namely, we say that the eigenvalue $\lambda_i(z)$ belongs to the z^l-group if $\lambda_i(z) = z^l \lambda_{il} + o(z^l)$, with $\lambda_{il} \neq 0$. We denote z^l-group by Λ_l.

Consider now the spectral representation of the perturbed reduced resolvent.

$$A^{\#}(z) = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i(z)} P_i(z) + \sum_{j=1}^{m_i-1} (-1)^j \frac{1}{\lambda_i^{j+1}(z)} D_i^j(z) \right\},$$

where $\{\lambda_i(z)\}_{i=1}^k$ is the 0-group. From the above formula one can see that in this case, the Laurent expansion for the reduced resolvent $A^{\#}(z)$ will possess terms with negative powers of z. Moreover, it turns out that under our assumptions, the z^k-group eigenvalues contribute to the terms of the Laurent expansion for $A^{\#}(z)$ with negative powers $-k, -k+1, ..., -1$ as well as to the regular part of the Laurent expansion.

The basic idea is to first treat the part of the perturbed operator corresponding to the eigenvalues which do not tend to zero as $z \to 0$. Then we treat subsequently the parts of the perturbed operator corresponding to the eigenvalues which belong to the z-group, z^2-group and so on.

What is helpful is that to treat the part of $A(z)$ corresponding to z^{k+1}-group we have to perform the same algorithm as for the part of the perturbed operator corresponding to the z^k-group. These steps constitute the (finite) reduction process.

Now we implement the general idea we have just briefly outlined above. Consider a fixed contour Γ_0 that encloses only the zero eigenvalue of the unperturbed operator A_0. Note that by continuity of eigenvalues the 0-group of eigenvalues of the perturbed operator $A(z)$ lies inside Γ_0 for z sufficiently small. Therefore, we may define the group reduced resolvent relative to the 0-group of eigenvalues as follows:

$$A^{\#\Omega}(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta, z) d\zeta \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} (A(z) - \zeta I)^{-1} d\zeta.$$

Since $A^{\#\Omega}(z)$ is an analytic function in some neighbourhood of the origin, it can be expanded as a power series

$$A^{\#\Omega}(z) \;=\; A_0^{\#\Omega} + \sum_{i=1}^{\infty} z^i A_i^{\#\Omega}. \tag{2.103}$$

Note that $A_0^{\#\Omega} = (A_0)^{\#}$ and from Theorem 19 it follows that the other coefficients $A_i^{\#\Omega}, i = 1, 2, ...$ can be calculated by the formula (2.100). We would like to emphasize that in general the group reduced resolvent $A^{\#\Omega}(z)$ is different from the reduced resolvent

$A^{\#}(z)$. The example of Subsection 2.5.5 demonstrates this phenomenon. However, we note that $A^{\#\Omega}(z)$ does coincide with $A^{\#}(z)$ in the case of regular perturbations.

Another operator that is used extensively in the reduction process is the *group projection*

$$P(z) \;=\; \frac{1}{2\pi i}\int_{\Gamma_0} R(\zeta,z)\,d\zeta,$$

which describes the subspace corresponding to the eigenvalues which split from zero. The group projection is an analytic function in some small neighbourhood of the origin (see e.g. [87]).

Next, as in the classical reduction process [87, 14], we define the restriction $B(z)$ of the operator $A(z)$ to the subspace determined by the group projection $P(z)$, that is,

$$B(z) \;:=\; \frac{1}{z}A(z)P(z) \;=\; \frac{1}{2\pi i z}\int_{\Gamma_0} \zeta R(\zeta,z)\,d\zeta,$$

where Γ_0 is some fixed contour enclosing only the zero eigenvalue of the unperturbed operator A_0. For the operator $B(z)$ to be analytic at zero, we need the following assumption.

Assumption S1 *The zero eigenvalue of the operator A_0 is semi-simple, that is the nilpotent operator D_0 corresponding to $\lambda_0 = 0$ is equal to zero.*

Note that this assumption is not too restrictive. For example, in the case of a self-adjoint perturbation operator, the zero eigenvalue of A_0 is semi-simple. This is in particular the case when one studies the generalized inverse of an analytically perturbed operator since it reduces to study a symmetric perturbation of the reduced resolvent (see Remark 15). Whenever Assumption S1 is satisfied, the operator $B(z)$ can be expressed as a power series [87]

$$B(z) \;=\; B_0 + \sum_{i=1}^{\infty} z^i B_i,$$

with $B_0 = P(0)A_1P(0)$, and

$$B_n \;=\; -\sum_{p=1}^{n+1}(-1)^p \sum_{\substack{\nu_1+\ldots+\nu_p=n+1\\ \mu_1+\ldots+\mu_{p+1}=p-1\\ \nu_j\ge 1,\,\mu_j\ge 0}} S_{\mu_1}A_{\nu_1}S_{\mu_2}\ldots A_{\nu_p}S_{\mu_{p+1}}, \tag{2.104}$$

where $S_0 := -P(0)$ and $S_k := ((A_0)^{\#})^k$.

Since the operator $B(z)$ is analytic in some neighbourhood of the origin, we can again construct the expansion for its group reduced resolvent

$$B^{\#\Omega}(z) \;=\; (B_0)^{\#} + \sum_{i=1}^{\infty} z^i B_i^{\#\Omega}. \tag{2.105}$$

The coefficients $B_i^{\#\Omega}, i = 1, 2, ...$ are calculated by the formula given in Theorem 19. This is the first reduction step. To continue, we must distinguish between two cases:

(i) If the splitting of the zero eigenvalue terminates, and consequently $B(z)$ is a *regular* perturbation of B_0, then $B^{\#\Omega}(z) = B^{\#}(z)$ and the reduced resolvent of the perturbed operator $A(z)$ is given by

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#}(z). \tag{2.106}$$

By substituting the series expansions (2.103) and (2.105) for $A^{\#\Omega}(z)$ and $B^{\#}(z)$ into (2.106), we obtain the Laurent series expansion for $A^{\#}(z)$, which has a simple pole at zero.

(ii) If the zero eigenvalue splits further, the expression

$$A^{\#\Omega\setminus\Lambda_1}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#\Omega}(z)$$

represents *only* the *group reduced resolvent* relative to the eigenvalues constituting the 0-group but not the z-group, and we have to continue the reduction process. In fact, we now consider $B(z)$ as a singular perturbation of B_0, and repeat the procedure with $B(z)$. The 0-group of eigenvalues of B_0 contains all the z^k-groups of $A(0)$ (with $k \geq 2$), but *not* the z-group. Specifically, we construct the next-step reduced operator

$$C(z) = z^{-1}B(z)Q(z),$$

where $Q(z)$ is the eigenprojection corresponding to the 0-group of the eigenvalues of $B(z)$. Again, to ensure that $C(z)$ is an analytic function of z, we assume

Assumption S2 *The zero eigenvalue of B_0 is semi-simple.*

We would like to emphasize that the subsequent reduction steps are totally identical with the first one. At each reduction step, we make the assumption Sk that the analogue of B_0 at step k has a semi-simple 0-eigenvalue. The final result is stated in the next theorem.

Theorem 21 *Let Assumptions Sk hold. Then, the reduction process terminates after a finite number of steps, say s, and the perturbed reduced resolvent $A^{\#}(z)$ has the following expression:*

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#\Omega}(z) + \frac{1}{z^2}C^{\#\Omega}(z) + ... + \frac{1}{z^s}Z^{\#}(z). \tag{2.107}$$

PROOF: Consider the first reduction step. Since $R(P(z))$ and $R(I - P(z))$ represent a direct decomposition of $\mathbb{R}^n$ and the subspace $R(P(z))$ is invariant under the operator $A(z)$, we can write

$$A^{\#}(z) = (A(z)[I - P(z)] + A(z)P(z))^{\#} = (A(z)[I - P(z)])^{\#} + (A(z)P(z))^{\#}$$

$$= A^{\#\Omega}(z) + z^{-1}(z^{-1}A(z)P(z))^{\#},$$

where Lemma 8 was used to get the first term of the right-hand-side. In view of Assumption $S1$, the operator $B(z) = z^{-1}A(z)P(z)$ is analytic in z and hence, one can apply the next reduction step. Similarly, Assumptions $Sk, k = 1, 2, \ldots$ guarantee that the reduction process can be carried out. Since the splitting of the zero eigenvalue has to terminate after a finite number of steps [87], we conclude that the reduction process has to terminate after a finite number of steps as well. Indeed, we successively eliminate the eigenvalues of the z-group, the z^2-group, etc ... Let $\lambda_i(z) = z^s\lambda_{is} + \ldots$ be the last eigenvalue which splits from zero. Then the corresponding reduced operator $Z(z)$ is regularly perturbed and the associated reduced resolvent $Z^{\#}(z)$ has the power series defined by Theorem 19. This completes the proof.

□

To obtain the Laurent series for $A^{\#}(z)$, it suffices to replace $A^{\#\Omega}(z), B^{\#\Omega}(z), \ldots$ in (2.107) by their respective power series (2.103) computed during the reduction process.

Moreover, applying formula (2.93), one can easily obtain the Laurent series for the Moore-Penrose generalized inverse of a perturbed operator. Since $\mathcal{A}_k, k = 0, 1, \ldots$ are symmetric (cf. Remark 15) the reduced operators are symmetric as well, and consequently, Assumptions Sk are satisfied.

Remark 18 *If the operator $A(z)$ has an inverse for $z \neq 0$, then the above algorithm can be used to calculate its Laurent expansion. Note that this problem, which was studied quite extensively [10, 22, 59, 61, 62, 73, 91, 96, 98, 126, 135, 145, 149] (see also Sections 2.2 and 2.3) is a particular case of our general approach.*

2.5.5 Example

As was mentioned in the introduction, the perturbation analysis of the reduced resolvent can be applied directly to the theory of singularly perturbed Markov chains. Namely, the reduced resolvent of the generator of a Markov chain is just with minus sign the deviation matrix of this chain. The deviation matrix plays a crucial role in the Markov chain theory. For example, it is used to obtain mean first-passage times. Taking into acount the above remark, we consider an example of a perturbed Markov chain. We have taken the example from Schweitzer [134] to demonstrate the validaty of our approach. Let us consider the following perturbed operator.

$$A(z) = A_0 + zA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} + z \begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ -4 & 3 & 1 \end{bmatrix}$$

Note that $-A(z)$ is the generator of a Markov chain. The zero eigenprojection and the reduced resolvent of the unperturbed matrix A_0 is given by

$$P(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \quad A_0^{\#} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

In this instance, the Laurent expansion for $A^{\#}(z)$ has a simple pole. Using Hassin and Haviv's method [63] for the determination of the singularity order of the perturbed Markov chains, one can check that

$$A^{\#}(z) = \frac{1}{z}H_{-1} + H_0 + zH_1 + \ldots$$

By applying the reduction process, we compute the singular coefficient H_{-1} and the first regular coefficient H_0. Since the zero eigenvalues of the reduced operators are always semi-simple in the case of perturbed Markov chains [40], we conclude from Theorem 21 that

$$H_{-1} = B_0^{\#}, \quad \text{and} \quad H_0 = A_0^{\#} + B_1^{\#\Omega}.$$

To compute $B_0^{\#}$ and $B_1^{\#\Omega}$, we need to calculate the first two terms of the expansion for the reduced operator $B(z)$. In particular,

$$B_0 = P(0)A_1P(0) = \begin{bmatrix} 2 & -1 & -1 \\ -3.5 & 1.75 & 1.75 \\ -3.5 & 1.75 & 1.75 \end{bmatrix}$$

$$B_1 = -(A_0^{\#}A_1P(0)A_1P(0) + P(0)A_1A_0^{\#}A_1P(0) + P(0)A_1P(0)A_1A_0^{\#}) = \frac{1}{8}\begin{bmatrix} 0 & 4 & -4 \\ -24 & 5 & 19 \\ 20 & -17 & -3 \end{bmatrix}$$

calculated with the help of (2.104). Next, we calculate the eigenprojection corresponding to the zero eigenvalue of the operator B_0, that is,

$$Q(0) = \frac{1}{22}\begin{bmatrix} 14 & 4 & 4 \\ 14 & 15 & -7 \\ 14 & -7 & 15 \end{bmatrix}.$$

Now using formula (2.100) from Theorem 19, we obtain

$$B_1^{\#\Omega} = Q(0)B_1(B_0^{\#})^2 - B_0^{\#}B_1B_0^{\#} + (B_0^{\#})^2B_1Q(0) = \frac{1}{2662}\begin{bmatrix} -16 & 52 & -36 \\ -236 & 41 & 195 \\ 248 & -201 & -47 \end{bmatrix}.$$

Thus, we finally obtain

$$H_{-1} = B_0^{\#} = \frac{1}{5.5^2}\begin{bmatrix} 1 \\ -1.75 \\ -1.75 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 8 & -4 & -4 \\ -14 & 7 & 7 \\ -14 & 7 & 7 \end{bmatrix}$$

and

$$H_0 = B_1^{\#\Omega} + A_0^{\#} = \frac{1}{1331}\begin{bmatrix} -8 & 26 & -18 \\ -118 & 686 & -568 \\ 124 & -766 & 642 \end{bmatrix}.$$

Note that the above H_{-1} coincides with the one given in [134]. Independently of the above calculations and using MATLAB symbolic toolbox, we have also expanded $A^{\#}(z)$ as a Laurent series and checked that H_0 is indeed correct.

The above method for the computation of the perturbed Markov chain deviation matrix has at least two advantages in comparison with the algebraic methods proposed in [8, 66, 132]. Fistly, this method can easily cope with the case where the perturbed Markov chain has several ergodic classes. Secondly, our approach does not require the knowledge of the series expansion for the ergodic projection, that is, the eigenprojection corresponding to the zero eigenvalue of the Markov chain generator. However, if we do have in hand the expansion for the ergodic projection, we can use the recursive formula (2.96) to compute the regular coefficients. Let us compute by the recursive formula the coefficient H_1 for our example. Using any method as in [29, 40, 66, 67, 91, 116, 132, 134], one can compute the coefficients for the expansion of the ergodic projection associated with z and z^2.

$$P_{01} = \frac{1}{121}\begin{bmatrix} 2 & -12 & 10 \\ 2 & -12 & 10 \\ 2 & -12 & 10 \end{bmatrix} \quad P_{02} = \frac{1}{1331}\begin{bmatrix} 32 & -192 & 160 \\ 32 & -192 & 160 \\ 32 & -192 & 160 \end{bmatrix}$$

Then, according to formula (2.96), we have

$$\begin{aligned} H_1 &= -(H_0A_1)H_0 - (P_{01}H_0 + P_{02}H_{-1}) - (H_{-1}P_{02} + H_0P_{01}) \\ &= \frac{1}{14641}\begin{bmatrix} -368 & 1856 & -1488 \\ -2128 & 12416 & -10288 \\ 1744 & -10816 & 9072 \end{bmatrix}. \end{aligned}$$

The latter coefficient also coincides with the one obtained with the help of MATLAB symbolic toolbox.

2.5.6 Asymptotics for $P_0(z)$

We know that the eigenprojection $P_0(z)$ of the perturbed operator corresponding to the identically zero eigenvalue is analytic in some (punctured) neighbourhood of $z = 0$ (see [13, 60] and Section 2.4), that is,

$$P_0(z) \;=\; P_{00} + \sum_{k=1}^{\infty} z^k P_{0k}, \quad \text{for} \quad z \neq 0. \tag{2.108}$$

For regular perturbations, P_{00} is just $P_0(0)$, and the group projection coincides with the eigenprojection. This is *not* the case for singular perturbations.

Therefore, an interesting question is how P_{00} in (2.108) relates to the original matrix $P_0(0)$ in the general case and how the power series (2.108) can be computed. The answers to these questions are provided below.

Proposition 2 *The coefficients of the power series (2.108) for the perturbed eigenprojection are given by*

$$P_{0k} = -\sum_{i=0}^{s+k} A_i H_{k-i}, \quad k = 1, 2, \ldots\ .$$

PROOF: The above formula is obtained by equating the terms with the same power of z in the identity (2.81) for the perturbed operators.

□

Corollary 4 *As $z \to 0$, the limit matrix P_{00} satisfies*

$$P_{00} \;=\; P_0(0)\,[I - \sum_{i=1}^{s} A_i H_{-i}]. \tag{2.109}$$

where $P_0(0)$ is the 0-eigenprojection of the unperturbed matrix A_0.

PROOF: When substituting the Laurent series expansion (2.91) into (2.81)-(2.82), and equating the terms of same power, one obtains in particular,

$$I - P_{00} \;=\; A_0 H_0 + A_1 H_{-1} + \ldots + A_s H_{-s}\ . \tag{2.110}$$

In addition, from $A(z)P_0(z) = 0$, we immediately obtain:

$$A_0 P_{00} \;=\; 0 \text{ so that } P_{00} \;=\; P_0(0)V, \tag{2.111}$$

for some matrix V. Moreover, as $P_0(0)^2 = P_0(0)$, we also have $P_0(0)P_{00} = P_0(0)^2 V = P_0(0)V = P_{00}$. Therefore, pre-multiplying both sides of (2.110) by $P_0(0)$, and using $P_0(0)A_0 = 0$, one obtains (2.109), the desired result.

□

Hence, (2.109) relates in a simple manner, the limit matrix P_{00} to the original 0-group $P_0(0)$, in terms of the perturbation matrices $A_k, k = 1, ..., s$, the original matrix $P_0(0)$, and the coefficients $H_{-k}, k = 1, ..., s$ of the singular part of $A(z)^{\#}$. This shows how the perturbed 0-eigenvectors compare to the unperturbed ones, for small z. Observe that in the case of a linear (or first-order) perturbation, only the singular term H_{-1} is involved. Finally, the regular case is obtained as a particular case, since then, $H_{-k}, k = 1, \dots, s$ vanish so that $P_{00} = P_0(0)$.

2.5.7 The proof of Theorem 18

To prove Theorem 18 we use the Cauchy contour integration and the residue technique [111, 87, 91]. First we present some auxiliary results.

Lemma 9 *Let Γ_1 and Γ_2 be two closed positively-oriented contours in the complex plane around zero and let $z_1 \in \Gamma_1, z_2 \in \Gamma_2$. Furthermore, assume that the contour Γ_2 lies inside the contour Γ_1. Then the following formulae hold:*

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{z_2^{-m-1}}{z_2 - z_1} dz_2 = -\eta_m z_1^{-m-1}, \tag{2.112}$$

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z_1^{-m-1}}{z_2 - z_1} dz_1 = -(1-\eta_n) z_2^{-m-1}, \tag{2.113}$$

with

$$\eta_m := \begin{cases} 0, & m < 0, \\ 1, & m \geq 0. \end{cases}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{z_2^{-m-1} P_0(z_2)}{z_2 - z_1} dz_2 = \begin{cases} 0, & m < 0, \\ -z_1^{-m-1}[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}], & m \geq 0. \end{cases} \tag{2.114}$$

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z_1^{-n-1} P_0(z_1)}{z_2 - z_1} dz_1 = \begin{cases} -z_2^{-n-1} P_0(z_2), & n < 0, \\ -[P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...], & n \geq 0. \end{cases} \tag{2.115}$$

Proof: The proof of formulae (2.112),(2.113) is given in the book of Kato [87] and in the book of Korolyuk and Turbin [91].

Let us establish the auxiliary integral (2.114). If $m < 0$, then the function $\frac{z_2^{-m-1} P_0(z_2)}{z_2 - z_1}$ is analytic inside the area enclosed by the contour Γ_2 and hence the auxiliary integral

(2.114) is equal to zero by the Cauchy Integral Theorem. To deal with the case $m \geq 0$, we first expand the function $\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1}$ as a Laurent series.

$$\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1} = -\frac{z_2^{-m-1}P_0(z_2)}{z_1(1-z_2/z_1)} = -z_1^{-1}z_2^{-m-1}[P_{00}+z_2P_{01}+z_2^2P_{02}+...][1+\frac{z_2}{z_1}+\frac{z_2^2}{z_1^2}+...] =$$

$$= z_2^{-m-1}(-z_1^{-1})P_{00}+...+z_2^{-1}(-z_1^{-1})[\frac{1}{z_1^m}P_{00}+\frac{1}{z_1^{m-1}}P_{01}+...+P_{0m}]+...$$

Then, according to the Residue Theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1}dz_2 &= (-z_1^{-1})[\frac{1}{z_1^m}P_{00}+\frac{1}{z_1^{m-1}}P_{01}+...+P_{0m}] \\ &= -z_1^{-m-1}[P_{00}+z_1P_{01}+...+z_1^mP_{0m}].\end{aligned}$$

Thus, we have calculated the integral (2.114). The same method is applied to calculate the auxiliary integral (2.115).

□

The proof of Theorem 18: Each coefficient of the Laurent series (2.88) can be represented by the contour integral formula

$$H_n = \frac{1}{2\pi i}\int_\Gamma z^{-n-1}A^\#(z)dz, \quad \Gamma \in D, \tag{2.116}$$

where Γ is a closed positively-oriented contour in the complex plane, which encloses zero but no other eigenvalues of A_0. Let us substitute (2.116) into the following expression

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1} = \sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\frac{1}{2\pi i}\int\limits_{\Gamma_1}z_1^{-n+i-1}A^\#(z_1)dz_1A_k\frac{1}{2\pi i}\int\limits_{\Gamma_2}z_2^{-m-i+k-2}A^\#(z_2)dz_2.$$

As in Lemma 9, we assume without loss of generality that the contour Γ_2 lies inside the contour Γ_1. Then, we can rewrite the above expressions as double integrals

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1} = \sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\left(\frac{1}{2\pi i}\right)^2\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}z_1^{-n+i-1}z_2^{-m-i+k-2}A^\#(z_1)A_kA^\#(z_2)dz_2dz_1$$

$$= \left(\frac{1}{2\pi i}\right)^2\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}z_1^{-n-1}z_2^{-m-1}\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}z_1^iz_2^{k-i-1}A^\#(z_1)A_kA^\#(z_2)dz_2dz_1$$

$$= \left(\frac{1}{2\pi i}\right)^2\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}\frac{z_1^{-n-1}z_2^{-m-1}}{z_2-z_1}\sum_{k=1}^{\infty}(z_2^k-z_1^k)A^\#(z_1)A_kA^\#(z_2)dz_2dz_1.$$

Using the resolvent-like identity (2.94), we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1} =$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \left[\frac{A^{\#}(z_1) - A^{\#}(z_2)}{z_2 - z_1} - \frac{A^{\#}(z_1)P_0(z_2)}{z_2 - z_1} + \frac{P_0(z_1)A^{\#}(z_2)}{z_2 - z_1}\right] dz_2 dz_1.$$

Thus, we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = I_1 - I_2 + I_3,$$

with:

$$\begin{aligned}
I_1 &:= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{A^{\#}(z_1) - A^{\#}(z_2)}{z_2 - z_1} dz_2 dz_1, \\
I_2 &:= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{A^{\#}(z_1)P_0(z_2)}{z_2 - z_1} dz_2 dz_1, \\
I_3 &:= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{P_0(z_1)A^{\#}(z_2)}{z_2 - z_1} dz_2 dz_1.
\end{aligned}$$

Let us separately calculate the integrals I_1, I_2 and I_3. The integral I_1 can be written as

$$\begin{aligned}
I_1 &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} A^{\#}(z_1) dz_2 dz_1 - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} A^{\#}(z_2) dz_2 dz_1 \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \left[\int_{\Gamma_2} \frac{z_2^{-m-1}}{z_2 - z_1} dz_2\right] z_1^{-n-1} A^{\#}(z_1) dz_1 \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2} \left[\int_{\Gamma_1} \frac{z_1^{-n-1}}{z_2 - z_1} dz_1\right] z_2^{-m-1} A^{\#}(z_2) dz_2
\end{aligned}$$

In the last equality we used Fubini Theorem to change the order of integration. Using the auxiliary integrals (2.112) and (2.113), we obtain

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\Gamma_1} (-\eta_m z_1^{-m-1}) z_1^{-n-1} A^{\#}(z_1) dz_1 - \frac{1}{2\pi i}\int_{\Gamma_2} (-(1-\eta_n) z_2^{-n-1}) z_2^{-m-1} A^{\#}(z_2) dz_2 \\
&= -\frac{\eta_n + \eta_m - 1}{2\pi i} \int_{\Gamma_1} z_1^{-n-m-2} A^{\#}(z_1) dz_1 = -(\eta_n + \eta_m - 1) H_{n+m+1},
\end{aligned}$$

where the second integral can be taken over Γ_1 by the principle of deformation of contours. We calculate the second integral I_2 as follows:

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i}\int_{\Gamma_1} z_1^{-n-1} A^{\#}(z_1) \frac{1}{2\pi i}\int_{\Gamma_2} \frac{z_2^{-m-1} P_0(z_2)}{z_2 - z_1} dz_2 dz_1 \\
&= \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_1} 0 z_1^{-n-1} dz_1, & m < 0, \\ -\frac{1}{2\pi i}\int_{\Gamma_1} z_1^{-n-1} A^{\#}(z_1) z_1^{-m-1} [P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}] dz_1, & m \geq 0 \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 0, & m < 0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-m-2}A^{\#}(z_1)[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}]dz_1, & m \geq 0 \end{cases} \\
&= \begin{cases} 0, & m < 0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-m-2}A^{\#}(z_1)[P_0(z_1) - z_1^{m+1}P_{0m+1} - z_1^{m+2}P_{0m+2} - ...]dz_1, & m \geq 0 \end{cases} \\
&= \begin{cases} 0, & m < 0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-m-2}A^{\#}(z_1)[z_1^{m+1}P_{0m+1} + z_1^{m+2}P_{0m+2} + ...]dz_1, & m \geq 0 \end{cases} \\
&= \begin{cases} 0, & m < 0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1} + z_1 P_{0m+2} + ...]dz_1, & m \geq 0 \end{cases}
\end{aligned}
$$

where, in the above expressions, the auxiliary integral (2.114) and the property $A^{\#}(z)P_0(z) = 0$ has been used. Now, we calculate the last integral I_3 with the help of the auxiliary integral (2.115).

$$
\begin{aligned}
I_3 &= \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{z_1^{-n-1}P_0(z_1)}{z_2 - z_1}dz_1 z_2^{-m-1}A^{\#}(z_2)dz_2 \\
&= \begin{cases} -\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-n-m-2}P_0(z_2)A^{\#}(z_2)dz_2, & n < 0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...]A^{\#}(z_2)dz_2, & n \geq 0, \end{cases} \\
&= \begin{cases} 0, & n < 0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...]A^{\#}(z_2)dz_2, & n \geq 0. \end{cases}
\end{aligned}
$$

Finally, summing up the three integrals I_1, I_2 and I_3, we obtain the relation (2.95).

□

2.6 Analytic perturbation of Sylvester matrix equation

2.6.1 Introduction

In this section of the thesis we study an analytic perturbation of the Sylvester matrix equation. Namely, we consider the perturbed Sylvester equation

$$A(z)X + XB(z) = C(z), \tag{2.117}$$

whose coefficient matrices $A(z) \in \mathbb{C}^{m\times m}$, $B(z) \in \mathbb{C}^{n\times n}$ and the right hand side $C(z) \in \mathbb{C}^{m\times n}$ depend analytically on a small complex parameter z. The above equation has several important applications in control and system theory (see e.g. [116, 150] and references therein). In particular, Wimmer [150] has considered equation (2.117) with coefficients $A(z), B(z)$ and $C(z)$ as matrix polynomials [60]. The latter is a particular instance of the more general analytic perturbation. Let us refer to z as the perturbation parameter. When $z = 0$, the equation

$$A(0)X + XB(0) = C(0) \tag{2.118}$$

is called the unperturbed Sylvester equation. Since the matrix-valued functions $A(z), B(z)$ and $C(z)$ are analytic in the neighbourhood of $z = 0$, they can be represented by the following power series

$$A(z) = \sum_{k=0}^{\infty} z^k A^{(k)}, \quad B(z) = \sum_{k=0}^{\infty} z^k B^{(k)}, \quad C(z) = \sum_{k=0}^{\infty} z^k C^{(k)}.$$

We distinguish between *regular* and *singular* perturbations. The perturbation is said to be regular if the unperturbed equation (2.118) has a unique solution, whereas the perturbation is said to be singular if the unperturbed equation (2.118) possesses none or multiple solutions, but the perturbed equation (2.117) has a unique solution for any $z \neq 0$ and sufficiently small.

The difference between regular and singular perturbations can also be explained in terms of the eigenvalues of $A(z)$ and $B(z)$. Let

$$\delta(z) := dist[\sigma(A(z)), \sigma(B(z))] = \min_{i,j} |\lambda_i(z) - \mu_j(z)|,$$

where $\lambda_i(z)$ and $\mu_i(z)$ are the respective eigenvalues of $A(z)$ and $B(z)$. Then, in the case of a regular perturbation, $\delta(0) \neq 0$ whereas in the case of a singular perturbation, $\delta(0) = 0$ but $\delta(z) > 0$ for $z \neq 0$ and sufficiently small. If $A(z)$ and $B(z)$ are normal matrices, then the following bounds are available [17, 18]:

$$||X(z)|| \leq \frac{c}{\delta(z)} ||C(z)||. \tag{2.119}$$

Thus, we can see that the bounds grow to infinity in the case of singular perturbations. This is another justification for using the term "singular perturbations". Apart from the above bounds there are many other bounds available for additive perturbations (e.g. see [69, 84, 115] and references therein). Since all of them are given in terms of some norm, they only indicate that in the singular case the solution may be very large, but they do not provide any direction for the deviation of the perturbed solution. By contrast, using a more specific analytic approach, we are able to exhibit the asymptotic behaviour of the perturbed solution. Moreover, as will be demonstrated in the sequel, if the unperturbed system has multiple solutions, then the perturbed solution may converge to some particular solution of the unperturbed problem. In the latter case, the normwise estimation

(2.119) does not reflect the real situation. Finally, we would like to mention the paper of Lancaster [97] that uses the special analytic perturbation $B(z) = B + zI$ to obtain feasibility conditions for the unperturbed Sylvester equation.

In the next two subsections we will recall some useful facts about analytic perturbation theory and solvability of the Sylvester equation.

2.6.2 Preliminaries on Sylvester matrix equation

Let A, B, C be complex-valued matrices in $\mathbb{C}^{m\times m}$, $\mathbb{C}^{n\times n}$, $\mathbb{C}^{m\times n}$ respectively and consider the Sylvester equation in $\mathbb{C}^{m\times n}$, that is:

$$AX + XB = C. \tag{2.120}$$

In this section, we first discuss the structure of the null space as well as the feasibility conditions of equation (2.120) in terms of eigen-elements of matrices A and B. Our exposition follows closely the paper of Kucera [92] (see also related work by Trampus [144]).

For any two vectors (x, y) in $\mathbb{C}^m$, $< x, y >$ denotes the usual scalar product, that is $< x, y >= y^* x$, where y^* denotes the conjugate transpose of y.

For any two matrices X, Y in $\mathbb{C}^{m\times n}$, $\langle X, Y\rangle$ denotes the usual scalar product $tr(Y^* X)$, where again Y^* denotes the conjugate transpose matrix of Y.

Let α_i be an eigenvalue of matrix A. Then $-\alpha_i$ is an eigenvalue of $-A$ and there exists a Jordan chain of generalized eigenvectors of $-A$ in $\mathbb{C}^m$ satisfying

$$(-A + \alpha_i I_m)x_{ir} = x_{i,r-1}, \quad r = 1, ..., \rho_i.$$

By convention, $x_{i0} = 0$. Let $y_{ir}, r = 1, ..., \rho_i$ be a system of generalized eigenvectors for the adjoint matrix $-A^*$. These vectors satisfy the following equations.

$$(-A^* + \alpha_i^* I_m)y_{ir} = y_{i,r-1}, \quad r = 1, ..., \rho_i.$$

In addition, the systems of vectors $\{x_{ir}\}$ and $\{y_{ir}\}$ are biorthogonal, that is:

$$< x_{ip}, y_{jr} >= 0, \quad \text{if} \quad i \neq j,$$

and if $i = j$,

$$< x_{ip}, y_{ir} >= \begin{cases} 1, & p + r = \rho_i + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.121}$$

Similarly, let β_j be an eigenvalue of the matrix B. Then there exists a system of eigenvectors $u_{js}, s = 1, ..., \sigma_j$ in $\mathbb{C}^n$, satisfying

$$(B - \beta_j I_n)u_{js} = u_{j,s-1}, \quad s = 1, ..., \sigma_j.$$

By convention, $u_{j0} = 0$. Let $v_{js}, s = 1, ..., \sigma_j$ be a system of the generalized eigenvectors for the adjoint matrix B^* such that

$$(B^* - \beta_j^* I_n) v_{js} = v_{j,s-1}, \quad s = 1, ..., \sigma_j,$$

with $v_{j0} = 0$. The vectors $\{u_{js}\}$ and $\{v_{js}\}$ also form a biorthogonal system. As above,

$$< u_{is}, v_{jl} >= 0, \quad \text{if} \quad i \neq j,$$

and if $i = j$,

$$< u_{is}, v_{il} >= \begin{cases} 1, & s + l = \sigma_i + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.122}$$

Now, consider the linear operator $L : \mathbb{C}^{m \times n} \to \mathbb{C}^{m \times n}$ defined by

$$X \mapsto LX := AX + XB \qquad X \in \mathbb{C}^{m \times n}.$$

All eigenvalues of L are of the form $\alpha_i + \beta_j$, where α_i and β_j are eigenvalues of the matrices A and B, respectively. Suppose that $\alpha_i + \beta_j = 0$ for some indicies i and j. Then, the operator L has a nontrivial null space spanned by the basis elements

$$X_{ijk} = \sum_{s=1}^{k} x_{i,k+1-s} v_{js}^*, \quad k = 1, ..., \mu_{ij}, \tag{2.123}$$

where $\mu_{ij} = \min(\rho_i, \sigma_j)$.

Similarly, the null space of the adjoint operator $L^* : \mathbb{C}^{m \times n} \to \mathbb{C}^{m \times n}$ defined by

$$Y \mapsto L^* Y := A^* Y + Y B^* \qquad Y \in \mathbb{C}^{m \times n}$$

is spanned by the basis elements

$$Y_{ijk} = \sum_{s=1}^{k} y_{i,k+1-s} u_{js}^*, \quad k = 1, ..., \mu_{ij}. \tag{2.124}$$

Then, the feasibility condition for the Sylvester equation (2.120), which can also be rewritten as $LX = C$, reads

$$\langle Y_{pqr}, C \rangle = tr(Y_{pqr}^* C) = 0 \ \forall p, q \text{ such that } \alpha_p + \beta_q = 0. \tag{2.125}$$

Translated in terms of eigenvectors of A and B, the feasibility condition (2.125) reads

$$\sum_{l=1}^{r} y_{p,r+1-l}^* C u_{ql} = 0, \quad r = 1, ..., \mu_{pq}, \tag{2.126}$$

with $\mu_{pq} = \min(\rho_p, \sigma_q)$.

2.6.3 Analytic perturbation under non-degeneracy assumption

In this subsection we briefly discuss the implementation of the algebraic reduction technique of Section 2.2 under the non-degeneracy assumption. For more details on the inversion of an analytically perturbed operator the interested reader is also referred to the following papers and books: [59, 60, 91, 116, 135, 145, 149].

Let $(X, \langle ., . \rangle)$ be a finite dimensional Hilbert space and consider a family of linear operators $L^{(i)} : X \to X$, $i = 0, 1, \ldots$. Let $L(z) : X \to X$ be the linear operator

$$L(z) := L^{(0)} + zL^{(1)} + z^2L^{(2)} + ..., \tag{2.127}$$

that is, $L(z)$ is an analytic perturbation of $L^{(0)}$.

Consider the linear equation $L^{(0)}x = c^{(0)}$, with $c^{(0)} \in X$, and its perturbed version

$$L(z)x(z) = c(z), \tag{2.128}$$

with $c(z) = c^{(0)} + zc^{(1)} + z^2c^{(2)} + ...$, and z a scalar parameter.

Suppose that the unperturbed operator $L^{(0)}$ is not invertible, whereas $L(z)$ is invertible for sufficiently small and nonzero z, which is the case of *singular* perturbations. Note that the perturbed solution $x(z)$ can have a singularity at $z = 0$. Let the vectors $\{u_i\}_{i=1}^p$ form a basis for the null space of $L^{(0)}$ and let the vectors $\{v_i\}_{i=1}^p$ form a basis for the null space of the adjoint operator $L^{(0)*}$. Following [104], consider the *non-degeneracy* assumption

$$\det(\{< v_i, L^{(1)}u_j >\}_{i,j=1}^n) \neq 0. \tag{2.129}$$

If (2.129) holds, then as one can conclude from the results of Section 2.2 the Laurent expansion for the perturbed solution $x(z)$ has at most a simple pole. That is,

$$x(z) = \frac{1}{z}x^{(-1)} + x^{(0)} + zx^{(1)} + ... \tag{2.130}$$

It turns out that the non-degenerate case, that is, when (2.129) holds, describes the most common case. Furthermore, the case of a higher order pole reduces to the case of a first order pole via a so-called "reduction technique" (see for example Sections 2.2 and 2.5 for a detailed development of this technique). Therefore, in the sequel, and for the sake of clarity of exposition, we voluntarily restrict ourselves to the non-degenerate case, that is, we will assume that (2.129) holds.

Substituting series (2.127) and (2.130) into equation (2.128) and equating terms with same powers of z, we obtain the following system of fundamental equations:

$$\begin{aligned} L^{(0)}x^{(-1)} &= 0 && (f0) \\ L^{(0)}x^{(0)} + L^{(1)}x^{(-1)} &= c^{(0)} && (f1) \\ L^{(0)}x^{(1)} + L^{(1)}x^{(0)} + L^{(2)}x^{(-1)} &= c^{(1)} && (f2) \\ &\vdots && \vdots \end{aligned}$$

Then, from equation (f0) we conclude that

$$x^{(-1)} = \sum_{j=1}^{p} \gamma_j u_j. \tag{2.131}$$

The coefficients γ_j can be found from the feasibility conditions of the second fundamental equation (f1), that is:

$$\langle v_i, c^{(0)} - L^{(1)} x^{(-1)} \rangle \; = \; 0, \quad i = 1, ..., p.$$

Substituting (2.131) into the above equations, the coefficients $\{\gamma_i\}$ must be solutions to the linear system of equations

$$\sum_{j=1}^{p} \langle v_i, L^{(1)} u_j \rangle \gamma_j \; = \; \langle v_i, c^{(0)} \rangle, \quad i = 1, ..., p. \tag{2.132}$$

In the non-degenerate case, the system (2.132) has a unique solution. In particular, if $\langle v_i, c^{(0)} \rangle = 0$ for all $i = 1, ..., p$, that is, unperturbed equation $L^{(0)} x = c^{(0)}$ is feasible, the singular term in the Laurent series (2.130) vanishes and the first regular coefficient $x^{(0)}$ corresponds to the best approximation for the perturbed problem.

Next, assuming that the coefficients $x^{(-1)}, ..., x^{(k-1)}$ are known, the next coefficient $x^{(k)}$ can be calculated by a recursive formula. Indeed, the (fk+1) fundamental equation

$$L^{(0)} x^{(k)} = c^{(k)} - \sum_{i=1}^{k+1} L^{(i)} x^{(k-i)}$$

has a general solution of the form

$$x^{(k)} = x_*^{(k)} + \sum_{j=1}^{p} \gamma_j^{(k)} u_j \tag{2.133}$$

where $x_*^{(k)}$ is any particular solution. For instance, one may choose $x_*^{(k)} := L^{(0)\dagger}(c^{(k)} - \sum_{i=1}^{k+1} L^{(i)} x^{(k-i)})$, where $L^{(0)\dagger}$ is the Moore-Penrose generalized inverse of $L^{(0)}$. Then, substituting again the expression (2.133) into the feasibility conditions for the next fundamental equation (fk+2), we obtain

$$\langle v_i, c^{(k+1)} - \sum_{s=1}^{k+2} L^{(s)} x^{(k+1-s)} \rangle \; = \; 0, \quad i = 1, ..., p.$$

which yields the linear system of equations

$$\sum_{j=1}^{p} \langle v_i, L^{(1)} u_j \rangle \gamma_j^{(k)} \; = \; \langle v_i, c^{(k+1)} - L^{(1)} x_*^{(k)} - \sum_{s=2}^{k+2} L^{(s)} x^{(k+1-s)} \rangle \quad i = 1, ..., p. \tag{2.134}$$

The latter system uniquely determines the coefficients $\{\gamma_i^{(k)}\}$.

It is worth noting that the linear systems of equations (2.132) and (2.134) have the same coefficient matrix in the left hand side. This fact leads to efficient computational procedures. For example, if a LU-decomposition is used for solving (2.132), it can be stored and re-used for solving (2.134).

2.6.4 Perturbation analysis of the Sylvester equation

First, note that that the perturbed Sylvester equation (2.117) can be viewed as the perturbed linear system $L(z)X(z) = C(z)$ with perturbed linear operator $L(z) : X \mapsto A(z)X + XB(z)$. The linear operator $L(z)$ acts on the Hilbert space of matrices $\mathbb{C}^{m\times n}$ endowed with the scalar product $\langle X, Y\rangle = tr(X^*Y)$. As already mentioned before, we only study the case of first-order singular perturbations, that is, when the Laurent series for the perturbed solution $X(z)$ has a simple pole at $z = 0$.

In this case, the system of fundamental equations for the Sylvester operator takes the form

$$\begin{array}{rcll} A^{(0)}X^{(-1)} + X^{(-1)}B^{(0)} & = & 0 & (fs0) \\ (A^{(0)}X^{(0)} + X^{(0)}B^{(0)}) + (A^{(1)}X^{(-1)} + X^{(-1)}B^{(1)}) & = & C^{(0)} & (fs1) \\ (A^{(0)}X^{(1)} + X^{(1)}B^{(0)}) + (A^{(1)}X^{(0)} + X^{(0)}B^{(1)}) + (A^{(2)}X^{(-1)} + X^{(-1)}B^{(2)}) & = & C^{(1)} & (fs2) \\ & \vdots & & \vdots \end{array}$$

To solve the above system, we will follow the general method briefly outlined in the previous section. From the fundamental equation (fs0), $X^{(-1)}$ can be written as the following linear combination

$$X^{(-1)} = \sum_{ijk} \gamma_{ijk}^{(-1)} X_{ijk},$$

where X_{ijk} is as in (2.123) and where the summation is taken over all pairs of indicies (i, j) such that $\alpha_i + \beta_j = 0$. The coefficients $\gamma_{ijk}^{(-1)}$ are solutions of the linear system of equations induced by the feasibility conditions of the second fundamental equation (fs1). This requires computing the quantities $\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle$, with $L^{(1)} : X \mapsto A^{(1)}X + XB^{(1)}$, and with X_{ijk}, Y_{pqr} as in (2.123) and (2.124).

$$\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle = \langle Y_{pqr}, A^{(1)}X_{ijk} + X_{ijk}B^{(1)}\rangle \tag{2.135}$$

The matrix of the linear system (2.132) in the unknown variables $\gamma_{ijk}^{(-1)}$ is filled-up with the above coefficients $\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle$ defined in (2.135). Under the non-degeneracy assumption (2.129), which now reads

$$\det\left[\langle Y_{pqr}, L^{(1)}X_{ijk}\rangle\right] \neq 0, \tag{2.136}$$

the matrix is non-singular so that there is a unique solution $\gamma_{ijk}^{(-1)}$.

We now develop $\langle Y_{pqr}, A^{(1)} X_{ijk}\rangle$ and $\langle Y_{pqr}, X_{ijk}B^{(1)}\rangle$ separately. Using the feasibility condition (2.126) given in terms of eigen-elements of A and B, and using the biorthogonality condition (2.122), we obtain

$$\begin{aligned}
\langle Y_{pqr}, A^{(1)} X_{ijk}\rangle &= \sum_{l=1}^{r} y^*_{p,r+1-l} A^{(1)} X_{ijk} u_{ql} \\
&= \sum_{l=1}^{r} y^*_{p,r+1-l} A^{(1)} \sum_{s=1}^{k} x_{i,k+1-s} v^*_{js} u_{ql} \\
&= \begin{cases} 0, & j \neq q, \\ \sum_{l=1}^{r} y^*_{p,r+1-l} A^{(1)} \sum_{s=1}^{k} x_{i,k+1-s} v^*_{qs} u_{ql}, & j = q, \end{cases} \\
&= \begin{cases} 0, & j \neq q, \\ \sum_{l=1}^{r} y^*_{p,r+1-l} A^{(1)} \sum_{1\le s\le k;\ s+l=\sigma_q+1} x_{i,k+1-s}, & j = q, \end{cases}
\end{aligned}$$

Similarly, using again (2.126) and biorthogonality condition (2.121), we get

$$\begin{aligned}
\langle Y_{pqr}, X_{ijk}B^{(1)}\rangle &= \sum_{l=1}^{r} y^*_{p,r+1-l} X_{ijk} B^{(1)} u_{ql} \\
&= \sum_{l=1}^{r} y^*_{p,r+1-l} \sum_{s=1}^{k} x_{is} v^*_{j,k+1-s} B^{(1)} u_{ql} \\
&= \begin{cases} 0, & i \neq p, \\ \sum_{l=1}^{r} \sum_{s=1}^{k} y^*_{p,r+1-l} x_{ps} v^*_{j,k+1-s} B^{(1)} u_{ql}, & i = p, \end{cases} \\
&= \begin{cases} 0, & i \neq p, \\ \sum_{l=1}^{r} \sum_{1\le s\le k;\ s+r+1-l=\rho_p+1} v^*_{j,k+1-s} B^{(1)} u_{ql}, & i = p, \end{cases}
\end{aligned}$$

We are now able to formulate our main result:

Theorem 22 *Under the non-degeneracy assumption (2.136), the linear system of equations in the unknown variables* $\gamma_{ijk}^{(-1)}$

$$\sum_{ik} \gamma_{iqk}^{(-1)} \left(\sum_{l=1}^{r} y^*_{p,r+1-l} A^{(1)} \sum_{1\le s\le k;\ s+l=\sigma_q+1} x_{i,k+1-s} \right) +$$

$$\sum_{jk} \gamma_{pjk}^{(-1)} \left(\sum_{l=1}^{r} \sum_{1\le s\le k;\ s+r+1-l=\rho_p+1} v^*_{j,k+1-s} B^{(1)} u_{ql} \right) = \sum_{l=1}^{r} y^*_{p,r+1-l} C^{(0)} u_{ql},\ \ r = 1, ..., \mu_{pq}. \tag{2.137}$$

has a unique solution and the perturbed solution of the Sylvester equation (2.117) has a Laurent series expansion with a simple pole

$$X(\varepsilon)=\frac{1}{\varepsilon}X^{(-1)}+X^{(0)}+\varepsilon X^{(1)}+... \tag{2.138}$$

where $X^{(-1)}=\sum_{ijk}\gamma_{ijk}^{(-1)}X_{ijk}$ *and the other coefficients can be calculated recursively by*

$$X^{(k)}=X_*^{(k)}+\sum_{ijk}\gamma_{ijk}^{(k)}X_{ijk},$$

where $X_*^{(k)}$ *is any particular solution of (fs k+1) and the coefficients* $\gamma_{ijk}^{(k)}$ *are determined by*

$$\sum_{ik}\gamma_{iqk}^{(-1)}\left(\sum_{l=1}^{r}y_{p,r+1-l}^{*}A^{(1)}\sum_{1\le s\le k;\ s+l=\sigma_q+1}x_{i,k+1-s}\right)+$$
$$\sum_{jk}\gamma_{pjk}^{(-1)}\left(\sum_{l=1}^{r}\sum_{1\le s\le k;\ s+r+1-l=\rho_p+1}v_{j,k+1-s}^{*}B^{(1)}u_{ql}\right)=$$
$$\sum_{l=1}^{r}y_{p,r+1-l}^{*}\left(C^{(k+1)}-A^{(1)}X_*^{(k)}-X_*^{(k)}B^{(1)}-\sum_{s=2}^{k+2}[A^{(s}X^{(k+1-s)}+X^{(k+1-s)}B^{(s)}]\right)u_{ql}. \tag{2.139}$$

□

Again we would like to emphasize that the linear systems (2.137) and (2.139) have the same coefficient matrix on the left hand side, which permits an efficient calculation of the regular terms in the Laurent series (2.138). Note also that if the unperturbed system (2.118) is feasible and the non-degeneracy assumption (2.129) holds, then the Laurent series (2.138) has only positive powers of ε and $X^{(0)}$ represents the limit solution as $\epsilon\to 0$, of the perturbed problem.

Consider now a particular but practically important case which yields a concise matrix representation. Suppose that there is only one pair of eigenvalues (α,β) such that $\alpha+\beta=0$, and in addition, assume that α and β are semisimple eigenvalues with respective multiplicities p and q. This case of a unique pair of semisimple eigenvalues was also considered in [116] in the context of a Lyapunov equation. Let us form the following matrices $X=[x_1,...,x_p]$ and $Y=[y_1,...,y_p]$, where x_i,y_i are the respective right and left eigenvectors of $-A$ corresponding to the eigenvalue α. Similarly, let the matrices U and V be formed with the right and left eigenvectors of B corresponding the eigenvalue β. Then, the results of Theorem 22 can be rewritten in a more elegant matrix form, as indicated below.

Corollary 5 *Let A and B possess only one pair of respective eigenvalues (α, β) such that $\alpha+\beta = 0$. In addition, assume that both α and β are semisimple with respective associated bases of right eigenvectors (X and U) and bases of left eigenvectors (Y and V). Under the non-degeneracy assumption (2.136), the* reduced *Sylvester equation*

$$(Y^* A^{(1)} X)\Gamma^{(-1)} + \Gamma^{(-1)}(V^* B^{(1)} U) = Y^* C^{(0)} U,$$

has a unique solution and the perturbed solution of the Sylvester equation (2.117) can be expanded in the Laurent series (2.138) with

$$X^{(-1)} = X\Gamma^{(-1)} V^*$$

and

$$X^{(k)} = X_*^{(k)} + X\Gamma^{(k)} V^*,$$

where $X_*^{(k)}$ is any particular solution of (fs k+1) and the matrix $\Gamma^{(k)}$ is a solution of the following Sylvester equation

$$(Y^* A^{(1)} X)\Gamma^{(k)} + \Gamma^{(k)}(V^* B^{(1)} U) =$$

$$= Y^* \left(C^{(k+1)} - A^{(1)} X_*^{(k)} - X_*^{(k)} B^{(1)} - \sum_{s=2}^{k+2} [A^{(s} X^{(k+1-s)} + X^{(k+1-s)} B^{(s)}] \right) U$$

□

Again, we would like to emphasize that to construct the Laurent series (2.138), we need to solve a series of new *reduced* Sylvester equations but with coefficient matrices of much smaller dimensions. This fact is especially useful in the case of higher order singularities. Namely, after each reduction step (see Section 2.2), we obtain new fundamental equations with coefficient matrices of smaller dimensions but the structure of these fundamental equations is essentially the same, that is, in performing the reduction process, one has to solve a series of Sylvester equations of smaller and smaller dimensions.

Chapter 3

Applications to Markov Chains and Markov Decision Processes

3.1 Background and literature review

First in this introductory section we review some known facts from Markov chains theory and the theory of Markov decision processes. Then, we briefly discuss the results that are obtained in the ensuing sections. We will also present the literature review on the perturbation analysis of Markov chains and Markov decision processes.

Since Markov chains theory is by now a classical subject, we will state the known results without the proof. A reader who might be interested in the detail development of the theory is referred to the following books: [79, 80, 89]. In the present thesis we are only concerned with the finite state space Markov chains. Namely,

Definition 7 *A sequence of random variables $\{X_t\}_{t \geq 0}$, whose values belong to a finite set $S = \{1, ..., n\}$ is said to be a (homogeneous finite)* Markov chain *with state space S, initial distribution $\alpha = \{\alpha_i\}_{i=1}^{n}$ and transition matrix $P = \{p_{ij}\}_{i,j=1}^{n}$ if and only if $P\{X_0 = i\} = \alpha_i, i \in S$ and*

$$P\{X_{t+1} = i_{t+1} | X_t = i_t, ..., X_0 = i_0\} = P\{X_{t+1} = i_{t+1} | X_t = i_t\} = p_{i_t, i_{t+1}} \tag{3.1}$$

for all $t \geq 0$ and $i_0, ..., i_{t+1} \in S$. The above equation is called Markov property.

Markov chains can also be defined in more general settings of denumerable state space [90] and continuous state space [68, 110, 122]. Homogeneity is introduced by the second equality in (3.1) which shows that the conditional probability of state i_{t+1} at time $t + 1$ given state i_t at time t has a prescribed value independent of t. If we denote the distribution of a discrete random variable X_t by $x_t \in R^{1 \times n}$, then the evolution of the process is given by the following matrix equation

$$x_{t+1} = x_t P = \alpha P^{t+1}, \tag{3.2}$$

where matrix P as in (3.1). If the Markov chain is aperiodic [89], then the powers of transition matrix P converge to a limit. In general, however, one has to consider the *Cesaro limit*

$$P^* = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} P^t, \tag{3.3}$$

which is known to exist always [79, 89]. The limit P^* is also referred to as *ergodic projection*. Indeed, P^* is an eigenprojection of transition matrix P corresponding to the eigenvalue one [87, 89]. We call the Markov chain *ergodic*, if for any two states there is a positive probability of moving from one state to another in a finite number of transitions. In the case of ergodic Markov chains, the Cesaro limit can be easily constructed. Namely, we first determine the *stationary distribution* or *invariant measure* as a solution of the linear system

$$\begin{cases} \mu P = \mu, \\ \mu \xi_n = 1, \end{cases}$$

where $\xi_n = [1 \cdots 1]^T \in R^{n\times 1}$. Then, the ergodic projection is given by

$$P^* = \xi_n \mu. \tag{3.4}$$

Note that P^* has identical rows. This demonstrates that in the ergodic case the starting state has no influence on the long run behaviour of the chain. However, this is not the case in general. In the general *multi-chain* case one can always renumerate the states in such an order that the transition matrix will take the following *canonical* form:

$$P = \begin{array}{c} \left[\begin{array}{cccc} P_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & P_n & 0 \\ R_1 & \cdots & R_n & S \end{array}\right] \end{array} \begin{array}{l} \} \, \Omega_1 \\ \quad \vdots \\ \} \, \Omega_n \\ \} \, \Omega_T, \end{array}$$

where the set of states Ω_i represents i-th ergodic class with transition matrix P_i and Ω_T represents the set of *transient* states. Let $N = |\Omega_1| + ... + |\Omega_n| + |\Omega_T|$ denote the total number of states in the Markov chain. Note that the elements of submatrix S are transition probabilities inside the transient set and the elements of R_i represent the one step probabilities of transition from the transient states to the ergodic states of class Ω_i. Often it is more convenient to use the Markov chain generator $G = P - I$ rather than the transition matrix itself. We will use the following notations for the generator in the canonical form:

$$G = \begin{bmatrix} A_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_n & 0 \\ R_1 & \cdots & R_n & T \end{bmatrix} \begin{array}{l} \} \, \Omega_1 \\ \vdots \\ \} \, \Omega_n \\ \} \, \Omega_T, \end{array}$$

where $A_i = P_i - I$ and $T = S - I$. In the multi-chain case the ergodic projection P^* can be given in terms of invariant measures of the ergodic classes Ω_i. However, the expression is more involved than the formula (3.4). First, we form the matrix of invariant mesures

$$M = \begin{bmatrix} m_1 \\ \cdots \\ m_n \end{bmatrix} \in R^{n \times N},$$

where $m_i = [0 \cdots 0 \ \mu_i \ 0 \cdots 0]$ with invariant measure μ_i placed in correspondence with the ergodic class Ω_i. Of course, $\mu_i \in R^{1 \times |\Omega_i|}$ can be found from the solution of the system

$$\begin{cases} \mu_i A_i = 0, \\ \mu_i \xi_{\Omega_i} = 1, \end{cases}$$

where ξ_{Ω_i} is a vector of ones with length $|\Omega_i|$. Next, we form the matrix of probabilities of absorption in one of the ergodic classes.

$$Q = [q_1 \cdots q_n] \in R^{N \times n},$$

where

$$q_i = \begin{bmatrix} 0 \\ \xi_{\Omega_i} \\ 0 \\ \varphi_i \end{bmatrix} \begin{array}{l} \\ \} \, \Omega_i \\ \\ \} \, \Omega_T \end{array} \tag{3.5}$$

where the j-th element of the vector q_i represents the probability that the process initiated in state j will be absorbed in the i-th ergodic class. The subvector φ_i can be calculated by the next formula

$$\varphi_i = (I - S)^{-1} R_i \xi_{\Omega_i} = -T^{-1} R_i \xi_{\Omega_i}. \tag{3.6}$$

Then, the ergodic projection is given by

$$P^* = QM. \tag{3.7}$$

Note that $MQ = I$. Clearly, the above formula is a generalization of (3.4).

In Section 3.2 we analyse the most general case of analytic perturbation of finite Markov chains. That is, the generator of the perturbed Markov chain $G(\varepsilon)$ is an analytic function of a perturbation parameter ε. Hence, it can be represented by power series

$$G(\varepsilon) = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + ...,$$

where G_0 is a generator of the unperturbed Markov chain. Even though the above power series can converge in some complex neighbourhood around $\varepsilon = 0$, we will only consider some real interval $[0, \varepsilon_{max}]$ where the elements of matrix $P(\varepsilon)$ are non-negative reals whose values are less or equal than one. We would like to emphasize that we make no assumption at all about the structure of the unperturbed and perturbed Markov chains. By contrast, most previous investigations were carried out under certain (e.g. ergodicity) assumptions. Next we provide a literature review on this topic, we also refer the reader to the recent surveys by Abbad and Filar [2] and Avrachenkov, Filar and Haviv [7].

The study of perturbed Markov chains was originated by Schweitzer in [131]. In that fundamental paper Schweitzer has rigorously analysed the regular perturbation of Markov chains. The regular perturbation is a perturbation that does not change the structure of the Markov chain, that is (a) the number of ergodic classes does not change with perturbation and (b) the states of the perturbed Markov chain belong to the same ergodic class as before perturbation. Schweitzer has studied the case of a linear perturbation $G(\varepsilon) = G_0 + \varepsilon G_1$, which is a particular case of more general analytic perturbation (3.17). In particular, he derived closed-form updating formulae for the left and right eigenvectors corresponding to the zero eigenvalue of the Markov chain generator. Recall that the left eigenvectors are invariant measures and the right eigenvectors provide probabilities to be absorbed in one of the ergodic classes. For example, the following updating formula [131] is available for the invariant measure of the perturbed Markov chain:

$$m(\varepsilon) = m(0)[I - \varepsilon G_1 Z(0)]^{-1}, \tag{3.8}$$

where $Z(0) = [I - P(0) + P^*(0)]^{-1}$ is a *fundamental matrix* [89] of the unperturbed Markov chain. In [131] Schweitzer, by presenting an illustrative example, has also pointed out a more challenging case of a singular perturbation. This example shows the essence of singular perturbation. Let us consider the perturbed Markov chain with the follow transition matrix

$$P(\varepsilon) = \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$$

Then, it is easy to see that the ergodic projection is given by

$$P^*(\varepsilon) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \varepsilon = 0 \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, & 0 < \varepsilon \leq 1 \end{cases}$$

From the above example, we can see that the ergodic projection has a discontinuity at $\varepsilon = 0$. The explanation for this fact is that the perturbed chain has less ergodic classes than the original chain. And hence, as follows from formula (3.7), the ergodic projection corresponding to the unperturbed chain has a larger rank than the one corresponding to the perturbed Markov chain.

Probably, the first motivation to study the singular perturbed Markov chains was given in the paper by Simon and Ando [137]. They demonstrated that several problems in econometrics lead to the mathematical model based on singularly perturbed Markov chains. The first rigorous theoretical developments of the singularly perturbed Markov chains have been carried out by Pervozvanski and Smirnov [117] and Gaitsgori and Pervozvanskii [53]. They have used so-called *aggregation* approach. The basic idea of this approach is to consider the ergodic classes of the original chain as states for some aggregated Markov chain which is responsible for rare transitions between these classes in the presence of perturbations. Similar ideas were developed in the work by Courtois and his co-authors [32, 33, 34]. As in [117, 53], they have also assumed that the unperturbed Markov chain consists of several ergodic classes without a set of transient states, and after the perturbation the Markov chain becomes ergodic. Since in this case the transition matrix of the unperturbed chain is block-diagonal, one often refers to this problem as the case of *nearly completely decomposable* Markov chains.

Delebecque and Quadrat [39] were first who have investigated a more general case, when the original Markov chain has a set of transient states in addition to the ergodic classes. Additional theoretical and computational development in this more general setting were given in [16, 91, 125, 146]. In particular, Korolyuk and Turbin [91] and later Bielecki and Stettner [16] have analysed the perturbation of Markov chains with transient states in the context of general Borel state space. In all papers reviewed above the authors considered a two time scale model. However, it turns out that in the presence of transient states the perturbed Markov chain exhibits multi-time scale behaviour. This phenomenon was thoroughly investigated in the fundamental paper of Delebecque [40]. In that paper, Delebecque has introduced the analytic perturbation of Markov chains and applied Kato's approach [87] based on spectral theory. Coderch *et al* [29, 30] have carried out similar development for continuous-time Markov processes. They have also demonstrated that their multi-scale approximations converge uniformly on the time interval $[0, \infty)$.

The study of *continuous-time* singularly perturbed Markov chains was originated in the paper by Phillips and Kokotovic [118] and then it proceeded pretty much in parallel with the developments of discrete-time model. The reader, who is interested in the analysis of singular perturbations for continuous-time Markov chains, is referred to the recent comprehensive book by Yin and Zhang [151]. In particular, Yin and Zhang [151] provide numerous applications of singularly perturbed Markov chains. In early 80's Schweitzer (see technical report [132] and later journal publication [134]) has generalized his updating formula (3.8) to the case of singular perturbations. The new updating formula has the same structure as (3.8)

$$m(\varepsilon) = m_0[I - \varepsilon G_1 Z_0]^{-1}, \tag{3.9}$$

with the unperturbed invariant mesure $m(0)$ changed to the *limiting* invariant measure $m_0 = \lim_{\varepsilon \to 0} m(\varepsilon)$ and the fundamental matrix $Z(0)$ of the unperturbed chain changed to the first regular coefficient Z_0 of the Laurent series expansion for $Z(\varepsilon)$, the fundamental matrix of the perturbed chain, at $\varepsilon = 0$.

Note that Schweitzer, as many other researchers, has assumed that the perturbed Markov chain is a unichain, that is after the perturbation there is only one ergodic class and possibly some transient states. The unichain assumption implies that the perturbed chain has a unique invariant measure. This assumption significantly facilitates the analysis. To the best of our knowledge, only Delebecque [40] has managed to overcome this assumption. However, his results do not provide a clear picture of the ergodic structure of the perturbed chain.

In the ensuing Section 3.2, following Delebecque [40], we analyse the perturbation of finite Markov chains in the most general setting, when the perturbed Markov chain has an arbitrary multi-chain structure. Our approach is based on algebraic techniques of Sections 2.2 and 2.4 and reveals the ergodic structure of the perturbed process. Our algebraic reduction technique is close to the one used by Haviv and Ritov [66, 67]. In particular, we investigate the case when some ergodic classes merge with the set of transient states.

In this literature review we also would like to mention the papers by Hunter [76, 77, 78] and Seneta [136], where the authors investigate the rank one perturbation of Markov chains and derive several updating formulae. Probably, the most general updating formula was obtained by Lasserre [99]. The paper [99] treats the case of singular *additive* perturbations, that is the perturbed transition matrix is given in the form $P' = P + D$. Note that the rank one perturbation is a particular case of additive perturbation with $D = ab^T$. As was shown in the paper of Abbad and Filar [1], there is no limit for the ergodic projection as $||D|| \to 0$ in the case of general additive perturbation.

Finally, we would like to mention the paper of Hassin and Haviv [63], where the authors proposed a combinatorial algorithm to deteremine the highest time scale needed to model the singularly perturbed Markov chain. Also many bounds and error estimations have been obtained for the approximation models of perturbed Markov chains [64, 105, 109, 138, 142].

In Section 3.3 we obtain several results on the asymptotic behaviour of the fundamental matrix of the singularly perturbed Markov chain. Then we apply these results to the perturbation analysis of mean first passage times. Before proceeding further, let us briefly review the known facts about the fundamental matrix and the mean first passage times. Let P be a transition matrix of a Markov chain and let P^* be the associated ergodic projection, then the *fundamental matrix* is defined as follows:

$$Z := [I - P + P^*]^{-1}.$$

Another equivalent definition of the fundamental matrix can be given in the form of matrix series

$$Z := \lim_{T\to\infty}(\mathrm{c}) \left[\sum_{t=0}^{T}(P - P^*)^t\right] = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \left[\sum_{n=0}^{t}(P - P^*)^n\right],$$

where lim(c) denotes a Cesaro limit. Of course, if the chain is aperiodic, we have the convergence in the usual sense. If P^* expresses the ergodic (long-run) behaviour of the chain, then, according to the second definition, matrix Z represents the transient (short-run) behaviour of the Markov chain. As was already mentioned above, the fundamental matrix is very useful in the perturbation analysis of Markov chains. Another important application of the fundamental matrix is the mean first passage times.

Definition 8 *If an ergodic Markov chain is initiated in state i, the expected number of transitions to reach state j for the first time is called the* mean first passage time *from i to j. It is denoted by m_{ij}.*

Obviously, the mean first passage time has a sensible definition only for the *ergodic* chains. Once we have in hand the fundamental matrix Z, the mean first passage time can be immediately computed by the following simple formula

$$m_{ij} = \frac{z_{jj} - z_{ij}}{\mu_j}, \tag{3.10}$$

where $\mu = [\mu_1 \cdots \mu_N]$ is an invariant measure of the Markov chain. By convention, $m_{ii} = 1/\mu_i$, which is the expected return time.

As was shown by Schweitzer [132, 134], the fundamental matrix of the singularly perturbed Markov chain can be expanded as a Laurent series

$$Z(\varepsilon) = [I - P(\varepsilon) + P^*(\varepsilon)]^{-1} = \frac{1}{\varepsilon^s} Z_{-s} + ... + \frac{1}{\varepsilon} Z_{-1} + Z_0 + \varepsilon Z_1 + ... \tag{3.11}$$

Schweitzer has considered the case of linear perturbation $P(\varepsilon) = P_0 + \varepsilon P_1$ and applied his early results on the inversion of singularly perturbed matrices (see also [133]). However, his approach seems to be excessively complicated. In particular, Schweitzer [132] has derived an intricate updating formula for the fundamental matrix of the singularly perturbed Markov chain. By contrast, we propose to use a natural *reduction* technique to compute the coefficients of the Laurent series (3.11). The reduction technique allows us to deal with the analytic perturbation as well as with the linear perturbation. We would like to note that our approach can be viewed as an extension of the results by Haviv and Ritov [66]. They have applied only one reduction step. Furthermore, for the case of linear (possibly singular) perturbation we have derived a *simple* updating formula for the regular part of $Z(\varepsilon)$. This formula is much simpler than the corresponding Schweitzer's updating formula [132] and it can be viewed as a generalization of another simple formula obtained by Schweitzer [131] for the case of regular perturbations.

Note that in Subsection 3.3.3 we prefer to obtain first the Laurent series for the deviation matrix

$$H(\varepsilon) = Z(\varepsilon) - P^*(\varepsilon)$$

rather than straight away for the fundamental matrix $Z(\varepsilon)$. There are several reasons for this. In particular, it is easier to implement the reduction process for the deviation matrix. Of course, once the Laurent series for deviation matrix is obtained, we can immediately calculate the Laurent series for the fundamental matrix. Moreover, note that the deviation matrix is the negative group inverse of the Markov chain generator. Actually, the theoretical basis for the computational procedure of Subsection 3.3.3 was already outlined in Section 2.4 where the perturbation of group inverses was considered. Alternatively, the results of Section 2.5 (see example there) based on spectral theory can be applied to obtain the Laurent series of the deviation matrix. However, to complete the development of Section 2.4 we have decided to pursue algebraic approach.

Finally, we would like to mention the work by Latouche and Louchard [100] and Latouche [101], where the authors have directly computed the expansions of mean first passage times for the singular perturbed Markov chain. However, they have assumed a block-diagonal structure of the unperturbed Markov chain. Because of this fact, the Laurent series for the first passage times can have only a simple pole. This is not the case if the original Markov chain has some transient states. By contrast, our approach does not impose any restrictions on the ergodic structure of the unperturbed chain and can treat a singularity of any order.

The example of Haviv and Ritov [66, 67] is used in Subsection 3.3.4 to illustrate the theoretical development and its application to the mean first passage times.

Since in the following sections we are going to apply the general results of analytic perturbation theory to Markov decision processes (MDPs, for short), let us briefly introduce MDPs and discuss different optimality criteria. An interested reader can find a more detailed study of MDP in the following books and surveys [45, 68, 72, 85, 86, 119, 148] and references therein.

In this thesis we consider a discrete-time MDP with a finite state space $\mathbb{X} = \{1, ..., N\}$ and a finite action space $\mathbb{A}(i) = \{1, ..., m_i\}$ for each state $i \in \mathbb{X}$. At any time point t the system is in one of the states $i \in \mathbb{X}$ and the controller or "decision-maker" chooses an action $a \in \mathbb{A}(i)$; as a result the following occur: (a) the controller gains an immediate reward $r(i, a)$, and (b) the process moves to a state $j \in \mathbb{X}$ with transition probability $p(j|i, a)$, where $p(j|i, a) \geq 0$ and $\sum_{j \in \mathbb{X}} p(j|i, a) = 1$.

A *decision rule* π_t at time t is a function which assigns a probability to the event that any particular action a is taken at time t. In general, π_t may depend on all history $h_t = (i_0, a_0, i_1, a_1, ..., a_{t-1}, i_t)$ up to time t. The distribution $\pi_t(a_t|h_t)$ defines the probability of selecting the action a_t at time t given the history h_t.

A *policy* (or *strategy*) is a sequence of decision rules $\pi = (\pi_0, \pi_1, ..., \pi_t, ...)$. A policy π is called *Markov* if $\pi_t(\cdot|h_t) = \pi_t(\cdot|i_t)$. If $\pi_t(\cdot|i) = \pi_{t'}(\cdot|i)$ for all $t, t' \in \mathbb{N}$ then the Markov policy π is called *stationary*. Furthermore, a *deterministic* policy π is a stationary policy whose single decision rule is nonrandomized. It can be defined by the function $f(i) = a, a \in \mathbb{A}(i)$.

Let Π, Π^S and Π^D denote the sets of all policies, all stationary policies and all deterministic policies, respectively. Note that often we can restrict ourselves to stationary or even deterministic policies [45, 47, 48].

For any stationary policy $\pi \in \Pi^S$ we can define the corresponding transition matrix $P(\pi) = \{p_{ij}(\pi)\}_{i,j=1}^N$ and the reward vector $r(\pi) = \{r_i(\pi)\}_{i=1}^N$

$$p_{ij}(\pi) := \sum_{a \in \mathbb{A}(i)} p(j|i, a)\pi(a|i), \quad r_i(\pi) := \sum_{a \in \mathbb{A}(i)} r(i, a)\pi(a|i).$$

The *expected average reward* $g_i(\pi)$ and the *expected discounted reward* $v_i^\lambda(\pi)$ can be expressed as follows:

$$g_i(\pi) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left[P^{t-1}(\pi) r(\pi)\right]_i$$

and

$$v_i^\lambda(\pi) := \sum_{t=1}^{\infty} \lambda^{t-1} \left[P^{t-1}(\pi) r(\pi)\right]_i = \left[(I - \lambda P(\pi))^{-1} r(\pi)\right]_i,$$

where $i \in \mathbb{X}$ is an initial state and $\lambda \in (0,1)$ is a discount factor. One can also use the interest rate $\rho = (1-\lambda)/\lambda \in (0,\infty]$ instead of the discount factor λ. Note that $\lambda \uparrow 1$ is equivalent to $\rho \downarrow 0$. Then the expected discount reward $v^\rho(\pi) := v^{\lambda(\rho)}(\pi)$ can be expanded as a Laurent series [112, 147, 95] for sufficiently small ρ

$$v^\rho(\pi) = (1+\rho)\left[\rho^{-1} y_{-1}(\pi) + \sum_{n=0}^{\infty} \rho^n y_n(\pi)\right], \tag{3.12}$$

where $y_{-1}(\pi) = P^*(\pi) r(\pi)$, $y_0 = H(\pi) r(\pi)$, and $y_n = (-1)^n H(\pi)^{n+1} r(\pi)$, $n = 1, 2, \ldots$. Note that the above series can be considered as a particular case of resolvent expansion [87, 129], which in turn can be viewed as a particular case of the inversion of singularly perturbed matrices. The *ergodic projection* $P^*(\pi)$ and the *reduced resolvent* (or *deviation matrix*) $H(\pi)$ associated with policy π are defined by

$$P^*(\pi) := \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P^{t-1}(\pi),$$

$$H(\pi) := (I - P(\pi) + P^*(\pi))^{-1} - P^*(\pi).$$

The next properties of these matrices will be used extensively.

$$P^*(\pi)P(\pi) = P(\pi)P^*(\pi) = P^*(\pi) \tag{3.13}$$

$$P^*(\pi)H(\pi) = H(\pi)P^*(\pi) = 0 \tag{3.14}$$

$$(I - P(\pi))H(\pi) = H(\pi)(I - P(\pi)) = I - P^*(\pi) \tag{3.15}$$

Note that in particular $y_{-1}(\pi) = g(\pi)$ and $y_0 = h(\pi)$, where $g(\pi)$ is the expected average reward vector defined above and $h(\pi)$ is so-called the expected bias reward vector.

Let us now discuss several optimality criteria [45, 85, 119, 147, 148]. The discount optimality and the average optimality are basic criteria in MDP models.

Definition 9 *The stationary policy π_* is called the discount optimal for fixed $\lambda \in (0,1)$ if $v_i^\lambda(\pi_*) \geq v_i^\lambda(\pi)$ for each $i \in \mathbb{X}$ and all $\pi \in \Pi^S$.*

Definition 10 *The stationary policy π_* is called the average optimal if $g_i(\pi_*) \geq g_i(\pi)$ for each $i \in \mathbb{X}$ and all $\pi \in \Pi^S$.*

It is well known that there exist deterministic discount and average optimal policies, which can be computed by a number of efficient algorithms [45, 85, 119]. The Laurent expansion (3.12) allows us to define more selective optimality criteria [112, 147, 148].

Definition 11 *We say that a policy $\pi^* \in \Pi^S$ is n-discount optimal for some integer $n \geq 0$ if*

$$[y_n(\pi^*)]_i \geq [y_n(\pi)]_i$$

for each $i \in \mathbb{X}$ and all policies π that are $(n-1)$-discount optimal. By convention, the average optimal policy is (-1)-discount optimal policy.

In particular, 0-discount optimal policy is called *bias optimal*.

Moreover, it is known that there exists a stationary (and even deterministic) policy that is n-discount optimal for all $n \geq -1$ [112, 147, 148]. The latter is referred to as the *Blackwell optimal policy* [19] and defined in the following way.

Definition 12 *We say that a policy π_* is Blackwell optimal if there exists some $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$ $v^\rho(\pi^*) \geq v^\rho(\pi)$ for all $\pi \in \Pi^S$.*

In other words, Blackwell optimal policy is the policy which is discount optimal policy for any discount factor sufficiently close to one. We shall give more detail discussion and literature review on n-discount optimality and Blackwell optimality in Section 3.5.

The results of analytic perturbation theory have several applications to Markov decision processes. The first example is the Laurent series (3.12) for the expected discount reward. Another important application is the singularly perturbed MDP. For the clarity of exposition we restrict ourselves to the case of a linear perturbation. Namely, we assume that the transition probabilities of the perturbed MDP are given by

$$p^\varepsilon(j|i,a) = p(j|i,a) + \varepsilon d(j|i,a) \geq 0, \tag{3.16}$$

where $p(j|i,a)$ are transition probabilities of the original unperturbed chain, $\sum_j d(j|i,a) = 0$, and ε is a "small" perturbation parameter. We are especially interested in the case of *singular* perturbations, that is when the perturbation changes the ergodic structure of the underlying Markov chain.

As the following example shows, the policies that are optimal for the unperturbed MDP may not coincide with the optimal policies for the perturbed MDP.

Example 1 *Let us consider an MDP model with $\mathbb{X} = \{1,2\}$, $\mathbb{A}(1) = \{a_1, b_1\}$, $\mathbb{A}(2) = \{a_2\}$ and*

$$p^\varepsilon(1|1,a_1) = 1, \quad p^\varepsilon(2|1,a_1) = 0;$$

$$p^\varepsilon(1|1,b_1) = 1-\varepsilon, \quad p^\varepsilon(2|1,b_1) = \varepsilon;$$

$$p^\varepsilon(1|2,a_2) = \varepsilon, \quad P^\varepsilon(2|2,a_2) = 1-\varepsilon;$$

$$r(1,a_1) = 1, \quad r(1,b_1) = 1.5, \quad r(2,a_2) = 0$$

There are only two deterministic policies, $u = [a_1, a_2]$ and $v = [b_1, a_2]$. For these policies one can easily calculate the average reward vectors. Namely,

$$g^\varepsilon(u) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \varepsilon = 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \varepsilon > 0 \end{cases} \quad \textit{and} \quad g^\varepsilon(v) = \begin{cases} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} & \varepsilon = 0 \\ \begin{bmatrix} 0.75 \\ 0 \end{bmatrix} & \varepsilon > 0 \end{cases}$$

Thus, we can see that for $\varepsilon = 0$ the average optimal policy is v, whereas for $\varepsilon > 0$ the average optimal policy is u.

Since often we do not know the exact value of the perturbation parameter ε, we are interested in finding the policy which is "close" to the optimal one for ε small but different from zero. We will call it a *suboptimal* policy or a *limit control optimal* policy. A strict definition will be given later in this section.

Most of the previous research was carried out with the assumption that the unperturbed MDP is *completely decomposable* and the perturbed process is ergodic. More precisely, one can introduce the following three assumptions:

A1) $\mathbb{X} = \cup_{k=1}^n \mathbb{X}_k$, where $\mathbb{X}_k \cap \mathbb{X}_l = \emptyset$ if $k \neq l$, $n > 1$, and $\sum_{k=1}^n card(\mathbb{X}_k) = N$

A2) $p(j|i, a) = 0$ whenever $i \in \mathbb{X}_k, j \in \mathbb{X}_l$ and $k \neq l$.

A3) For every $i = 1, ..., n$ the unperturbed MDP associated with the subspace $\mathbb{X}_k$ is ergodic.

A4) The transition matrix $P^\varepsilon(\pi)$ is irreducible for any $\pi \in \Pi^S$ and any ε sufficiently small but different from zero, that is the perturbed MDP is ergodic.

The structure defined by assumptions A1)-A4) has a clear interpretation. The perturbed MDP model can be viewed as a complex system consisting of n "weakly-interacting" subsytems associated with $\mathbb{X}_k, k = 1, ..., n$. Note that perturbation $\varepsilon d(j|i, a)$, where i and j are the states of different subsystems $\mathbb{X}_k$ and $\mathbb{X}_l$ respectively, represents the probability of rare transitions between the subsystems, which are independent in the unperturbed chain.

This model of singularly perturbed MDP was first studied by Pervozvanski and Gaitsgory [116]. They proposed an *aggregation-disaggregation* algorithm for the computation of suboptimal policies for the perturbed MDP with average and discount optimality criteria. Furthermore, they have shown that in the case of discount optimality criterion an optimal policy for the original problem is also suboptimal for the perturbed problem. The latter is not true in general if one uses average optimality criterion. In particular, if the perturbation is singular, the suboptimal policy of the perturbed problem can be quite different from the optimal policies of the unperturbed problem. The explanation for this phenomenon is that the interaction between weakly-coupled subsytems in the singularly perturbed process makes an effect only after sufficiently long time interval, and, if one uses discounting, the end of the trajectory has no contribution to the expected discount reward. To investigate this phenomenon, Delebecque and Quadrat [39] proposed to consider the discount optimality criterion, where the interest rate goes to zero (equivalently, the discount factor goes to one) as the perturbation parameter vanishes. In particular, in [39] Delebecque and Quadrat assumed that $\rho = \rho(\varepsilon) = \mu\varepsilon$. The latter allows to exhibit the two time scale behaviour of the singularly perturbed MDP. In addition, the model in [39] can admit a set of transient states. Delebecque and Quadrat solved the dynamic programming equation for the perturbed model by using an approach based on lexicographical ordering. The concept of lexicographical ordering was first used in MDP by Veinott [147] to calculate a Blackwell optimal policy. Moreover, as one can see from the futher discussion, the singular perturbed MDPs have a close relation with Blackwell optimality [19, 112, 147, 119, 148] and Markov branching decision chains (for comprehensive study of Markov branching chains an interested reader is refered to the paper by Huang and Veinott [74]). Later, based on the results of Delebecque [40], Quadrat [120] extended the results of [39] to include the possibility of several time scales. The key point in [40] and [120] is the use of the following dependency of the interest rate on the perturbation parameter $\rho = \rho(\varepsilon) = \mu\varepsilon^l$, where l represents the order of a time scale. Altman and Gaitsgory [4] have generalized the results of [116] to constrained Markov decision processes.

Abbad, Bielecki and Filar [1, 3, 15, 2] formulated the *limit control principle*, which provides a formal setting for the concept of suboptimal policies. First, we note that for any stationary policy $\pi \in \Pi^S$, there exists a limiting ergodic projection

$$\hat{P}^*(\pi) := \lim_{\varepsilon \to 0} P^{*,\varepsilon}(\pi).$$

The average reward optimization problem for the perturbed MDP can be written in the form

$$g_i^{opt,\varepsilon} = \max_{\pi \in \Pi^S} [P^{*,\varepsilon}(\pi) r(\pi)]_i \quad (L^\varepsilon).$$

The limit control principle says that instead of the above singular program one can consider a well-defined *limit Markov control problem*:

$$\hat{g}_i^{opt} = \max_{\pi \in \Pi^S} [\hat{P}^*(\pi) r(\pi)]_i \quad (L).$$

It is natural to expect that an optimal strategy, if it exists, to (L) could be approximately optimal for the perturbed problem (L^ε), when the perturbation parameter is small. Bielecki and Filar [15] have shown the this fact indeed takes place. Namely, let π_* be any maximizer in (L), then

$$\lim_{\varepsilon\to 0}\max_{i\in\mathbb{X}}|g_i^\varepsilon(\pi_*)-g^{opt,\varepsilon}|=0$$

Abbad and Filar [1] proved that there exists a deterministic policy that solves the limit control problem (L). Furthermore, in [3] Abbad, Filar and Bielecki provided two methods for the computation of optimal policy for (L). Their first algorithm is based on linear programming approach and the second algorithm is similar to the one constructed by Pervozvanski and Gaitsgory [116]. Both algorithms essentially use the *disaggregation-aggregation* approach. Recently, Bielecki and Stettner [16] have generalized the limit control principle to MDPs with general Borel state spaces and compact action spaces.

As a result of the solution of problem (L), one gets *limit control optimal* policy $f_*\in\Pi^D$ such that for any other deterministic policy $f\in\Pi^D$ the following holds

$$\lim_{\varepsilon\to 0}(g_i^\varepsilon(f_*)-g_i^\varepsilon(f))\geq 0,\quad i\in\mathbb{X}.$$

However, as was noted in [1], a policy that solves (L), in general, is only suboptimal. Remarkably, there exists a policy that is optimal for all sufficiently small ε but is not necessarily optimal for $\varepsilon=0$. This policy is called *uniform optimal* [5] and it satisfies the following inequality

$$g_i^\varepsilon(f_*)\geq g_i^\varepsilon(f),\quad i\in\mathbb{X},$$

for any deterministic policy f and all ε sufficiently small but different from zero. Again, we would like to emphasize that a uniform optimal policy is limit control optimal, but the converse need not hold, as the following example illustrates.

Example 2 *Consider* $\mathbb{X}=\{1,2\}$, $\mathbb{A}(1)=\{a_1,b_1\}$, $\mathbb{A}(2)=\{a_2\}$; *let*

$$\begin{array}{ll} p^\varepsilon(1|1,a_1)=1, & r(1,a_1)=10\\ p^\varepsilon(1|1,b_1)=1-\varepsilon, & r(1,b_1)=10\\ p^\varepsilon(2|1,b_1)=\varepsilon, & \\ p^\varepsilon(1|2,a_2)=1, & r(2,a_2)=0 \end{array}$$

Then the stationary policy $u(1)=a_1, u(2)=a_2$ *is uniformly optimal with expected average reward* $g_i^\varepsilon(u)=10$. *The stationary policy* $v(1)=b_1, v(2)=a_2$ *is limit control optimal as* $\lim_{\varepsilon\to 0}g_i^\varepsilon(v)=10$, *but for every* $\varepsilon>0$,

$$g_i^\varepsilon(v)=\frac{10}{1+\varepsilon}<g_i^\varepsilon(u).$$

Abbad and Filar [1] have also proposed an heuristic procedure to determine the uniform optimal policy. However, their method has two disadvantages. First, they need to consider a series of problems (L^{ε_k}), where the sequence $\{\varepsilon_k\}_{k=1}^\infty$ converges to zero.

And hence they have to deal with a series of problems that become more and more ill-conditioned. Second, it is not possible to say exactly when one has to terminate their procedure. In contrast, we propose in Section 3.4 a well-defined computational procedure which is based on asymptotic linear programming. The *asymptotic linear programming* was first introduced by Jeroslow [81, 82] and later refined by Hordijk, Dekker and Kallenberg [71]. We would like to note that instead of asymptotic linear programming one can use *asymptotic simplex method*, which will be discussed in detail in the next chapter. Moreover, if the asymptotic linear programming or asymptotic simplex method is used, then we can drop Assumptions A1)-A4) and consider the most general situation.

As can be concluded from the above overview of Markov decision processes, there is an intrinsic relationship between the Blackwell optimality and singular perturbations. The work of Delebecque and Quadrat [39, 40, 120] clearly emphasize this connection. Therefore we devote Section 3.5 to Blackwell optimality criterion. In Section 3.5 we propose an algorithm based on the system of nested linear programs. This algorithm is an extension of the procedure proposed by Denardo [42, 44] for bias optimality criterion and it can be related to the algorithm proposed by Huang and Veinott [74] for branching Markov decision chains. It can be considered as an alternative approach to the algorithm based on nested policy improvements, which is proposed by Puterman [119]. A reader will find a more detail discussion of literature review for the Blackwell optimality in the special Subsection 3.5.1.

Moreover, in the ensuing Section 4.2, using the asymptotic simplex method, we demonstrate how Blackwell optimality, singularly perturbed MDPs and branching Markov decision chains can be considered in a unified framework.

3.2 Multichain singularly perturbed Markov chains

3.2.1 Preliminaries

In this section we are going to treat the most general case of analytic perturbation of Markov chains. Namely, we only assume that the generator of the perturbed chain is an analytic function and hence that it can be expanded as a power series at $\varepsilon = 0$

$$G(\varepsilon) = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + ..., \quad 0 \le \varepsilon \le \varepsilon_{max}, \tag{3.17}$$

where G_0 is a generator of the unperturbed Markov chain. We would like to emphasize that we take *no assumption* about the ergodic structures of the perturbed and unperturbed chains. That is, both the perturbed and unperturbed model can have several ergodic classes and sets of transient states.

The starting point of our analysis is to rearrange the states of the perturbed Markov chain in such a way that the generator $G(\varepsilon)$ can be written in the canonical form

$$G(\varepsilon) = \begin{bmatrix} A_1(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_n(\varepsilon) & 0 \\ R_1(\varepsilon) & \cdots & R_n(\varepsilon) & T(\varepsilon) \end{bmatrix} \begin{array}{l} \} \, \Omega_1 \\ \quad \vdots \\ \} \, \Omega_n \\ \} \, \Omega_T. \end{array} \tag{3.18}$$

This can be done for example by using the efficient classification algorithm of Fox and Landi [52] (see also related *condensation* technique of Balcer and Veinott [9]).

Now note that all invariant measures $m_i(\varepsilon)$ of the perturbed Markov chain can be immediately constructed from the invariant measures of the ergodic classes associated with stochastic subgenerators $A_i(\varepsilon), i = 1, ..., n$. Namely, $m_i(\varepsilon) = [0 \cdots 0 \; \mu_i(\varepsilon) \; 0 \cdots 0]$, where $\mu_i(\varepsilon)$ is uniquely determined by the system

$$\begin{cases} \mu_i(\varepsilon) A_i(\varepsilon) = 0, \\ \mu_i(\varepsilon) \xi_{\Omega_i} = 1. \end{cases}$$

This is exactly the perturbation problem under the ergodicity assumption. Even though it has already been thoroughly analysed by many authors (see the literature review of Section 3.1), to make the presentation self-contained, we show in the next subsection how results of Section 2.4 can be applied to this problem.

The last challenging task that we face in the general case of multi-chain perturbation is the computation of power series for the right eigenvectors of the perturbed Markov chains. The pertubed right eigenvectors contain the probabilities to be absorbed in one of ergodic classes after perturbation. These quantities exhibit the multi-chain ergodic properties of the perturbed chain. This will be done in Subsection 3.2.3 with the help of the results on the inversion of analytically perturbed matrices from Section 2.2.

Finally, using formula (3.7), one can easily construct an asymptotic expansion for the ergodic projection of the perturbed Markov chain.

3.2.2 Ergodic case

To make the present section of the thesis self-contained, let us briefly outline the calculation of the asymptotic expansion for the perturbed invariant measure in the case of an ergodic perturbed chain. One can consider this problem as the perturbation of the left null space of the generator matrix. Therefore the results of Section 2.4 are immediately applicable. Of course, as was noted in the introductory Section 3.1, there are several other different approaches to this problem.

Let us substitute the power series $\mu(\varepsilon) = \mu_0+\varepsilon\mu_1+\varepsilon^2\mu_2...$ and $A(\varepsilon) = A_0+\varepsilon A_1+\varepsilon^2 A_2...$ into the system

$$\begin{cases} \mu(\varepsilon)A(\varepsilon) = 0, \\ \mu(\varepsilon)\xi_N = 1. \end{cases}$$

and equate terms with the same power of ε. The latter results in the system of *fundamental* equations.

$$\begin{array}{rcll} \mu_0 A_0 & = & 0 & (MF0) \\ \mu_1 A_0 + \mu_0 A_1 & = & 0 & (MF1) \\ \mu_2 A_0 + \mu_1 A_1 + \mu_0 A_2 & = & 0 & (MF2) \\ & \vdots & & \vdots \\ \mu_k A_0 + \mu_{k-1} A_1 + \cdots + \mu_1 A_{k-1} + \mu_0 A_k & = & 0 & (MFk) \\ & \vdots & & \vdots \end{array}$$

and the system of normalization conditions

$$\begin{array}{rcll} \mu_0\xi_N & = & I & (MN0) \\ \mu_1\xi_N & = & 0 & (MN1) \\ \vdots & & & \vdots \\ \mu_k\xi_N & = & 0 & (MNk) \\ \vdots & & & \vdots \end{array}$$

Now we can reduce the above system to another equivalent system with the matrix coefficients of smaller dimensions.

Proposition 3 *A solution of the fundamental equations (MF) together with the normalization conditions (MN) is given by the folowing recursive formulae*

$$\mu_0 = \mu_0^{(1)} M \tag{3.19}$$

$$\mu_k = \mu_k^{(1)} M + \sum_{j=1}^{k} \mu_{k-j} A_j H, \quad k \geq 1, \tag{3.20}$$

where the auxiliary sequence $\mu_k^{(1)}$, $k \geq 0$, is a unique solution to the next system of reduced fundamental equations

$$\begin{array}{rcll} \mu_0^{(1)} A_0^{(1)} & = & 0 & (RMF0) \\ \mu_1^{(1)} A_0^{(1)} + \mu_0^{(1)} A_1^{(1)} & = & 0 & (RMF1) \\ & \vdots & & \vdots \\ \mu_k^{(1)} A_0^{(1)} + \mu_{k-1}^{(1)} A_1^{(1)} + \cdots + \mu_1^{(1)} A_{k-1}^{(1)} + \mu_0^{(1)} A_k^{(1)} & = & 0 & (RMFk) \\ & \vdots & & \vdots \end{array}$$

coupled with new reduced normalization conditions

$$\begin{array}{rcll} \mu_0^{(1)}\xi_n & = & I & (RMN0) \\ \mu_1^{(1)}\xi_n & = & 0 & (RMN1) \\ & \vdots & & \vdots \\ \mu_k^{(1)}\xi_n & = & 0 & (RMNk) \\ & \vdots & & \vdots \end{array}$$

where the coefficient matrices $A_j^{(1)} \in R^{n\times n}$, $j \geq 0$, *are given by the formula*

$$A_j^{(1)} = M\left(\sum_{p=1}^{k+1}(-1)^{p-1}\sum_{\nu_1+\cdots+\nu_p=k+1}(\Pi_{j=1}^{p-1}A_{\nu_j}H)A_{\nu_p}\right)Q,$$

and where $M \in R^{n\times N}$ *is a marix whose rows are invariant measures of the unperturbed Markov chain,* $Q \in R^{N\times n}$ *is a matrix of eigenvectors corresponding to the zero eigenvalue of the unperturbed generator (it can be calculated by formula (3.5)) and* $H = [P^* - A]^{-1} - P^*$ *is a deviation matrix of the unperturbed chain.*

Note that the dimension of the coefficients $A_j^{(1)}$, $j \geq 0$ is equal to n, the number of ergodic classes of the unperturbed Markov chain, which is usually much smaller than N, the number of states in the original Markov chain. Moreover, matrix $A_0^{(1)}$ can be considered as a generator of the aggregated Markov chain whose states represent the ergodic classes of the original chain [40, 116]. Now we have several options available. Since the reduced system (RMF) has essentialy the same structure as the original fundamental system (MF), we can perform a series of reduction steps. We terminate the reduction process, say after s steps, when the system

$$\begin{cases} \mu_0^{(s)}A_0^{(s)} = 0, \\ \mu_0^{(s)}\xi_{n_s} = 1, \end{cases}$$

has a unique solution. This approach is similar to the one discussed in the paper by Delebecque [40]. In particular, we obtain the folowing representation for the *limiting* invariant measure

$$\mu_0 = \mu_0^{(s)}M^{(s-1)}\cdots M^{(1)}M,$$

where $M^{(k)}$ is a matrix of invariant measures for the aggregated chain at the k-th reduction step. And the solution to the final step reduced system is given by the recursive formula

$$\mu_k^{(s)} = \sum_{j=1}^{k}\mu_{k-j}^{(s)}A_j^{(s)}H, \quad k \geq 1.$$

Alternatively, we can stop after the first reduction step and then solve the system (RMF) with the help of generalized inverses and augmented matrices (for more detail see Section 2.4.3). The latter approach is in line with the work of Haviv and Ritov [66, 67]. Of course, one can make any number of reduction steps between 1 and s and then apply the approach based on the generalized inverses and augmented matrices.

3.2.3 Multi-chain case

Now we are in a position to analyse the most general case of analytic perturbation when the perturbed chain has a multi-chain structure. As was shown in Subsection 3.2.1, after we have obtained the invariant measures $\mu_1(\varepsilon), ..., \mu_n(\varepsilon)$ for the ergodic classes $\Omega_1, ..., \Omega_n$ of the perturbed Markov chain, we can immediately construct the invariant measures of the entire perturbed chain. They are simply of the form $m_i(\varepsilon) = [0 \cdots 0 \;\; \mu_i(\varepsilon) \;\; 0 \cdots 0]$, where $\mu_i(\varepsilon)$ is put in the place of states which constitute the ergodic class Ω_i.

Now we demonstrate how one can calculate the asymptotic expansions for the right 0-eigenvectors of the perturbed Markov chain generator. The elements of the i-th eigenvector are probability to be absorbed in the ergodic class Ω_i. According to formula (3.5), the right 0-eigenvectors of the perturbed chain can be written in the form

$$q_i(\varepsilon) = \begin{bmatrix} 0 \\ \xi_{\Omega_i} \\ 0 \\ \varphi_i(\varepsilon) \end{bmatrix} \begin{array}{l} \\ \} \; \Omega_i \\ \\ \} \; \Omega_T \end{array} \tag{3.21}$$

where subvector $\varphi_i(\varepsilon)$ is given by

$$\varphi_i(\varepsilon) = -T^{-1}(\varepsilon) R_i(\varepsilon) \xi_{\Omega_i}. \tag{3.22}$$

Note that if some ergodic classes become transient sets after the perturbation, then the matrix valued function $T^{-1}(\varepsilon)$ has a singularity at $\varepsilon = 0$. To explain this phenomenon, let us consider the first term in the perturbation series $T(\varepsilon) = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + ...$

$$T_0 = \begin{bmatrix} \tilde{A}_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \tilde{A}_m & 0 \\ \tilde{R}_1 & \cdots & \tilde{R}_n & \tilde{T} \end{bmatrix}.$$

Blocks $\tilde{A}_1, ..., \tilde{A}_m$ represent the ergodic classes of the original Markov chain which merge with the transient set after the perturbation. Since each of $\tilde{A}_1, ..., \tilde{A}_m$ is a Markov chain generator, we conclude that matrix T_0 has at least m zero eigenvalues and, of course, it is not invertible. However, matrix $T(\varepsilon)$ is invertible for $\varepsilon \neq 0$ and sufficiently small. From the discussion of Sections 2.2 and 2.3, it follows that one can expand $T^{-1}(\varepsilon)$ as a Laurent series at $\varepsilon = 0$

$$T^{-1}(\varepsilon) = \frac{1}{\varepsilon^s}U_{-s} + ... + \frac{1}{\varepsilon}U_{-1} + U_0 + \varepsilon U_1 + ... \tag{3.23}$$

One can also use the methods of Sections 2.2 and 2.3 to calculate the coefficients of the above series. Substituting the power series $R_i(\varepsilon) = R_{i0} + \varepsilon R_{i1} + \varepsilon^2 R_{i2} + ...$ and the Laurent series (3.23) into formula (3.22), we obtain the asymptotic expansion for $\varphi_i(\varepsilon)$. Since the elements of $\varphi_i(\varepsilon)$ are probabilities, the function $\varphi_i(\varepsilon)$ is bounded and hence the singular terms of (3.23) satisfy the following conditions

$$\sum_{j=-s}^{k} U_j R_{k-j} \xi_{\Omega_i} = 0, \quad k = -s, ..., -1 \tag{3.24}$$

and $\varphi_i(\varepsilon)$ is expanded as series with non-negative powers of ε

$$\varphi_i(\varepsilon) = \varphi_{i0} + \varepsilon\varphi_{i1} + \varepsilon^2\varphi_{i2} + ... \ , \tag{3.25}$$

where

$$\varphi_{ik} = -\sum_{j=-s}^{k} U_j R_{k-j} \xi_{\Omega_i}, \quad k \geq 0. \tag{3.26}$$

The above formulae are valid in the most general setting. Now we would like to discuss several important particular cases. First we discuss the situation when no ergodic classes merge with the transient set. In other words, $T_0 = \tilde{T}$, where $\tilde{T}$ is a proper substochastic matrix. The latter implies that T_0 has an inverse and the asymptotic expansion for $\varphi_i(\varepsilon)$ can be immediately constructed, using Neumann expansion for $T^{-1}(\varepsilon)$, that is

$$\varphi_{i0} = -T_0^{-1} R_{i0} \xi_{\Omega_i},$$

$$\varphi_{ik} = -T_0^{-1}\left[R_{ik}\xi_{\Omega_i} + \sum_{j=1}^{k} T_j \varphi_{ik-j}\right].$$

This case is interesting, since, even if the perturbation can be singular, the calculation of the asymptotic expansions for the right 0-eigenvectors is quite trivial.

Now let us consider an important particular case of a *linear* perturbation, that is $T(\varepsilon) = T_0 + \varepsilon T_1$. In this case, using results of Schweitzer and Stewart [135], we can constract an elegant updating formula for the perturbed subvector $\varphi_i(\varepsilon)$. Towards this end we consider the *singular* and *regular* parts of the Laurent series (3.23)

$$U^S(\varepsilon) = \frac{1}{\varepsilon^s}U_{-s} + ... + \frac{1}{\varepsilon}U_{-1},$$

and

$$U^R(\varepsilon) = U_0 + \varepsilon U_1 + ...,$$

respectively. Note that, using the results of Schweitzer and Stewart [135], we conclude that the regular part $U^R(\varepsilon)$ can be written in the close analytic form

$$B^R(\varepsilon) = (I + \varepsilon U_0 T_1)^{-1} U_0 \tag{3.27}$$

Then, $\varphi_i(\varepsilon) = -U_{-1}R_{i1}\xi_{\Omega_i} - U^R(\varepsilon)R_i(\varepsilon)\xi_{\Omega_i}$ can be calculated by the following updating formula

$$\varphi_i(\varepsilon) = -U_{-1}R_{i1}\xi_{\Omega_i} - (I + \varepsilon U_0 T_1)^{-1} U_0 R_i(\varepsilon)\xi_{\Omega_i},$$

or in terms of the limiting value φ_{i0}

$$\varphi_i(\varepsilon) = \varphi_{i0} - \varepsilon[U_0 R_{i1} - (I + \varepsilon U_0 T_1)^{-1} U_0 T_1 U_0 R_i(\varepsilon)]\xi_{\Omega_i}.$$

We would like to emphasize that the above updating formulae are computationally stable for small values of ε in contrast to the original formula (3.22), where $T^{-1}(\varepsilon)$ is ill-conditioned when ε is close to zero.

Next we consider the case of first order singular perturbations. By this we mean that the Laurent series (3.23) has a simple pole. According to our experience, in general it is quite unlikely that the Laurent series (3.23) has negative powers of ε less than -1. In particular, this setting permits us to derive a nice expression for the limiting value of $\varphi_i(\varepsilon)$ as ε goes to zero. Recall that we have deduced conditions (3.24) from the probabilistic interpretation. In the case of first order singularity it is easy to demonstrate by algebraic methods that the asymptotic expansion for $\varphi_i(\varepsilon)$ does not have a singular part. Towards this end, we write $\varphi_i(\varepsilon)$ as

$$\varphi_i(\varepsilon) = \frac{1}{\varepsilon}\varphi_{i,-1} + \varphi_{i0} + \varepsilon\varphi_{i1} + \ldots$$

and show that $\varphi_{i,-1} = 0$. Upon substitution of the above series and the series for $T(\varepsilon)$ and $R_i(\varepsilon)$ into equation

$$T(\varepsilon)\varphi_i(\varepsilon) = -R_i(\varepsilon)\xi_{\Omega_i}, \tag{3.28}$$

we obtain the following system of equations

$$T_0\varphi_{i,-1} = 0 \tag{3.29}$$

$$T_0\varphi_{i0} + T_1\varphi_{i,-1} = -R_{i0}\xi_{\Omega_i} \tag{3.30}$$

$$\cdots$$

From equation (3.29) we conclude that

$$\varphi_{i,-1} = \tilde{Q}c_{-1} \tag{3.31}$$

where $\tilde{Q}$ is a matrix of right 0-eigenvectors of matrix T_0 (which can be calculated by (3.5)) and $c_{-1} \in R^{m\times 1}$ is a vector of coefficients which can be determined from the feasibility condition of equation (3.30). The feasibility condition for equation (3.30) can be written in the form

$$\tilde{M}T_1\varphi_{i,-1} = -\tilde{M}R_{i0}\xi_{\Omega_i}, \tag{3.32}$$

where $\tilde{M}$ is a matrix whose rows $\tilde{m}_k, k = 1, ..., m$ are "invariant measures" of the substochastic matrix T_0. Namely, $\tilde{m}_k = [0\cdots 0\ \tilde{\mu}_k\ 0\cdots 0]$, where $\tilde{\mu}_k$ is an invariant measure associated with stochastic matrix generator $\tilde{A}_k$. Next we substitute (3.31) into equation (3.32) to get

$$\tilde{M}T_1\tilde{Q}c_{-1} = -\tilde{M}R_{i0}\xi_{\Omega_i}.$$

Note that in the case of first order singularity the matrix $\tilde{M}T_1\tilde{Q}$ is invertible (This is a so-called nondegenerated case [104], for more detail see Section 2.6.3). Therefore $c_{-1} = 0$ (and thereby $\varphi_{i,-1} = 0$) provided that $\tilde{M}R_{i0} = 0$. However, the latter is true, since $\tilde{\Omega}_1, ..., \tilde{\Omega}_m$ correspond to ergodic classes when $\varepsilon = 0$ and submatrix R_{i0} must have zeros in places corresponding to those ergodic classes. Namely, R_{i0} has the following structure

$$R_{i0} = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \rho_{i0} \end{bmatrix} & \begin{array}{l} \}\tilde{\Omega}_1 \\ \vdots \\ \}\tilde{\Omega}_m \\ \}\tilde{\Omega}_{\tilde{T}} \end{array} \end{array}. \tag{3.33}$$

Now we can write the asymptotic expansion for $\varphi_i(\varepsilon)$ in the form

$$\varphi_i(\varepsilon) = \varphi_{i0} + \varepsilon\varphi_{i1} + \varepsilon^2\varphi_{i2} + ...$$

Again from (3.28), we obtain the system of fundamental equations for the coefficients $\varphi_{ik}, k \geq 0$.

$$T_0\varphi_{i0} = -R_{i0}\xi_{\Omega_i} \tag{3.34}$$

$$T_0\varphi_{i1} + T_1\varphi_{i0} = -R_{i1}\xi_{\Omega_i} \tag{3.35}$$

$$\cdots$$

The general solution of equation (3.34) can be written in the form

$$\varphi_{i0} = \tilde{Q}c_0 + \varphi_{part}, \tag{3.36}$$

where φ_{part} is any particular solution of (3.34). It is interesting that φ_{part} can be expressed in closed form, as follows

$$\varphi_{part} = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\tilde{T}^{-1}\rho_{i0}\xi_{\Omega_i} \end{bmatrix} & \begin{array}{l} \}\tilde{\Omega}_1 \\ \vdots \\ \}\tilde{\Omega}_m \\ \}\tilde{\Omega}_{\tilde{T}} \end{array} \end{array}. \tag{3.37}$$

Next we substitute the general solution (3.36) into the feasibility condition for the next fundamental equation (3.35).

$$\tilde{M}T_1\varphi_0 = -\tilde{M}R_{i1}\xi_{\Omega_i}$$

$$\tilde{M}T_1\tilde{Q}c_0 + \tilde{M}T_1\varphi_{part} = -\tilde{M}R_{i1}\xi_{\Omega_i}$$

Because of the first order singularity assumption, the matrix $\tilde{M}T_1\tilde{Q}$ is invertible and we have got

$$c_0 = -(\tilde{M}T_1\tilde{Q})^{-1}\tilde{M}(T_1\varphi_{part} + R_{i1}\xi_{\Omega_i}),$$

and finally,

$$\varphi_{i0} = -\tilde{Q}(\tilde{M}T_1\tilde{Q})^{-1}\tilde{M}(T_1\varphi_{part} + R_{i1}\xi_{\Omega_i}) + \varphi_{part} \tag{3.38}$$

Now let us demonstrate the above theoretical development with the help of the following example.

Example

Consider a Markov chain with the perturbed transition matrix

$$P(\varepsilon) = \begin{array}{c} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \varepsilon & 1-2\varepsilon & \varepsilon \\ \varepsilon & 0 & 0 & 1-\varepsilon \end{array}\right] \end{array} \begin{array}{l} \} \,\Omega_1 \\ \} \,\Omega_2 \\ \} \,\tilde{\Omega}_1 \\ \} \,\tilde{\Omega}_2 \end{array}$$

In this example we have got

$$T_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$R_{10} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R_{20} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

Next we conclude that

$$\tilde{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\varphi_{part} = 0$ for both ergodic classes Ω_1 and Ω_2, since there is no submatrix $\tilde{T}$, which represents the states that are transient in the perturbed chain as well as in the unperturbed chain. Then, using the formula (3.38), we get

$$\varphi_{10} = -\tilde{Q}(\tilde{M}T_1\tilde{Q})^{-1}\tilde{M}R_{11}\xi_{\Omega_1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

and

$$\varphi_{20} = -\tilde{Q}(\tilde{M}T_1\tilde{Q})^{-1}\tilde{M}R_{21}\xi_{\Omega_2} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

The above result is rather interesting. Of course, it is apparent that if the process initiated in the second transient state $\tilde{\Omega}_2$ then it will be absorbed in the first ergodic class Ω_1 with probability one. However, it is a bit surprising that if the process is initiated in the first transient state $\tilde{\Omega}_1$, then it will enter the two ergodic states with equal probabilities. Since to enter the first ergodic class Ω_1 from the first transient state takes two steps and to enter the second ergodic class Ω_2 from the same transient state takes only one step, one might have expected the probabilities to be absorbed in these two ergodic classes to be different. Nevertheless, the above analysis shows that this is not the case.

3.3 Fundamental matrix of a singularly perturbed Markov chain

3.3.1 Preliminaries

Even though perturbed Markov chains have been discussed thouroughly in the previous section, we would like to restate here some basic facts and definitions.

Here as in Section 3.2 we consider the perturbed finite Markov chain with transition matrix $P(\varepsilon)$ depending on a small parameter ε. Two types of perturbation are discussed. The first type is a linear perturbation

$$P(\varepsilon) = P_0 + \varepsilon B, \tag{3.39}$$

where $P_0 \in \mathbb{R}^{N\times N}$ (N – the number of states) is the transition matrix of the unperturbed chain and εB is the perturbation term. The second type is an analytic perturbation

$$P(\varepsilon) = P_0 + \varepsilon G_1 + \varepsilon^2 G_2 + ...$$

or

$$G(\varepsilon) = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + ..., \tag{3.40}$$

where $G(\varepsilon) = P(\varepsilon) - I \in \mathbb{R}^{N\times N}$, $G_0 = P_0 - I \in \mathbb{R}^{N\times N}$ are the generators of the perturbed and unperturbed chains respectively.

Let ξ_n be the vector $[1 \cdots 1]^T \in \mathbb{R}^{n\times 1}$. Since each row of a transition matrix is a probability distribution, we immediately have

$$P(\varepsilon)\xi_N = P_0\xi_N = \xi_N \quad \text{and} \quad G(\varepsilon)\xi_N = G_0\xi_N = 0. \tag{3.41}$$

For the same reasons, $B\xi_N = 0$ and $G_k\xi_N = 0, k \geq 1$.

Let us denote by P^* and $P^*(\varepsilon)$ the *ergodic projections* of the unperturbed and perturbed chains, respectively.

There are a number of algorithms for computing P^* [79, 89] and the asymptotic expansion for $P^*(\varepsilon)$ (see [40, 30, 33, 116, 125, 132, 134] and Section 3.2).

$$P^*(\varepsilon) = P_0^* + \varepsilon P_1^* + \varepsilon^2 P_2^* + ..., \quad \varepsilon \neq 0 \tag{3.42}$$

The constant term P_0^* will be called the *limiting ergodic projection.* Note that in general $P_0^* \neq P^*$ [131]. In particular, the discontinuity occures in the case of a singular perturbation. In Section 3.2 we have analysed the asymptotic behaviour of the ergodic projection without any assumption about the structure of the chain. Since the concept of mean first passage times has sense only for the ergodic Markov chains, in this section we make the following standard assumption.

ASSUMPTION A. *For any $\varepsilon > 0$ sufficiently small, the perturbed matrix $P(\varepsilon)$ is a unichain transition matrix, that is, the perturbed chain has a unique ergodic class and possibly, a set of transient states.*

It will be seen that it does not create significant complications to make the above slightly more general *unichain* assumption rather than to assume that the perturbed chain is ergodic. In particular, the above assumption still implies that $P^*(\varepsilon) = \xi_N \mu(\varepsilon)$ for $\varepsilon \neq 0$, where the *stationary probability distribution* or *invariant measure* $\mu(\varepsilon)$ of the perturbed chain is the unique solution of the system

$$\begin{cases} \mu(\varepsilon)P(\varepsilon) = \mu(\varepsilon), \\ \mu(\varepsilon)\xi_N = 1 \end{cases}$$

and $\mu(\varepsilon)$ has a Taylor series expansion

$$\mu(\varepsilon) = \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + ... , \quad \varepsilon \neq 0. \tag{3.43}$$

In the case of a linear perturbation (cf. (3.39)), Schweitzer [132, 134] proved that $\mu(\varepsilon)$ can be expressed by the *updating* formula

$$\mu(\varepsilon) = \mu_0[I - \varepsilon BZ_0]^{-1}, \quad \varepsilon \neq 0, \tag{3.44}$$

where Z_0 is the constant term in the fundamental matrix Laurent series expansion (cf. (3.47) below). Hence, the coefficients $\mu_k, k = 0, 1, ...$ in (3.43) form a *power* sequence

$$\mu_k = \mu_0(BZ_0)^k, \quad k = 0, 1, ...$$

The *fundamental matrix* of the original chain is defined by the formula [79, 89]

$$Z \stackrel{def}{=} (I - P_0 + P^*)^{-1} = (P^* - G_0)^{-1}. \tag{3.45}$$

Of course, the same formula holds for the perturbed chain

$$Z(\varepsilon) \stackrel{def}{=} (I - P(\varepsilon) + P^*(\varepsilon))^{-1} = (P^*(\varepsilon) - G(\varepsilon))^{-1}. \tag{3.46}$$

In this paper, we are mainly concerned with the characterization of the fundamental matrix $Z(\varepsilon)$ of the perturbed chain.

In the case of a singular perturbation, $Z(\epsilon)$ also has a discontinuity at $\epsilon = 0$. Moreover, in this case, $||Z(\varepsilon)|| \to \infty$ as $\varepsilon \to 0$ and $Z(\varepsilon)$ admits a Laurent series expansion [132, 134, 66, 67]

$$Z(\varepsilon) = \frac{1}{\varepsilon^s} Z_{-s} + ... + \frac{1}{\varepsilon} Z_{-1} + Z_0 + \varepsilon Z_1 + ..., \varepsilon \neq 0. \tag{3.47}$$

Note that s, the order of the pole, is finite and $s \leq N$. We denote the singular and the regular parts of the fundamental matrix expansion (3.47) by $Z^S(\varepsilon)$ and $Z^R(\varepsilon)$, respectively. Schweitzer has also obtained formulae for the matrices Z_k. However, they are rather complicated and their computation requires to handle large size matrices (cf. [132]).

In sequel, we propose a different approach that allows us to compute more efficiently the coefficients of the Laurent series (3.47). This approach can be considered as a particular realisation of the general scheme proposed in Section 2.4 for the perturbation analysis of group inverses. Our method leads to the operations with matrices of small dimensions. For example, immediately after the first stage of the reduction process one handles *aggregated* Markov chains with no more than m states (m being the number of ergodic classes). This would, typically, constitute a drastic reduction of the dimension. In addition, for the case of a linear perturbation we provide a simple formula for the regular part $Z^R(\epsilon)$, which readily simplifies to the usual formula [131] if the perturbation is regular.

We introduce the *deviation matrices* (or *reduced resolvents*) H and $H(\varepsilon)$ for the original and perturbed chains:

$$H \stackrel{def}{=} Z - P^* \quad \text{and} \quad H(\varepsilon) \stackrel{def}{=} Z(\varepsilon) - P^*(\varepsilon).$$

We immediately have

$$H^S(\varepsilon) = Z^S(\varepsilon) \quad \text{and} \quad H^R(\varepsilon) = Z^R(\varepsilon) - P^*(\varepsilon). \tag{3.48}$$

The fundamental matrix and the reduced resolvent are known to satisfy [79, 89] the useful identities:

$$HG_0 = G_0 H = ZG_0 = G_0 Z = P^* - I, \tag{3.49}$$

$$HP^* = P^* H = 0, \quad ZP^* = P^* Z = P^*, \tag{3.50}$$

(with G_0 as in (3.40)) and similarly, for the perturbed chain

$$H(\varepsilon)G(\varepsilon) = G(\varepsilon)H(\varepsilon) = Z(\varepsilon)G(\varepsilon) = G(\varepsilon)Z(\varepsilon) = P^*(\varepsilon) - I, \tag{3.51}$$

$$H(\varepsilon)P^*(\varepsilon) = P^*(\varepsilon)H(\varepsilon) = 0, \quad Z(\varepsilon)P^*(\varepsilon) = P^*(\varepsilon)Z(\varepsilon) = P^*(\varepsilon). \tag{3.52}$$

The following additional identities will also be useful

$$Z\xi_N = Z(\varepsilon)\xi_N = P^*\xi_N = P^*(\varepsilon)\xi_N = \xi_N, \tag{3.53}$$

$$H\xi_N = H(\varepsilon)\xi_N = 0. \tag{3.54}$$

3.3.2 The regular part of the fundamental matrix

In this subsection we obtain a closed form analytic formula for the regular part $Z^R(\varepsilon)$ of the fundamental matrix expansion (3.47). We use arguments similar to those in Schweitzer's elegant technique introduced in [134]. We also show that Schweitzer's formula for regular perturbations is in fact a particular case of our formula, which handles both regular and singular perturbations.

Theorem 23 *Let $P(\varepsilon) = P_0 + \varepsilon B$ be the transition matrix of a linearly perturbed Markov chain and let Assumption A hold. Then the regular part $Z^R(\epsilon)$ of the fundamental matrix $Z(\epsilon)$ is given by*

$$Z^R(\varepsilon) = \{[I - P^*(\varepsilon)]Z_0 + P_0^*\}[I - \varepsilon BZ_0]^{-1} - P^*(\varepsilon)Z^S(\varepsilon). \tag{3.55}$$

Proof: For arbitrary $0 < \epsilon_1,\ \epsilon_2$, we have the following identity (see e.g. [99]; this identity is also a particular case of the more general identity (2.94))

$$Z(\varepsilon_1) - Z(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)Z(\varepsilon_1)BZ(\varepsilon_2) + Z(\varepsilon_1)P^*(\varepsilon_2) - P^*(\varepsilon_1)Z(\varepsilon_2).$$

Under Assumption A, and using(3.53), we have $Z(\varepsilon_1)P^*(\varepsilon_2) = Z(\varepsilon_1)\xi_N\pi(\varepsilon_2) = \xi_N\pi(\varepsilon_2) = P^*(\varepsilon_2)$. Hence,

$$Z(\varepsilon_1) - Z(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)Z(\varepsilon_1)BZ(\varepsilon_2) + P^*(\varepsilon_2) - P^*(\varepsilon_1)Z(\varepsilon_2) \tag{3.56}$$

With ε_2 fixed in (3.56), the regular parts with respect to ε_1 satisfy:

$$Z^R(\varepsilon_1) - Z(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)Z^R(\varepsilon_1)BZ(\varepsilon_2) + Z_{-1}BZ(\varepsilon_2) + P^*(\varepsilon_2) - P^*(\varepsilon_1)Z(\varepsilon_2) \tag{3.57}$$

Letting $\varepsilon_1 = \varepsilon_2$ in (3.57) yields

$$Z^R(\varepsilon_2) - Z(\varepsilon_2) = Z_{-1}BZ(\varepsilon_2) + P^*(\varepsilon_2) - P^*(\varepsilon_2)Z(\varepsilon_2) \tag{3.58}$$

and since $P^*(\varepsilon) = P^*(\varepsilon)Z(\varepsilon)$, we have

$$Z^R(\varepsilon_2) - Z(\varepsilon_2) = Z_{-1}BZ(\varepsilon_2)$$

so that

$$Z^S(\varepsilon) = -Z_{-1}BZ^S(\varepsilon) \tag{3.59}$$

and

$$0 = Z_{-1}BZ^R(\varepsilon). \tag{3.60}$$

If instead, we fix ε_1 in (3.56) and consider the regular parts with respect to ε_2, we obtain

$$Z^S(\varepsilon) = -Z^S(\varepsilon)BZ_{-1} \tag{3.61}$$

and

$$Z^R(\varepsilon)BZ_{-1} = P^*(\varepsilon) - P^*(\varepsilon)Z^R(\varepsilon) = P^*(\varepsilon)Z^S(\varepsilon). \tag{3.62}$$

Taking the regular parts in (3.57) with respect to ε_2 (with of course, ε_1 fixed) yields

$$\begin{aligned} Z^R(\varepsilon_1) - Z^R(\varepsilon_2) &= (\varepsilon_1 - \varepsilon_2)Z^R(\varepsilon_1)BZ^R(\varepsilon_2) - Z^R(\varepsilon_1)BZ_{-1} + Z_{-1}BZ^R(\varepsilon_2) + P^*(\varepsilon_2) \\ &\quad -P^*(\varepsilon_1)Z^R(\varepsilon_2). \end{aligned}$$

The term $Z_{-1}BZ^R(\varepsilon_2)$ vanishes in view of (3.60). Then with $\varepsilon_1 := \varepsilon$ and letting $\varepsilon_2 \to 0$, one gets

$$Z^R(\varepsilon) - Z_0 = \varepsilon Z^R(\varepsilon)BZ_0 - Z^R(\varepsilon)BZ_{-1} + P_0^* - P^*(\varepsilon)Z_0.$$

$$Z^R(\varepsilon)[I - \varepsilon BZ_0] = [I - P^*(\varepsilon)]Z_0 + P_0^* - Z^R(\varepsilon)BZ_{-1}.$$

Using (3.62), we get

$$Z^R(\varepsilon)[I - \varepsilon BZ_0] = [I - P^*(\varepsilon)]Z_0 + P_0^* - P^*(\varepsilon)Z^S(\varepsilon). \tag{3.63}$$

We shall now show that

$$P^*(\varepsilon)Z^S(\varepsilon) = P^*(\varepsilon)Z^S(\varepsilon)[I - \varepsilon BZ_0]. \tag{3.64}$$

It suffices to prove that $Z^S(\varepsilon)BZ_0 = 0$. Indeed, from (3.60) and (3.61),

$$Z^S(\varepsilon)BZ_0 = -Z^S(\varepsilon)BZ_{-1}BZ_0 = 0.$$

Finally, substituting (3.64) into (3.63) and multiplying (3.63) by $[I - \varepsilon BZ_0]^{-1}$ from the right-hand-side, we get the required formula (3.55).

□

Two useful corollaries follow directly from Theorem 23. First, consider the term $P^*(\varepsilon)Z^S(\varepsilon)$. This term is regular, despite the fact that it is the product of the perturbed ergodic projection and the *singular* part of the fundamental matrix. The next corollary shows the explicit regular structure of this product.

Corollary 6 *The formula (3.55) is equivalent to*

$$Z^R(\varepsilon) = \{[I - P^*(\varepsilon)]Z_0 + P_0^*\}[I - \varepsilon BZ_0]^{-1} - P^*(\varepsilon)\sum_{i=1}^{s}(BZ_0)^i Z_{-i}. \tag{3.65}$$

PROOF: In view of Assumption (A) and (6) we have $P^*(\varepsilon) = \xi_N \pi(\varepsilon) = P_0^*[I - \varepsilon BZ_0]^{-1}$ and hence $P_k^* = P_0^*(BZ_0)^k, k = 0, 1, ...$ in the power series (3.42).

Note that *only* the regular part of $P^*(\varepsilon)Z^S(\varepsilon)$ contributes to equation (3.55), since $Z^R(\varepsilon)$ and $\{[I - P^*(\varepsilon)]Z_0 + P_0^*\}[I - \varepsilon BZ_0]^{-1}$ are both regular. Then,

$$[P^*(\varepsilon)Z^S(\varepsilon)]^R = [\sum_{k=1}^{\infty} P_0^*(\varepsilon BZ_0)^k \sum_{i=1}^{s} \varepsilon^{-i}Z_{-i}]^R = \sum_{i=1}^{s}\sum_{k=i}^{\infty} P_0^*(BZ_0)^k Z_{-i}\varepsilon^{k-i} =$$

$$= \sum_{i=1}^{s}\sum_{k=0}^{\infty} P_0^*(\varepsilon BZ_0)^k (BZ_0)^i Z_{-i} = \sum_{k=0}^{\infty} P_0^*(\varepsilon BZ_0)^k \sum_{i=1}^{s}(BZ_0)^i Z_{-i} = P^*(\varepsilon)\sum_{i=1}^{s}(BZ_0)^i Z_{-i}$$

This yields formula (3.65).

□

By using (3.48) and (3.44), one easily obtains the counterpart of (3.55) for the deviation matrix $H(\varepsilon)$.

Corollary 7 *The regular part of the deviation matrix $H(\varepsilon)$ is given by*

$$H^R(\varepsilon) = [I - P^*(\varepsilon)]H_0[I - \varepsilon BZ_0]^{-1} - P^*(\varepsilon)H^S(\varepsilon).$$

Remark 19 *Schweitzer's formula for regular perturbations [131, 134]*

$$Z(\varepsilon) = \{[I - P^*(\varepsilon)]Z(0) + P^*\}[I - \varepsilon BZ(0)]^{-1}$$

is a particular case of (3.55) (since then, $Z^S(\epsilon) = 0$).

Remark 20 *The matrices Z_k, $k \geq 1$ are easily obtained from (3.65), with*

$$Z_k = Z_0(BZ_0)^k - P_0^*\sum_{j=0}^{k}(BZ_0)^j Z_0(BZ_0)^{k-j} + P_0^*(BZ_0)^k\{I - \sum_{i=1}^{s}(BZ_0)^i Z_{-i}\}. \tag{3.66}$$

Of course, in practical applications, a more efficient (and numerically stable) computational scheme for the above matrices $Z_k, k \geq 1$ is via recurrent formulae, in the same spirit as in [91, 132]. For instance, if we define the two k-dependent expressions

$$U_k = (BZ_0)^k \quad \textit{and} \quad W_k = \sum_{j=0}^{k}(BZ_0)^j Z_0(BZ_0)^{k-j},\ k = 1, \ldots,$$

then (3.66)) is equivalent to

$$Z_k = Z_0U_k - P_0^*[W_k - U_k\{I - \sum_{i=1}^{s}(BZ_0)^i Z_{-i}\}],\ k = 1, \ldots, \tag{3.67}$$

with U_k and W_k computed recursively in (3.68) below.

$$U_{k+1} = U_k(BZ_0), \quad W_{k+1} = W_k(BZ_0) + U_{k+1}Z_0, \quad k = 1, \dots \ . \tag{3.68}$$

Remark 21 *Since the deviation matrix $H(\varepsilon)$ ia a minus group inverse of the perturbed Markov chain generator $G(\varepsilon)$, we can use the recursive formula (2.96) from Section 2.5.3. In particular, this formula allows us to deal with analytic perturbations. Here (2.96) takes the following form*

$$\begin{aligned} H_{k+1} \;=\; & \sum_{i=0}^{k+s}\left(\sum_{j=0}^{s} H_{-j}G_{i+j+1}\right)H_{k-i} - \sum_{i=1}^{m} P^*_{k+1-i}H_i \\ & -(P^*_{m+1}H_0 + \dots + P^*_{m+1+s}H_{-s}) \end{aligned} \tag{3.69}$$

where $m \geq 0$ and $P^*_k, k \geq 0$ are coefficients of the Taylor series for the ergodic projection $P^*(\varepsilon)$ of the perturbed Markov chain. Note that the term $(H_{-s}P^*_{m+1+s} + \dots + H_0P^*_{m+1})$ in (2.96) vanishes, since according to Assumption A $H_k\xi_N = 0$ and $P^*_k = \xi_N\mu_k$.

3.3.3 The singular part and the reduction process

Note that once the matrices $Z_{-k}, k = 1, \dots, s$ and Z_0 have been computed, then it follows from Remarks 20 and 21 that the matrices $Z_k, k \geq 1$ can be obtained directly via the recurent formulae (3.67), (3.68) for the linear perturbation case and via (3.69) for the analytic case. Note also that Z_0 is a prerequisite in Schweitzer's updating formula (3.44).

Therefore, our aim in this section is to provide an efficient algorithm for computing $Z_{-k}, k = 0, \dots, s$ (including Z_0). We shall use the reduction process to compute the first regular term Z_0 and the singular part of the fundamental matrix expansion (3.47). It turns out that the reduction process is easier for the deviation matrix $H(\varepsilon)$ than for the fundamental matrix $Z(\varepsilon)$. In particular, since the deviation matrix is the negative group inverse of the Markov chain generator, the general approach for the calculation of the Laurent series for perturbed group inverses (see Section 2.4.5) can be applied to the present problem. In this section we choose to pursue the *algebraic* reduction technique. Alternatively, one can perform the reduction process based on the tools of spectral theory (for more detail see Section 2.5). Of course, once the matrices $H_{-k}, k = 1, \dots, s$ and H_0 are computed, one can immediately recover the matrices $Z_{-k}, k = 0, \dots, s$ from (3.48), that is

$$Z_{-k} = H_{-k}, \ k = \overline{1, s}; \quad Z_0 = H_0 + P^*_0. \tag{3.70}$$

The reduction process for analytic perturbations has practically the same level of difficulty as for linear perturbations. Therefore, we consider the general case of analytic perturbations (3.40).

Under Assumption A, the deviation matrix $H(\varepsilon)$ of the perturbed Markov chain is uniquely defined by the following equations (cf. [79, 66])

$$H(\varepsilon)G(\varepsilon) = P^*(\varepsilon) - I, \tag{3.71}$$

$$H(\varepsilon)\xi_N = 0. \tag{3.72}$$

To simplify subsequent summations, we introduce a new notation for the coefficients H_k. Let us denote $X_k \stackrel{def}{=} H_{k-s}, k = 0, 1, \ldots$, so that $H(\varepsilon) = (X_0 + \varepsilon X_1 + \ldots)/\varepsilon^s = X(\varepsilon)/\varepsilon^s$ and (3.71)-(3.72) become

$$X(\varepsilon)G(\varepsilon) = \varepsilon^s[P^*(\varepsilon) - I], \tag{3.73}$$

$$X(\varepsilon)\xi_N = 0. \tag{3.74}$$

Then, substitute (3.40) and (3.42) into (3.73) and equate the terms with the same power of ε to get

$$\begin{array}{ll}
X_0 G_0 = 0 & (FH.0) \\
X_1 G_0 + X_0 G_1 = 0 & (FH.1) \\
\cdots & \\
X_{s-1} G_0 + ... + X_0 G_{s-1} = 0 & (FH.s-1) \\
X_s G_0 + ... + X_0 G_s = P_0^* - I & (FH.s) \\
X_{s+1} G_0 + ... + X_0 G_{s+1} = P_1^* & (FH.s+1) \\
\cdots &
\end{array}$$

the system of *fundamental equations*. It follows from (3.74) that each equation (FH.k) is coupled with the normalization condition

$$X_k \xi_N = 0, \; k \geq 0. \tag{3.75}$$

Before applying the reduction process, let us formally introduce the n-th step reduced system of fundamental equations:

$$\begin{array}{ll}
X_0^{(n)} G_0^{(n)} = 0 & (FHn.0) \\
X_1^{(n)} G_0^{(n)} + X_0^{(n)} G_1^{(n)} = 0 & (FHn.1) \\
\cdots & \\
X_{s-n-1}^{(n)} G_0^{(n)} + ... + X_0^{(n)} G_{s-n-1}^{(n)} = 0 & (FHn.s-n-1) \\
X_{s-n}^{(n)} G_0^{(n)} + ... + X_0^{(n)} G_{s-n}^{(n)} = A_0^{(n)} & (FHn.s-n) \\
X_{s-n+1}^{(n)} G_0^{(n)} + ... + X_0^{(n)} G_{s-n+1}^{(n)} = A_1^{(n)} & (FHn.s-n+1) \\
 & \cdots
\end{array}$$

With $n = 0$, we retrieve the original system of fundamental equations (FH). Namely, $X_k^{(0)} = X_k, G_k^{(0)} = G_k$ and $A^{(0)} = P_0^* - I, A_k^{(0)} = P_k^*, k \geq 1$. The n-th step reduced equations (FHn) are also coupled with the normalization conditions

$$X_k^{(n)} \xi_{m_n} = 0, \quad k = 0, 1, ... \tag{3.76}$$

The matrix $G_0^{(n)} \in \mathrm{IR}^{\mathrm{m_n \times m_n}}$ can be interpreted as the generator of the n-th step aggregated Markov chain [30, 40]. Further, m_n is the number of ergodic classes of the $(n-1)$-th step aggregated Markov chain and in particular, m_1 is the number of ergodic classes in the original chain. The corresponding aggregated ergodic projection $P^{*(n)}$ satisfies

$$P^{*(n)} G_0^{(n)} = G_0^{(n)} P^{*(n)} = 0. \tag{3.77}$$

The aggregated ergodic projection is constructed by the following decomposition [15, 40]

$$P^{*(n)} = Q^{(n)} M^{(n)}, \tag{3.78}$$

where

$$Q^{(n)} \in \mathrm{IR}^{\mathrm{m_n \times m_{n+1}}} : \mathrm{G_0^{(n)} Q^{(n)} = 0, Q^{(n)} \xi_{m_{n+1}} = \xi_{m_n}}; \tag{3.79}$$

$$M^{(n)} \in \mathrm{IR}^{\mathrm{m_{n+1} \times m_n}} : \mathrm{M^{(n)} G_0^{(n)} = 0, M^{(n)} \xi_{m_n} = \xi_{m_{n+1}}}. \tag{3.80}$$

In addition,

$$M^{(n)} Q^{(n)} = I_{m_{n+1}}.$$

The n-th step deviation matrix $H^{(n)}$ is computed via the formula

$$H^{(n)} = (P^{*(n)} - G_0^{(n)})^{-1} - P^{*(n)}. \tag{3.81}$$

Of course, this matrix satisfies (3.49)-(3.50), that is,

$$H^{(n)} G_0^{(n)} = G_0^{(n)} H^{(n)} = P^{*(n)} - I, \tag{3.82}$$

$$H^{(n)} P^{*(n)} = P^{*(n)} H^{(n)} = 0. \tag{3.83}$$

We may now formulate the main result of this section, which allows us to solve the system (FH), step-by-step.

Theorem 24 *Let the n-th step $(n < s)$ reduced fundamental system (FHn) with the normalization conditions (3.76) be given. Then the unknown $X_k^{(n)}$ satisfy:*

$$X_k^{(n)} = X_k^{(n+1)} M^{(n)} + \begin{cases} \sum_{i=0}^{k-1} X_i^{(n)} G_{k-i}^{(n)} H^{(n)}, & k < s - n, \\ \sum_{i=0}^{k-1} X_i^{(n)} G_{k-i}^{(n)} H^{(n)} - A_{k+n-s}^{(n)} H^{(n)}, & k \geq s - n, \end{cases} \tag{3.84}$$

where $X_k^{(n+1)} \in \mathbb{R}^{\mathrm{m_n \times m_{n+1}}}, \mathrm{k} = 0, 1, \dots$ *are solutions of the next* $(n+1)$*-th step reduced fundamental equations*

$$\begin{aligned}
X_0^{(n+1)} G_0^{(n+1)} &= 0 && (FHn+1.0) \\
X_1^{(n+1)} G_0^{(n+1)} + X_0^{(n+1)} G_1^{(n+1)} &= 0 && (FHn+1.1) \\
&\cdots \\
X_{s-n-2}^{(n+1)} G_0^{(n+1)} + \dots + X_0^{(n+1)} G_{s-n-2}^{(n+1)} &= 0 && (FHn+1.s-n-2) \\
X_{s-n-1}^{(n+1)} G_0^{(n+1)} + \dots + X_0^{(n+1)} G_{s-n-1}^{(n+1)} &= A_0^{(n+1)} && (FHn+1.s-n-1) \\
X_{s-n}^{(n+1)} G_0^{(n+1)} + \dots + X_0^{(n+1)} G_{s-n}^{(n+1)} &= A_1^{(n+1)} && (FHn+1.s-n) \\
&\cdots
\end{aligned}$$

The matrices $G_k^{(n+1)}$ *and the right-hand-sides* $A_k^{(n+1)}$ *are given by*

$$G_k^{(n+1)} = M^{(n)} \sum_{p=1}^{k+1} \sum_{\nu_1+\dots+\nu_p=k+1} G_{\nu_1}^{(n)} H^{(n)} G_{\nu_2}^{(n)} \cdots H^{(n)} G_{\nu_p}^{(n)} Q^{(n)}, \tag{3.85}$$

$$A_k^{(n+1)} = \sum_{i=0}^{k} A_i^{(n)} \sum_{p=1}^{k-i} \sum_{\nu_1+\dots+\nu_p=k-i} G_{\nu_1}^{(n)} H^{(n)} G_{\nu_2}^{(n)} \cdots H^{(n)} G_{\nu_p}^{(n)} Q^{(n)}, \quad k = 0, 1, \dots . \tag{3.86}$$

The new reduced equations (FHn+1) are also coupled with the normalization conditions

$$X_k^{(n+1)} \xi_{m_{n+1}} = 0, \quad k = 0, 1, \dots . \tag{3.87}$$

Proof: Define the new unknowns

$$X_k^{(n+1)} \stackrel{def}{=} X_k^{(n)} Q^{(n)}. \tag{3.88}$$

The first fundamental equation (FHn.0) implies

$$X_0^{(n)} = X_0^{(n)} P^{*(n)}. \tag{3.89}$$

Substituting the decomposition of the ergodic projection (3.78) into (3.89), we get

$$X_0^{(n)} = X_0^{(n)} Q^{(n)} M^{(n)} = X_0^{(n+1)} M^{(n)}. \tag{3.90}$$

Multiply (38.1) by $Q^{(n)}$ from the right and use (3.79) to obtain

$$X_0^{(n)} G_1^{(n)} Q^{(n)} = 0. \tag{3.91}$$

Then, substitute (3.90) into (3.91) to get

$$X_0^{(n+1)} M^{(n)} G_1^{(n)} Q^{(n)} = 0$$

or

$$X_0^{(n+1)} G_0^{(n+1)} = 0,$$

where $G_0^{(n+1)} \stackrel{def}{=} M^{(n)} G_1^{(n)} Q^{(n)}$. The above equation is the first required equation (FHn+1.0). The n-th step deviation matrix $H^{(n)}$ plays a crucial role in obtaining the next reduced equations. Indeed, consider the following decomposition of $X_1^{(n)}$

$$\begin{aligned} X_1^{(n)} &= X_1^{(n)} P^{*(n)} + X_1^{(n)}(I - P^{*(n)}) \\ &= X_1^{(n)} Q^{(n)} M^{(n)} - X_1^{(n)} G_0^{(n)} H^{(n)} \\ &= X_1^{(n+1)} M^{(n)} - X_1^{(n)} G_0^{(n)} H^{(n)}. \end{aligned} \tag{3.92}$$

In the above the definition of $X_1^{(n+1)}$ (3.88) and the property (3.82) of the deviation matrix $H^{(n)}$ have been used. Now, using (FHn.1) for $X_1^{(n)} G_0^{(n)}$ and substituting into (3.92) yields

$$\begin{aligned} X_1^{(n)} &= X_1^{(n+1)} M^{(n)} + X_0^{(n)} G_1^{(n)} H^{(n)} \\ &= X_1^{(n+1)} M^{(n)} + X_0^{(n+1)} M^{(n)} G_1^{(n)} H^{(n)}. \end{aligned} \tag{3.93}$$

Note that we have expressed the n-th step unknown $X_1^{(n)}$ in terms of new unknowns $X_0^{(n+1)}, X_1^{(n+1)}$. Similar expressions are obtained for $X_k^{(n)}, k \geq 2$.

Now substitute (3.90) and (3.93) into (FHn.2) to get

$$[X_1^{(n+1)} M^{(n)} + X_0^{(n+1)} M^{(n)} G_1^{(n)} H^{(n)}] G_1^{(n)} + X_0^{(n+1)} M^{(n)} G_2^{(n)} = 0.$$

Multiplying (FHn.2) by $Q^{(n)}$ from the right and using (3.79) yields

$$X_1^{(n+1)} M^{(n)} G_1^{(n)} Q^{(n)} + X_0^{(n+1)} M^{(n)} (G_1^{(n)} H^{(n)} G_1^{(n)} + G_2^{(n)}) Q^{(n)} = 0,$$

or, equivalently,

$$X_1^{(n+1)} G_0^{(n+1)} + X_0^{(n+1)} G_1^{(n+1)} = 0,$$

with $G_1^{(n+1)} \stackrel{def}{=} M^{(n)} (G_1^{(n)} H^{(n)} G_1^{(n)} + G_2^{(n)}) Q^{(n)}$. Thus, we have the second step reduced equation (FHn+1.1) and an expression for $G_1^{(n+1)}$. The other next step reduced equations are obtained with similar arguments. The general formulae (3.85) and (3.86) can be proved by induction.

Note that if $X_k^{(n+1)}, k \geq 0$ are known, then the coefficients $X_k^{(n)}, k \geq 0$ are easily calculated by the recurrent formula (3.84). Indeed,

$$\begin{aligned} X_k^{(n)} &= X_k^{(n)} P^{*(n)} + X_k^{(n)}(I - P^{*(n)}) \\ &= X_k^{(n+1)} M^{(n)} - X_k^{(n)} G_0^{(n)} H^{(n)} \\ &= X_k^{(n+1)} M^{(n)} + \left\{ \begin{array}{ll} \sum_{i=0}^{k-1} X_i^{(n)} G_{k-i}^{(n)} H^{(n)}, & k < s-n, \\ \sum_{i=0}^{k-1} X_i^{(n)} G_{k-i}^{(n)} H^{(n)} - A_{k+n-s}^{(n)} H^{(n)}, & k \geq s-n, \end{array} \right. . \end{aligned}$$

Finally, we show that the normalization condition (3.87) holds. To prove this, we need the second identity in (3.80) and the property of the ergodic projection $P^{*(n)}\xi_{m_n} = \xi_{m_n}$. For example, consider the case $k < s - n$.

$$\begin{aligned} 0 = X_k^{(n)}\xi_{m_n} &= X_k^{(n+1)}M^{(n)}\xi_{m_n} + \sum_{i=0}^{k-1} X_i^{(n)}G_{k-i}^{(n)}H^{(n)}\xi_{m_n} \\ &= X_k^{(n+1)}\xi_{m_{n+1}} + \sum_{i=0}^{k-1} X_i^{(n)}G_{k-i}^{(n)}H^{(n)}P^{*(n)}\xi_{m_n} \\ &= X_k^{(n+1)}\xi_{m_{n+1}}, \end{aligned}$$

since $H^{(n)}P^{*(n)} = 0$.

□

Note that the $(n+1)$-th step reduced system (FHn+1) has a very similar structure to the n-th step reduced system (FHn). The only one, but important, difference between the structures of these two systems is that the system (FHn+1) has **less** equations with null right-hand-sides. Thus, after s reduced steps, the system (below) of reduced equations has non-zero right-hand-sides.

$$\begin{aligned} X_0^{(s)}G_0^{(s)} &= A_0^{(s)} && (FHs.0) \\ X_1^{(s)}G_0^{(s)} + X_0^{(s)}G_1^{(s)} &= A_1^{(s)} && (FHs.1) \\ &\cdots \\ X_k^{(s)}G_0^{(s)} + ... + X_0^{(s)}G_k^{(s)} &= A_k^{(s)} && (FHs.k) \\ &\cdots \end{aligned}$$

The next proposition gives the simple recurrent formulas for the solution of the (final step) reduced system (FHs).

Proposition 4 *The solutions $X_k^{(s)}, k = 0, 1, ...$ of the system (FHs) are given by*

$$X_0^{(s)} = -A_0^{(s)}H^{(s)};\ X_k^{(s)} = [\sum_{i=0}^{k-1} X_i^{(s)}G_{k-i}^{(s)} - A_k^{(s)}]H^{(s)}, \quad k \geq 1\ . \tag{3.94}$$

PROOF: The final-step aggregated generator $G_0^{(s)}$ has the same number of ergodic classes as the perturbed chain described by the generator $G(\varepsilon), \varepsilon > 0$ [40]. Hence, in view of Assumption A, the aggregated generator $G_0^{(s)}$ is a unichain generator and the corresponding ergodic projection is just $P^{*(s)} = \xi_{m_s}\mu_0^{(s)}$, where $\mu_0^{(s)} \in \mathbb{R}^{1\times m_s}$ is a unique stationary distribution vector of the aggregated generator $G_0^{(s)}$.

Of course, the final-step reduced system (FHs) is coupled with the normalization conditions $X_k^{(s)}\xi_{m_s} = 0, k = 0, 1, \ldots$. Multiplying by $\mu_0^{(s)}$, we get

$$X_k^{(s)}\xi_{m_s}\mu_0^{(s)} = X_k^{(s)}P^{*(s)} = 0, \quad k = 0, 1, \ldots\ . \tag{3.95}$$

Now, using the modified normalization conditions (3.95) and the decomposition of $X_k^{(s)}$ into subspaces $\mathcal{R}(P^{*(s)})$ and $\mathcal{R}(I - P^{*(s)})$, we obtain the recurrent formulas (3.94)

$$\begin{aligned} X_k^{(s)} &= X_k^{(s)} P^{*(s)} + X_k^{(s)}(I - P^{*(s)}) \\ &= X_k^{(s)}(I - P^{*(s)}) = -X_k^{(s)} G_0^{(s)} H^{(s)} \\ &= [\sum_{i=0}^{k-1} X_i^{(s)} G_{k-i}^{(s)} - A_k^{(s)}] H^{(s)}, \end{aligned}$$

□

Using Theorem 24 and Proposition 4, we are now able to outline a practical algorithm for the computation of matrices $Z_{-k}, k = 0, ..., s$.

Computational algorithm:

1. Determine the order of singularity s, using for instance, the method proposed in [63] (knowledge of the order of singularity makes the algorithm more straightforward and economical).

2. By using the formulae in Theorem 24, make s reduction steps (i.e. successively calculate the matrices $G_k^{(n)}, k = 0, ..., 2s-n$ and the right-hand-sides $A_k^{(n)}, k = 0, ..., s$ for $n = 1, ..., s$). As a result, one gets the final system of reduced fundamental equations (FHs).

3. Compute $X_k^{(s)}, k = 0, ..., s$ in (FHs) using the recurrent formulae (3.94).

4. Using (3.84), restore successively all the $X_k^{(n)}, k = 0, ..., s$ from $n = s - 1$ down to $n = 0$. In particular,

$$H_k = X_{k+s} = X_{k+s}^{(0)}, \quad k = -s, ..., 0.$$

5. Finally, via (3.70), compute the matrices of the fundamental matrix expansion

$$Z_{-k} = H_{-k}, k = 1, ..., s; \quad Z_0 = H_0 + P_0^* \ .$$

Now, let us make several remarks about the reduction process.

Remark 22 *To compute the matrices $G_k^{(n+1)}$ and $A_k^{(n+1)}$, instead of using (3.85) and (3.86), one may also use recurrent formulas that are more efficient. In the spirit of Remark 20 and [91, 132], define*

$$V_k^{(n)} = \sum_{p=1}^{k+1} \sum_{\nu_1 + ... + \nu_p = k+1} G_{\nu_1}^{(n)} H^{(n)} G_{\nu_2}^{(n)} \cdots H^{(n)} G_{\nu_p}^{(n)},$$

which can be calculated by the simple recurrent formula

$$V_k^{(n)} = G_{k+1}^{(n)} + \sum_{i=1}^{k} G_i^{(n)} H^{(n)} V_{k-i}^{(n)}, \quad k = 0, 1, \dots \; . \tag{3.96}$$

We then have

$$G_k^{(n+1)} = M^{(n)} V_k^{(n)} Q^{(n)} \quad and \quad A_k^{(n+1)} = \sum_{i=0}^{k} A_i^{(n)} V_{k-i-1}^{(n)} Q^{(n)}. \tag{3.97}$$

Remark 23 *One may obtain* every *matrix $Z_k, k = -s, -s+1, \dots$ by the reduction process. At first sight, it seems that if we have got $Z_{-s}, \dots, Z_k$ and want to obtain the next coefficient Z_{k+1}, we need to calculate a lot of additional reduced coefficients $G_k^{(n)}$. Fortunately, we can avoid this. From Subsection 2.4.5 (see also Section 2.2), we know that once $X_0, \dots, X_k$ are computed, the next coefficient X_{k+1} is uniquely determined by the next finite subsystem of fundamental equations (FH)*

$$\begin{gathered} X_k G_0 = R_k - \textstyle\sum_{i=1}^{k} X_{k-i} G_i \\ X_{k+1} G_0 + X_k G_1 = R_{k+1} - \textstyle\sum_{i=1}^{k} X_{k-i} G_{i+1} \\ \cdots \\ X_{k+s} G_0 + \dots + X_k G_s = R_{k+s} - \textstyle\sum_{i=1}^{k} X_{k-i} G_{i+s} \end{gathered}$$

where

$$R_k = \begin{cases} 0, & k < s, \\ P_0^* - I, & k = s, \\ P_{k-s}^*, & k > s. \end{cases}$$

plus the corresponding normalization conditions. Note that the above system can be efficiently solved by the same reduction process as before. Moreover, we only need to recompute the right hand sides $A_k^{(n)}$. The coefficient matrices $G_k^{(n)}, k = 0, \dots, s-n, n = 1, \dots, s$ computed before can be used again. By doing this, one can even accelerate the computational procedure for $Z_k, k = -s, \dots, 0$ outlined above.

However, despite the above elegant modification of the reduction process, we still recommend to calculate the regular part of the Laurent series by using recurrent formula (3.67), (3.68) and (3.69).

Finally, we discuss the computational complexity of the above algorithm. Obviously, Steps 2 and 4 have the highest computational burden. In fact, Step 2 even prevails. Therefore, it suffices to estimate the number of arithmetic operations in Step 2 to obtain the computational complexity of the reduction process.

Step 2 consists of s reduction steps. Note that the first reduction step is the most demanding from a computational point of view, since it reduces the determining system from the full state space into the aggregated chain subspace with dimension m_1 equal to the number of ergodic classes in the original unperturbed chain. It is not difficult to see that the number of operations in this procedure is $O(s^2N^3)$. Indeed, multiplying two $N \times N$-matrices requires $O(N^3)$ operations and the recurrent formulas (3.96),(3.97) for $k = 0, ..., 2s - 1$ require $O(s^2)$ such multiplications. After this crucial first step, we only deal with matrices whose dimension does not exceed m_1. The complexity of the other reduction steps can be estimated as $O(s^3m_1^3)$. Thus, Step 2 requires $O(s^2N^3 + s^3m_1^3)$ operations.

Let us now discuss this evaluation. In most practical applications $m_1 << N$ and $s << N$, that is, the number of ergodic classes and the order of singularity are much less than N, the number of states of the original chain. Therefore, the complexity of the algorithm is in fact not much worse than $O(N^3)$ (or $O(N^4)$, see remark below about the determination of s). However, if $m_1 \sim N$, then the complexity of the algorithm is $O(s^3N^3)$. The latter may increase significantly (even up to $O(N^6)$), if s is of the same order of magnitude as N. However, we are convinced that the cases of large s are quite rare.

Finally, note that the determination of s by the method proposed in [63] requires $O(N^4)$ operations. However, if s has not been determined, then the above reduction-process algorithm can be run with $s := 1, 2, \dots$ until $G_0^{(s)}$ has rank $m_s - 1$, in which case s is the order of the pole. Therefore, assuming that $m_1 \ll N$, the computational complexity to determine both s and the singular part Z^s is just $O(s^3N^3)$. When $s \ll N$ (as one may expect in practice), compare with $O(N^4)$ just for getting s in [63].

3.3.4 Asymptotic expansions for mean first passage times

As was noticed in the introductory Section 3.1, the fundamental matrix expansion (3.47) can be employed to obtain a close approximation of mean first passage times of the singularly perturbed Markov chain. In particular, it gives the "time-scales" to link states in different ergodic classes (of the unperturbed chain), which is valuable information (e.g. for simulation).

For an ergodic Markov chain, the *mean first passage time* between states i and j are given by the formula [79, 89]

$$m_{ij} = \frac{z_{jj} - z_{ij}}{\mu_j}, \tag{3.98}$$

where $Z := \{z_{ij}\}_{i,j=1}^N$ and $\mu := \{\mu_j\}_{j=1}^N$ are the fundamental matrix and the stationary distribution of the Markov chain, respectively.

Since by Assumption (A) the singularly perturbed Markov chain is regular, (3.98) is valid, so that

$$m_{ij}(\varepsilon) = \frac{z_{jj}(\varepsilon) - z_{ij}(\varepsilon)}{\mu_j(\varepsilon)}, \quad \varepsilon \neq 0. \tag{3.99}$$

Substituting the expansions (3.43) and (3.47) into (3.99), we immediately obtain the Laurent series expansion

$$m_{ij}(\varepsilon) = \frac{1}{\varepsilon^{s_{ij}}} m_{ij}^{(-s_{ij})} + \ldots + \frac{1}{\varepsilon} m_{ij}^{(-1)} + m_{ij}^{(0)} + \varepsilon m_{ij}^{(1)} + \ldots, \quad \varepsilon \neq 0.$$

Note that the series inversion $1/\mu_j(\varepsilon)$ is an elementary task, since $\mu_j(\varepsilon) = \mu_j^{(0)} + \varepsilon \mu_j^{(1)} + \ldots$ is the *scalar* Taylor series. Note that in general, the order of singularity s_{ij} of $m_{ij}(\varepsilon)$, is different from the order of singularity s of the fundamental matrix, and depends on the particular pair of states i, j considered.

Hassin and Haviv [63], Haviv and Ritov [66, 67] give a nice elementary example of a singularly perturbed Markov chain with second-order singularity. Our reduction process and the Laurent series expansion for mean first passage times are illustrated on this elementary example.

Consider the singularly perturbed Markov chain with following transition matrix

$$P(\varepsilon) = P_0 + \varepsilon G_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

It is a second order perturbation, that is $s = 2$ [66, 63]. Hence, the reduction process consists of just two steps. Using standard definitions [79, 89] we get the matrices of normalized left and right eigenvectors M, Q, the ergodic projection P^* and the reduced resolvent H of the unperturbed chain, and using [40] (or a method of Subsection 3.2.2), the Taylor series expansion of the perturbed ergodic projection $P^*(\varepsilon) = P_0^* + \varepsilon P_1^* + \ldots$ as shown below

$$M^{(0)} = M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; Q^{(0)} = Q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix};$$

$$P^{*(0)} = P^* = QM$$

$$H^{(0)} = H = (I - P_0 + P^*)^{-1} - P^* = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_0^{(0)} = P_0^* - I = \begin{bmatrix} -1 & 0.5 & 0 & 0.5 \\ 0 & -0.5 & 0 & 0.5 \\ 0 & 0.5 & -1 & 0.5 \\ 0 & 0.5 & 0 & -0.5 \end{bmatrix}; A_1^{(0)} = P_1^* = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \end{bmatrix};$$

$$A_2^{(0)} = P_2^* = \begin{bmatrix} -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix}.$$

Now, after the first step of the reduction process, we have

$$G_0^{(1)} = MG_1Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; G_1^{(1)} = MG_1HG_1Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$G_2^{(1)} = MG_1HG_1HG_1Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; G_3^{(1)} = MG_1HG_1HG_1HG_1Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$A_0^{(1)} = A_0^{(0)}Q = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix}; A_1^{(1)} = (A_1^{(0)} + A_0^{(0)}HG_1)Q = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix};$$

$$A_2^{(1)} = (A_2^{(0)} + A_1^{(0)}HG_1 + A_0^{(0)}HG_1HG_1)Q = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

In this particular example, $M^{(1)} = Q^{(1)} = I_2, P^{*(1)} = I_2$ and $H^{(1)} = 0$, so that the next reduction step is straightforward. Namely,

$$G_0^{(2)} = G_1^{(1)} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}; G_1^{(2)} = G_2^{(1)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; G_2^{(2)} = G_3^{(1)} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

and $A_k^{(2)} = A_k^{(1)}, k = 0, 1, 2$.

The above matrices are used in the (final step) reduced system (FHs) that can be solved easily via the recurrent formulas (3.94).

$$P^{*(2)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, H^{(2)} = H = (P^{*(2)} - G_0^{(2)})^{-1} - P^{*(2)} = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix};$$

$$X_0^{(2)} = -A_0^{(2)}H^{(2)} = \begin{bmatrix} 0.25 & -0.25 \\ 0.25 & -0.25 \\ -0.25 & 0.25 \\ -0.25 & 0.25 \end{bmatrix}; X_1^{(2)} = (X_0^{(2)}G_1^{(2)} - A_1^{(2)})H^{(2)} = \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \\ 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix};$$

$$X_2^{(2)} = (X_0^{(2)} G_2^{(2)} + X_1^{(2)} G_1^{(2)} - A_2^{(2)}) H^{(2)} = 0$$

Then, we use the recurrent formulas (3.84) to successively restore $X_k^{(1)}, k = 0, 1, 2$ and $X_k^{(0)}, k = 0, 1, 2$. Again, due to $M^{(1)} = I_2$ and $H^{(1)} = 0$, we have $X_k^{(1)} = X_k^{(2)}, k = 0, 1, 2$.

$$X_0^{(0)} = X_0^{(1)} M^{(0)} = \begin{bmatrix} 0 & 0.25 & 0 & -0.25 \\ 0 & 0.25 & 0 & -0.25 \\ 0 & -0.25 & 0 & 0.25 \\ 0 & -0.25 & 0 & 0.25 \end{bmatrix};$$

$$X_1^{(0)} = X_1^{(1)} M^{(0)} + X_0^{(0)} G_1 H^{(0)} = \begin{bmatrix} 0.25 & -0.5 & -0.25 & 0.5 \\ 0.25 & 0 & -0.25 & 0 \\ -0.25 & 0.5 & 0.25 & -0.5 \\ -0.25 & 0 & 0.25 & 0 \end{bmatrix};$$

$$X_2^{(0)} = X_2^{(1)} M^{(0)} + (X_1^{(0)} G_1 - A^{(0)}) H^{(0)} = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, from (3.70), we get

$$Z_{-2} = X_0^{(0)},\ Z_{-1} = X_1^{(0)},\ Z_0 = X_2^{(0)} + P_0^* = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix}.$$

Using MATLAB symbolic toolbox for a direct expansion of $(I - P(\varepsilon) + P^*(\varepsilon))^{-1}$, one may check that both results coincide. Schweitzer's formula (3.44) also permits us to check the validity of the Taylor's series expansion of $\mu(\epsilon)$. Indeed,

$$\mu_k = \mu_0 (G_1 Z_0)^k, \quad k = 0, 1, \dots .$$

Substituting the expression obtained for Z_0 into the above equation, we get

$$\mu_k = (-1)^k [\ -0.5 \ \ 0.5 \ \ -0.5 \ \ 0.5\ ], \quad k = 1, 2, \dots ,$$

which coincides with the expression for $\mu_k, k \geq 1$ given in [66].

Now, let us apply the coefficients of the fundamental matrix expansion to compute mean first-passage times via formula (3.99). Due to the symmetry of this particular example, one only needs to compute the mean first passage times to states 1 and 2.

$$m_{21}(\varepsilon) = \frac{z_{11}(\varepsilon) - z_{21}(\varepsilon)}{\mu_1(\varepsilon)} = [(\frac{0.25}{\varepsilon} + 0.5 + \dots) - (\frac{0.25}{\varepsilon} + 0 + \dots)]/[0.5\varepsilon - 0.5\varepsilon^2 + \dots] = \frac{1}{\varepsilon} + \dots,$$

$$m_{31}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + ..., \quad m_{41}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + ...,$$
$$m_{12}(\varepsilon) = \frac{1}{\varepsilon} + ..., \quad m_{32}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{0}{\varepsilon} + ..., \quad m_{42}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + ...$$

The mean first passage times exhibit different time scales. Namely, the mean first passage times $m_{21}(\varepsilon), m_{12}(\varepsilon)$ between states in the same ergodic class are of order ε^{-1} whereas the mean first passage times $m_{31}(\varepsilon), m_{41}(\varepsilon), m_{32}(\varepsilon), m_{42}(\varepsilon)$, that is, between states in different ergodic classes, are of order ϵ^{-2}. It confirms the intuition that a transition between states in different ergodic classes takes much more time to occur than a transition between states in the same class. Finally, we would like to emphasize that the present approach permits us to handle perturbations with *any* order of singularity. By contrast, the method of Latouche and Louchard [100] is restricted to the case of first-order singularity.

3.4 Uniform optimality in singularly perturbed Markov decision processes

3.4.1 Preliminaries

In this section we study the Markov decision processes with perturbed transition probabilities (3.16). Using asymptotic linear programming [81, 82, 71], we obtain the uniform optimal policy $\pi_* \in \Pi^S$ such that

$$g_i^\varepsilon(\pi_*) \geq g_i^\varepsilon(\pi), \quad i \in \mathbb{X},$$

for any other stationary policy $\pi \in \Pi^S$. As was shown in Example 2, the uniform optimality is a more strong criterion than limit optimality.

Here we provide two algorithms to obtain a uniform optimal policy. The first algorithm is based on linear programming approach [41, 70, 85, 86] and the second algorithm is based on policy iteration procedure [72, 45, 119]. Both algorithms essentially use the concept of ordering in the field of rational functions [81, 82].

We would like to emphasize that in contrast to the previous research the present method does not need any assumptions about the ergodic structure of the perturbed and the unperturbed MDPs.

The following notation is used below: δ_{ij} is the Kronecker delta, $|B|$ is the cardinality of a set B, that is, the number of elements in B, and $\mathbb{K} = \{(i,a) : i \in \mathbb{X}, a \in \mathbb{A}(i)\}$. For two vectors c and d of appropriate dimensions, $c \cdot d$ denotes the summation over common indices (scalar product).

3.4.2 Linear programming for average optimality

Let us fix ε and consider the solution to the perturbed problem (L^ε) (see p.98) based on linear programming approach [41, 70]. Given the initial distribution β over $\mathbb{X}$, define $\mathbb{F}_\beta^\varepsilon$ to be the set of $\{(z,\zeta)\}$, $z,\zeta \in \mathrm{I\!R}^{|\mathbb{K}|}$, that satisfy

$$\sum_{i\in\mathbb{X}}\sum_{a\in\mathbb{A}(i)}(\delta_{ij}-p(j|i,a)-\varepsilon d(j|i,a))z(i,a)=0, \qquad \forall j\in\mathbb{X} \tag{3.100}$$

$$\sum_{a\in\mathbb{A}_j} z(j,a)+\sum_{i\in\mathbb{X}}\sum_{a\in\mathbb{A}(i)}(\delta_{ij}-p(j|i,a)-\varepsilon d(j|i,a))\zeta(i,a)=\beta(j), \qquad \forall j\in\mathbb{X} \tag{3.101}$$

$$z\geq 0, \qquad \zeta\geq 0. \tag{3.102}$$

Remark 24 *(i) Every $z(\cdot,\cdot)\in\mathbb{F}_\beta^\varepsilon$ satisfies $\sum_{i,a}z(i,a)=1$. This can be seen by summing equation (3.101) over all $j\in\mathbb{X}$.*
(ii) We may delete one of the constraints among (3.100). This follows from the fact that the coefficients of $z(i,a)$ in (3.100) sum to 0.

Next we consider the perturbed linear program with fixed ε: Find $z,\zeta\in\mathrm{I\!R}^{|\mathbb{K}|}$, such that

$$\max\{r\cdot z\} \qquad (LP^\varepsilon)$$

subject to

$$(z,\zeta)\in\mathbb{F}_\beta^\varepsilon.$$

This linear program (LP^ε) is related to (L^ε) in the following way. Given any $(z,\zeta)\in\mathbb{F}_\beta^\varepsilon$, define the stationary policy $\pi\in\Pi^S$ by

$$\pi(a|i)=\begin{cases}\dfrac{z(i,a)}{\sum_{a'\in\mathbb{A}(i)}z(i,a')}, & \textit{if } \sum_{a'\in\mathbb{A}(i)}z(i,a')>0\\[2ex] \dfrac{\zeta(i,a)}{\sum_{a'\in\mathbb{A}(i)}\zeta(i,a')}, & \textit{if } \sum_{a'\in\mathbb{A}(i)}z(i,a')=0 \textit{ and } \sum_{a'\in\mathbb{A}(i)}\zeta(i,a')>0\\[2ex] \textit{arbitrary}, & \textit{otherwise.}\end{cases} \tag{3.103}$$

This construction was introduced in [41] by Denardo and Fox and later was developed by Hordijk and Kallenberg [70]. The following lemma is an immediate corollary of the results in [70], for $\varepsilon>0$ and fixed.

Lemma 10 *(i) Fix $\varepsilon>0$. The optimal values of the perturbed MDP (L^ε) and of (LP^ε) are equal. (ii) Suppose that $(z^*(\varepsilon),\zeta^*(\varepsilon))$ is an optimal solution of $LP(\varepsilon)$ with an associated policy π_* constructed via (3.103), then π_* is an average optimal policy for (L^ε).*

However, the results in [70] do not permit us to find a uniformly optimal deterministic policy. The latter is a more difficult problem both from a theoretical point of view and due to the fact that the rank of the coefficient matrix of (LP^{ε}) can change at $\varepsilon = 0$ (the case of the singular perturbation). This can create numerical problems when $\varepsilon > 0$ is small.

3.4.3 Application of asymptotic linear programming

A key step in our method is to find a basis for (LP^{ε}), which is optimal for all ε sufficiently small. In order to achieve this we use an *asymptotic linear programming* approach due to Jeroslow [81, 82]. This approach allows to find an optimal solution to the parametric linear program

$$\max c(\varepsilon)x$$

subject to

$$A(\varepsilon)x = b(\varepsilon), \quad x \geq 0,$$

where the elements of $A(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ are rational functions of ε. The solution obtained by the method of [81, 82] is optimal for all ε sufficiently small but different from zero. Note that our problem (LP^{ε}) is a parametric linear program, where only elements of the coefficient matrix $A(\varepsilon)$ are linear functions of the perturbation parameter ε.

The asymptotic linear programming of Jeroslow [81, 82] was previously applied to MDPs by Hordijk et al. [71] to obtain a Blackwell optimal policy and by Huang and Veinott [74] in the context of Markov branching decision chains.

The main idea of asymptotic linear programming is to consider the entries of the coefficient matrix of (LP^{ε}) not as the elements of the usual Archimedean ordered field of the real numbers, but as elements of the non-Archimedean ordered field $\mathbb{F}(\mathbb{R})$ of rational functions with real coefficients. An order relation between two rational functions will be defined as in [81, 82]. We denote the ordering and equality in this field by " $>_l$ " and " $=_l$ ", respectively (for more detail see auxiliary Subsection 3.4.7).

The logical steps of asymptotic linear programming method (ALP) are:

- In order to identify an initial feasible basis, we may add the artificial variables $\rho(i), \xi(i), i \in \mathbb{X}$, one variable for each of the constraints in (3.100) and (3.101), respectively. We note that

$$(z, \zeta, \rho, \xi) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \beta)$$

is feasible. A feasible initial basis is an identity matrix corresponding to the artificial variables.

- One may follow the standard or the revised simplex method for solving the linear program $\max\{r \cdot \mathrm{x}\}$ $s.t.$ $A(\varepsilon)\mathrm{x} =_l b, \mathrm{x} \geq 0$, but with respect to the order and equality relations of $\mathbb{F}(\mathbb{R})$. Note that after adding the artificial variables, $A(\varepsilon)$ is of full rank. A basis is optimal if and only if the reduced costs $\overline{r}_N(\varepsilon) := r_N - r_B B^{-1}(\varepsilon)N(\varepsilon) \leq_l 0$. Here, r_B (r_N, resp.) is the vector of rewards corresponding to the elements in the basis (not in the basis, resp.); $B(\varepsilon)$ is a basic submatrix of $A(\varepsilon)$, and $N(\varepsilon)$ is the corresponding nonbasic submatrix. Let us use the simplex tableaux. We shall write in the simplex tableaux only the numerators of the rational functions which are the coefficients of variables in (LP^ε). This is possible because the entries of the simplex tableaux can be assumed to have common denominators. Let y_{ik} denote the ik-th entry of the current tableau. In the first column we indicate the basic variables. The reduced costs are written in the bottom row of the tableau, and $B^{-1}b$ appears in the right most column. We denote the i-th element of the latter by y_{i0}.

- The entry and exit rules are natural extensions of the usual simplex rules:
  (i) First, we determine the entering non-basic column. Any column (not in the basis) whose corresponding reduced cost is strictly positive (with respect to "$>_l$") can be chosen to enter. As usual, we can choose the column with the largest nonnegative reduced cost to enter.
  (ii) The column p to leave the basis is one that minimizes (with respect to "$>_l$") the ratio $y_{i0}(\varepsilon)/y_{ik}(\epsilon)$ among those $i's$ satisfying $y_{ik}(\varepsilon) >_l 0$ (k is the index of the entering column).
  We now perform the pivot operation and construct the new tableau. This employs algebraic operations over the field of polynomials $P(\mathbb{R})$.

- The order of the polynomials appearing in both the numerators and denominators never exceeds the number of constraints, that is, twice the number of states. This follows from the fact that (i) each coefficient of the constraints $A(\varepsilon)$ is linear in ε, and (ii) the elements in the simplex tableau are given by $B^{-1}(\varepsilon)b$, where $A(\varepsilon)$ is the matrix of constraints, and b is the vector of the right hand side coefficients.

3.4.4 Numerical examples

Consider the following illustrative examples.

Example 3 Consider an MDP with $\mathbb{X} = \{1, 2\}$, $\mathbb{A}(1) = \{a_1, b_1\}$, and $\mathbb{A}(2) = \{a_2, b_2\}$; let

$$\begin{array}{ll}
p^\varepsilon(1|1, a_1) = 1, & r(1, a_1) = 10 \\
p^\varepsilon(1|1, b_1) = 1 - \varepsilon, & r(1, b_1) = 10 \\
p^\varepsilon(2|1, b_1) = \varepsilon, & \\
p^\varepsilon(1|2, a_2) = 1, & r(2, a_2) = 0 \\
p^\varepsilon(1|2, b_2) = p^\varepsilon(2|2, b_2) = 0.5, & r(2, b_2) = 5.
\end{array}$$

We take $\beta(1) = \beta(2) = 0.5$.

By adding the artificial variables, the linear program (LP^ε) becomes:

$$Max\,\{10z(1, a_1) + 10z(1, b_1) + 5z(2, b_2) - 100\xi(1) - 100\xi(2) - 100\rho(1) - 100\rho(2)\}$$

subject to the constraints

$$\begin{array}{lllll}
+\varepsilon z(1, b_1) - z(2, a_2) - 0.5z(2, b_2) & & +\xi(1) & & = 0 \\
-\varepsilon z(1, b_1) + z(2, a_2) + 0.5z(2, b_2) & & +\xi(2) & & = 0 \\
+z(1, a_1) + z(1, b_1) & +\varepsilon\zeta(1, b_1) - \zeta(2, a_2) - 0.5\zeta(2, b_2) & & +\rho(1) & = 0.5 \\
\quad +z(2, a_2) + z(2, b_2) & -\varepsilon\zeta(1, b_1) + \zeta(2, a_2) + 0.5\zeta(2, b_2) & & +\rho(2) & = 0.5
\end{array}$$

$$z(1, a_1), z(1, b_1), z(2, a_2), z(2, b_2), \zeta(1, a_1), \zeta(1, b_1), \zeta(2, a_2), \zeta(2, b_2), \xi(1), \xi(2), \rho(1), \rho(2) \geq 0.$$

A reader familiar with MDPs will note that this example is of the so-called "unichain" model (for $\varepsilon > 0$). Consequently, a simpler version of (LP^ε) could have been used (e.g., see Kallenberg [85] p.132). However, the present version of (LP^ε) applies generally and hence is better suited for demonstrating this new technique.

We added a penalty term for the artificial variables to ensure that they exit the basis. We shall delete the 2nd constraint (and will thus not use $\xi(2)$, see Remark 24 (ii)).

The first simplex tableau is given in Table 3.1. The common denominator is 1. We then choose the 1st column $z(1, a_1)$ to enter. The row/variable to exit is the 2nd one, $\rho(1)$. In all the tableaux the pivoting element is underlined.

The second simplex tableau is given in Table 3.2. The common denominator is again one. The column that enters the basis is $\zeta(1, a_1)$ for which the reduced cost 110 is the largest. The column to exit is $\rho(2)$.

The third and fourth simplex tableaux are given in Tables 3.3, and 3.4. The common denominator in Table 3.3 is 1, and in 3.4 it is ε.

At this stage we have obtained an optimal solution over the field of rational functions $\mathbb{F}(\mathbb{R})$ with real coefficients (see auxiliary Subsection 3.4.7). A uniformly optimal policy uses actions a_1 and a_2 in states 1 and 2,respectively, as follows from (3.103). Note that it is uniformly optimal for all $\varepsilon > 0$ and sufficiently small. The value of this MDP is 10

	artif. variables			z *variables*				ζ *variables*				
Basis	$\xi(1)$	$\rho(1)$	$\rho(2)$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	r.h.s.
$\xi(1)$	1	0	0	0	ε	-1	-0.5	0	0	0	0	0
$\rho(1)$	0	1	0	$\underline{1}$	1	0	0	0	ε	-1	-0.5	0.5
$\rho(2)$	0	0	1	0	0	1	1	0	$-\varepsilon$	1	0.5	0.5
red. cost	0	0	0	110	$110 + 100\varepsilon$	0	55	0	0	0	0	

Table 3.1: First tableau of Example 3

	artif. variables			z *variables*				ζ *variables*				
Basis	$\xi(1)$	$\rho(1)$	$\rho(2)$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	r.h.s.
$\xi(1)$	1	0	0	0	ε	-1	-0.5	0	0	0	0	0
$z(1, a_1)$	0	1	0	1	1	0	0	0	ε	-1	-0.5	0.5
$\rho(2)$	0	0	1	0	0	1	1	0	$-\varepsilon$	$\underline{1}$	0.5	0.5
red. cost	0	-110	0	0	100ε	0	55	0	-110ε	110	55	

Table 3.2: Second tableau of Example 3

	artif. variables			z *variables*				ζ *variables*				
Basis	$\xi(1)$	$\rho(1)$	$\rho(2)$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	r.h.s.
$\xi(1)$	1	0	0	0	$\underline{\varepsilon}$	-1	-0.5	0	0	0	0	0
$z(1, a_1)$	0	1	1	1	1	1	1	0	0	0	0	1
$\zeta(2, a_2)$	0	0	1	0	0	1	1	0	$-\varepsilon$	1	0.5	0.5
red. cost	0	-110	-110	0	100ε	-110	-55	0	0	0	0	

Table 3.3: Third tableau of Example 3

	artif. variables			z *variables*				ζ *variables*				
Basis	$\xi(1)$	$\rho(1)$	$\rho(2)$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	$1a_1$	$1b_1$	$2a_2$	$2b_2$	r.h.s.
$z(1, b_1)$	1	0	0	0	ε	-1	-0.5	0	0	0	0	0
$z(1, a_1)$	-1	ε	ε	ε	0	$1 + \varepsilon$	$0.5 + \varepsilon$	0	0	0	0	ε
$\zeta(2, a_2)$	0	0	ε	0	0	ε	ε	0	$-\varepsilon^2$	ε	0.5ε	0.5ε
red. cost	-100ε	-110 ε	-110ε	0	0	-10ε	-5ε	0	0	0	0	

Table 3.4: Fourth tableau of Example 3

(independently of ε). The stationary deterministic policies that choose action b_1 in state 1 are optimal for the limit problem but are not optimal for any positive ε.

Now consider the second example, which exposes the singularity of the problem.

Example 4 *Consider an MDP with* $\mathbb{X} = \{1, 2, 3\}$ *and* $\mathbb{A}(1) = \{a_1, b_1\}$, $\mathbb{A}(2) = \{a_2\}$, $\mathbb{A}(3) = \{a_3\}$. *And let*

$$\begin{array}{ll} p^\varepsilon(1|1,a_1) = 1-\varepsilon, p^\varepsilon(3|1,a_1) = \varepsilon, & r(1,a_1) = 10 \\ p^\varepsilon(1|1,b_1) = 1-\varepsilon, p^\varepsilon(2|1,b_1) = \varepsilon, & r(1,b_1) = 10 \\ p^\varepsilon(2|2,a_2) = 1-\varepsilon, p^\varepsilon(3|2,a_2) = \varepsilon, & r(2,a_2) = 10 \\ p^\varepsilon(1|3,a_3) = p^\varepsilon(2|3,a_3) = 0.5, & r(3,a_3) = 0. \end{array}$$

Since the perturbed Markov chain is ergodic for all stationary policies, we may use a simplified form of (LP^ε) with only the z-variables (see Kallenberg [85], p.128). Thus our (LP^ε) problem is

$$\max\{\sum_{i,a} r(i,a)z(i,a) | \mathcal{A}(\varepsilon)z = b, z \geq 0\},$$

where

$$\mathcal{A}(\varepsilon) = \begin{pmatrix} \varepsilon & \varepsilon & 0 & -0.5 \\ 0 & -\varepsilon & \varepsilon & -0.5 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We shall start with a basis consisting of the columns that correspond to the policy (a, a, a). Hence

$$B(\varepsilon) = \begin{pmatrix} \varepsilon & 0 & -0.5 \\ 0 & \varepsilon & -0.5 \\ 1 & 1 & 1 \end{pmatrix}.$$

We have

$$B^{-1}(\varepsilon) = (\varepsilon + \varepsilon^2)^{-1} \begin{pmatrix} 0.5+\varepsilon & -0.5 & 0.5\varepsilon \\ -0.5 & 0.5+\varepsilon & 0.5\varepsilon \\ -\varepsilon & -\varepsilon & \varepsilon^2 \end{pmatrix}.$$

Note that $B^{-1}(\varepsilon)$ has a singularity at $\varepsilon = 0$. Now,

$$B^{-1}(\varepsilon)b = (1+\varepsilon)^{-1} \begin{pmatrix} 0.5 \\ 0.5 \\ \varepsilon \end{pmatrix},$$

and finally

$$B^{-1}(\varepsilon)\mathcal{A}_{[1,b]}(\varepsilon) = (1+\varepsilon)^{-1} \begin{pmatrix} 1.5+\varepsilon \\ -0.5-\varepsilon \\ \varepsilon \end{pmatrix}.$$

	z variables				
Basis	$(1, a_1)$	$(1, b_1)$	$(2, a_2)$	$(3, a_3)$	r.h.s.
$z(1, a_1)$	$1 + \varepsilon$	$\underline{1.5 + \varepsilon}$	0	0	0.5
$z(2, a_2)$	0	$-0.5 - \varepsilon$	$1 + \varepsilon$	0	0.5
$z(3, a_3)$	0	ε	0	$1 + \varepsilon$	ε
red. cost	0	$10\ \varepsilon$	0	0	

Table 3.5: First tableau of Example 4

	z variables				
Basis	$(1, a_1)$	$(1, b_1)$	$(2, a_2)$	$(3, a_3)$	r.h.s.
$z(1, b_1)$	$1 + \varepsilon$	$1.5 + \varepsilon$	0	0	0.5
$z(2, a_2)$	$0.5 + \varepsilon$	0	$1.5 + \varepsilon$	0	1
$z(3, a_3)$	$-\varepsilon$	0	0	$1.5 + \varepsilon$	ε
red. cost	- 10 ε	0	0	0	

Table 3.6: Second tableau of Example 4

The reduced reward coefficients corresponding to the element $(1, b)$ is

$$r(1, b) - (r^B B^{-1}(\varepsilon)\mathcal{A}_{[1,b_1]}(\varepsilon)) = 10\varepsilon(1 + \varepsilon)^{-1},$$

where r^B is the vector of rewards corresponding to the elements in the basis.

In the first simplex tableau (Table 3.5) we only write the numerators. The denominator which is common to all entries is $(1 + \varepsilon)$. The expected long-run average reward corresponding to this basis is $10(1 + \varepsilon)^{-1}$.

To obtain the second tableau, the second column enters the basis and the first row exits the basis. This simply corresponds to the policy (b, a, a). The common denominator to all entries in the second Tableau is $(1.5 + e)$, which we have omitted.

The reward corresponding to the new basis is $15(1.5 + \varepsilon)^{-1}$ and the policy (b, a, a) is uniformly optimal.

3.4.5 Asymptotic policy iteration

The ordering for rational functions proposed in [81, 82] can also be used in a policy iteration type method to obtain a uniformly optimal policy. The steps of *Asymptotic Policy Iteration Method* (API) are as follows:

1. Start with an arbitrary stationary deterministic policy $f \in \Pi^D$.

2. Solve the equations with unknowns g, h, w, which are all $|\mathbb{X}|$-dimensional vectors whose entries are elements in the field $\mathbb{F}(\mathbb{R})$ of rational functions (in ε) with real coefficients:

$$P^{\epsilon}(f)g =_l g \tag{3.104}$$

$$(P^{\epsilon}(f) - I)h =_l g - r(f) \tag{3.105}$$

$$(P^{\epsilon}(f) - I)w =_l h \tag{3.106}$$

   Note that, over the reals, g and h are uniquely determined (e.g., see [119] p. 452), and it can be shown that, over the field of rational functions, there exist unique $g =_l g^{\varepsilon}(f), h =_l h^{\varepsilon}(f)$ satisfying (3.104)-(3.106).

3. Choose a stationary deterministic policy f' that selects an action $f'(i) = a$ in state i as follows: If there is an action a such that

$$\sum_j p^{\varepsilon}(j|i,a)g_j >_l g_i,$$

   then $f'(i) = a$. If such an action does not exist, but there is an action a for which

$$\sum_j p^{\varepsilon}(j|i,a)g_j =_l g_i \quad \text{and} \quad r(i,a) + \sum_j p^{\varepsilon}(j|i,a)h_j >_l g_i + h_i,$$

   then set $f'(i) := a$. If there are no actions satisfying either of the above conditions, then the action at this state is unchanged (we set $f'(i) := f(i)$).

4. If $f' = f$ then stop. Otherwise, set $f := f'$, and return to Step 1.

3.4.6 Finite convergence of ALP and API

The next proposition demonstrates that both the (ALP) and (API) methods indeed find a uniformly optimal policy.

Proposition 5 *Consider a singularly perturbed MDP and the associated optimal control problem L^{ε}. Apply either the (ALP) or the (API) method to this problem. Both methods terminate in finitely many iterations and yield a uniformly optimal control.*

PROOF: (i) For the (API) method.
In Steps 2 and 3 of (API) the relations " $>_l$ " and " $=_l$ " are with respect to the ordering of rational function with real coefficients, as in [81, 71] (see exact definitions given in Subsection 3.4.7). The above algorithm finds an optimal policy in finitely many steps. This follows from the fact that each step in the algorithm yields a strict improvement of the

long-run expected average reward criterion with respect to the ordering over $\mathbb{F}(\mathbb{R})$. Now, the uniformity follows from the following reasoning. Consider a rational function p/q, where p and q are polynomials with rational coefficients. It is called positive (nonnegative, resp.) if $p/q >_l (\geq_l) p_0$, respectively. Then p/q is positive if and only if there exists $\varepsilon_0 > 0$ such that $p(\epsilon)/q(\epsilon) > 0$ for all $\varepsilon \in (0, \varepsilon_0]$. The latter implies that we find a policy which is optimal for all sufficiently small ε, namely, a uniform optimal policy.

(ii) For the (ALP) method.

The proof of the convergence of the general asymptotic linear programming is given by Jeroslow [81, 82]; see also Theorem 28 in Section 4.2.

□

3.4.7 The field of rational functions with real coefficients

In order for this thesis to be self contained, we include a brief description of the field of rational functions (see e.g. [81, 71]). Let $P(\mathbb{R})$ denote the set of polynomials with real coefficients of the form

$$p(x) = a_0 + a_1 x + ... + a_n x^n, \qquad n \in \mathbb{N}.$$

Let p_0 and p_1 denote the polynomials $p_0 \triangleq 0, p_1 \triangleq 1$, respectively. Let $d(p)$ denote the dominating (leading) coefficient of the polynomial p; it is a_k, where k is the smallest integer with $a_k \neq 0$.

The field $\mathbb{F}(\mathbb{R})$ of rational functions with real coefficients consists of elements of the form p/q, where $p, q \in P(\mathbb{R})$. A polynomial p is identified with $p/1$. We say that $p/q =_l r/s$ if $ps = qr$, $p, q, r, s \in P(\mathbb{R})$. The addition and multiplication of elements in $\mathbb{F}(\mathbb{R})$ are defined by

$$\frac{p}{q} + \frac{r}{s} =_l \frac{ps + rq}{qs}, \qquad \frac{p}{q} \cdot \frac{r}{s} =_l \frac{pr}{qs}, \qquad p, q, r, s \in P(\mathbb{R}).$$

p_0 and p_1 are the identities with respect to addition and multiplication, respectively. A complete ordering on $\mathbb{F}(\mathbb{R})$ is obtained by defining

$$p/q >_l p_0 \quad \text{if and only if} \quad d(p)d(q) > 0.$$

With these definitions it can be shown (see [46] for the general methodology) that many of the properties that hold for unperturbed MDP's over $\mathbb{R}$, easily extend to the perturbed MDP's when considered over the field $\mathbb{F}(\mathbb{R})$. For instance, the uniqueness of the solution to the parametric equations (3.104)-(3.106).

3.5 Blackwell optimality via nested linear programs

3.5.1 Preliminaries and literature review

The discount optimal policy takes into account the behaviour of the MDP during the early stages of the process, whereas the average optimality takes care only of the long-run behaviour of the chain. In many problems, the asymptotic and transient behaviours of the chain are both important. In these cases, one can use sensitive discount optimality approach. Recall that the policy is called n-discount optimal, if it maximizes all terms up to the n-th in the Laurent series (3.12). Moreover, there exists a policy that is n-discount optimal for all $n \geq -1$. This policy is called *Blackwell* optimal. Note that the Blackwell optimal policy is the policy which is average optimal and at the same time discount optimal for all discount factors sufficiently close to 1. The latter means that a Blackwell optimal policy simultaneously takes into account the transient and the asymptotic behaviours of the chain. The following important result says that in order to find a Blackwell optimal policy it is sufficient to find n-discount optimal policy for some finite n [95, 148].

Theorem 25 *Suppose π_* is $(|S| - r - 1)$-discount optimal, where r is the number of ergodic classes in the Markov chain induced by transition matrix $P(\pi_*)$. Then π_* is Blackwell optimal. And moreover π_* is n-discount optimal for all $n \geq -1$.*

The bias optimality (in other words, 0-discount optimality (see pp.95-96)) and Blackwell optimality were first introduced in the pioneering work of Blackwell [19]. Later Veinott defined n-discount optimality [147] that provides a unified treatment of average optimality, bias optimality as well as Blackwell optimality. Moreover, in [147] he proposed an algorithm for the computation of n-discount optimal policy based on lexicographic ordering. Veinott developed further his algorithm in [148], where the proof of Theorem 25 can also be found.

Recently Puterman [119] proposed a method to obtain an n-discount optimal policy based on the succesive solution of n-discount optimality equations. To solve n-discount optimality equations Puterman used a policy iteration algorithm. The system of n-discount optimality equations appears implicitly in Veinott [147] and explicitly in Dekker [37] and in Dekker and Hordijk [38].

Furthermore, Hordijk *et. al.* [71] used the asymptotic linear programming approach of [81, 82] to calculate a Blackwell optimal policy. Lamond [93, 94] calculated a Blackwell optimal policy by using an asymptotic simplex method, which is a further modification of asymptotic linear programming. A reader will find more discussion on asymptotic simplex method in the ensuing Section 4.2. Unfortunately, the methods of [71] and [93, 94] cannot be used (at least at present) for the calculation of n-discount optimal policies that are not Blackwell optimal. Huang and Veinott [74] proposed an algorithm for Markov branching decision chains based on the series of parametric linear programs where only the objective function and the right hand side depend on the interest rate but not the coefficient matrix. The later fact facilitates the application of the asymptotic linear programming. However, one needs to use augmented matrices of dimension $(n+2)|\mathbb{X}|$ to determine an n-discount optimal policy. The authors of [74] pointed out that their method can be applied to Blackwell optimality.

In [42] (see also [44]) Denardo proposed an algorithm for determining the bias optimal policy via nested MDPs with truncated state space. For each truncated MDP one has to solve an average optimality problem with a specific immediate reward. One can solve MDPs either by policy iteration method or linear programming. Later Denardo [43] showed that his results can be extended to semi-Markov decision chains. The algorithm of [42] for bias optimality was further refined by Kallenberg [85, 86]. Note that the procedure of Puterman [119] can be considered as an extension of [42] to n-discount optimality with application of policy iteration technique. Also Cavazos-Cadena and Lasserre [26] proposed a similar approach to MDPs with denumerable state space.

Our algorithm for determining the n-discount optimal policy and Blackwell optimal policy is a generalization of the results in [42, 85] to n-discount optimality via nested linear programs. This approach can be considered as a counterpart to Puterman's method [119]. We would like to note that Denardo [42] has already pointed out the possibility of such an extension. We feel that the method of Huang and Veinott [74] is strongly related to our procedure. The authors of [74] also hinted on the connection of their method to the results of Denardo [40]. However, at present we cannot show explicitly the relationship between these two approaches. We would like to note that our algorithm can be used for the computation of Blackwell optimal as well as n-discount optimal policies. Moreover, since it is based on linear programming, the algorithm can be extended to the MDP model with general Borel state space [68, 152].

Sensitive discount optimality has several important applications in the control of queueing systems, inventory problem, investment management and immigration research [65, 74, 119, 148].

Here we restrict ourselves only to the case of a completely ergodic MDP model. Namely,

Assumption (E) *For any stationary policy π the Markov chain induced by transition matrix $P(\pi)$ has single ergodic class and there is no transient states.*

The above assumption implies that the ergodic projection $P^*(\pi)$ associated with any stationary policy π has identical rows. The latter implies that the expected average reward vector $g(\pi) = P^*(\pi)r(\pi)$ has identical elements as well. Therefore we shall use the notation $g(\pi)$ not only for the average reward vector but also for any its component. Since in every particular case it is clear whether we use a vector or a scalar, this notation should not cause any confusion.

The reminder of this section is organized as follows: In the next two subsections we briefly review the known results on average optimality and bias optimality [42, 85]. In Subsection 3.5.4 we present our general approach to n-discount optimality. The formal algorithm for the computation of Blackwell optimal as well as n-discount optimal policies via nested linear programs are given in Subsection 3.5.5.

3.5.2 Average optimality: (-1)-discount optimality

In this subsection we recall the well-known algorithm [45, 85, 119] for finding an average optimal policy for the completely ergodic Markov decision process (see Assumption (E)). This algorithm is based on the pair of dual linear programs. The primal program is

$$\min \ \tilde{g} \tag{3.107}$$

$$\text{s.t.} \qquad \tilde{g} + \sum_j (\delta_{ij} - p(j|i,a))\tilde{h}_j \geq r(i,a), i \in \mathbb{X}, a \in \mathbb{A}(i)\}$$

and the corresponding dual program is

$$\max \sum_{i,a} r(i,a)x_{ia} \tag{3.108}$$

$$\text{s.t.} \quad \sum_{i,a}(\delta_{ij} - p(j|i,a))x_{ia} = 0, \quad \sum_{i,a} x_{ia} = 1, \quad x_{ia} \geq 0.$$

Let (g_*, h_*) be an optimal solution of the linear program (3.107), then $g_* = g(\pi_*) = \max\{g(\pi)|\pi \in \Pi^S\}$, where π_* is an average optimal policy. An average optimal policy can be obtained from the dual linear program (3.108). Suppose that x_* is a basic optimal solution, then deterministic average optimal policy $f \in \Pi^D$ can be obtained as follows:

$$f(i) = a_i, \quad \text{such that} \quad x_{*ia_i} > 0. \tag{3.109}$$

Note that the above procedure defines a one-to-one correspondence between the sets of all basic optimal solutions of (3.108) and the set of all average optimal deterministic policies [45, 85, 119].

3.5.3 Bias optimality: 0-discount optimality

Now let us discuss the approach for the computation of the bias optimal policy based on the system of two nested linear programs [42, 85]. Recall that a bias optimal policy is an average optimal policy f which maximizes the bias vector $h(f) = H(f)r(f)$. The exposition of this section follows closely the book of Kallenberg [85, Ch.5].

We first solve the pair of dual linear programs (3.107) and (3.108) for the computation of the stationary average optimal policy. As a result, we have the optimal solutions (g_*, h_*) and x_* of the primal and dual linear programs respectively. Recall that g_* is an optimal value of the average reward. Let us define the following set

$$\mathbb{A}_0(i) \triangleq \{a \in \mathbb{A}(i) \mid g_* + \sum_j (\delta_{ij} - p(j|i,a))h_{*j} = r(i,a)\}. \tag{3.110}$$

Remark 25 *The above set can be determined in two ways: either by a direct search procedure among all actions $a \in \mathbb{A}(i)$ or from the set of optimal basic feasible solutions of the dual problem* (3.108)*, namely, if x_* is an optimal basic feasible solution, then by complementary slackness $a \in \mathbb{A}_0(i)$ whenever $x_{*ia} > 0$. In particular, the second approach will be justified by the next proposition.*

Let us remind that all statements of Section 3.5 are made under Assumption (E) .

Proposition 6 *Let f be any deterministic and stationary average optimal policy and (g_*, h_*) be an optimal solution to* (3.107)*. Then,*
(i) $f(i) \in \mathbb{A}_0(i)$, $i \in \mathbb{X}$, and
(ii) the corresponding bias vector $h(f) = H(f)r(f)$ can be calculated by the formula

$$h(f) = h_* - P^*(f)h_*.$$

Proof: See [42, 44, 85].

According to Definition 11, a stationary policy d is bias optimal or 0-discount optimal if $h_i(d) = \max_f\{h_i(f)|g(f) = g_*\}$. In other words an average optimal policy, which supplies the optimal value to the bias vector, is a bias optimal policy. It follows from the second part of the Proposition 6 that the maximization of $h(f) = h_* - P^*(f)h_*$ is equivalent to the maximization of $-P^*(f)h_*$ over all average optimal policies f.

Note that $g^{(0)} := -P^*(f)h_*$ is the expected average reward vector for the Markov decision process with transition matrix $P(f)$ and immediate reward vector $r^{(0)}(i,a) := -h_{*i}$. Thus we can consider another Markov decision process with state space S and *truncated* action set $\mathbb{A}_0(i)$. Note that the part (i) of Proposition 6 guarantees that $\mathbb{A}_0(i) \neq \emptyset$ for every state $i \in \mathbb{X}$. Moreover, according to Assumption (E), all stationary policies induce Markov chains with single ergodic class (without transient states). Hence all policies constructed from the truncated action sets $\mathbb{A}_0(i),\ i \in \mathbb{X}$ also induce irreducible Markov chains. The latter implies that the new *truncated* MDP model is also completely ergodic and we can use the linear programming approach from Subsection 3.5.2 again.

Let us now present the dual and primal linear programs for the truncated Markov decision process. The primal program is

$$\min\ \tilde{g}^{(0)} \tag{3.111}$$

$$\text{s.t.}\quad \tilde{g}^{(0)} + \sum_j (\delta_{ij} - p(j|i,a))\ \tilde{h}_j^{(0)} \geq -h_{*i}, i \in \mathbb{X},\ a \in \mathbb{A}_0(i).$$

where h_* is an optimal solution to (3.107). An optimal solution of this program $g_*^{(0)}$ is the optimal average reward for the truncated model. The dual program is of the form

$$\max \sum_i (-h_{*i}) \sum_{a \in \mathbb{A}_0(i)} x_{ia}^{(0)} \tag{3.112}$$

$$\text{s.t.}\quad \sum_{i,a} (\delta_{ij} - p(j|i,a)) x_{ia}^{(0)} = 0,\ \sum_{i,a} x_{ia}^{(0)} = 1,\ x_{ia}^{(0)} \geq 0,$$

$$i \in S, \quad a \in \mathbb{A}_0(i).$$

Suppose that $x_*^{(0)}$ is a basic optimal solution for the dual program (3.112), then any deterministic policy f, such that

$$f_*(i) = a_i, \quad \text{if} \quad x_{*ia_i}^{(0)} > 0,$$

is an average optimal policy for the truncated model. The next theorem states that f_* is also a bias optimal policy for the original MDP.

Theorem 26 *Under the ergodicity Assumption (E), let f_* be any average optimal policy for the truncated Markov decision process. Then,*

$$h(f_*) = h_{opt},$$

where h_{opt} is the optimal value of the bias vector. Namely, the policy f_* is bias optimal.

PROOF: See [42, 85].

□

3.5.4 The generalization to n-discount optimality

In this subsection we propose an algorithm for constructing n-discount optimal policy based on the system of nested linear programs. This is a generalization of the approach discussed in the preceding subsection. We recall (see pp.95-96) that n-discount optimality

$(n \geq 1)$ is more selective criterion which implies average optimal as well as bias optimal criteria.

We will present an iterative approach for calculating the n-discount optimal policies $(n \geq 1)$. Namely, the linear program for $(n+1)$-discount optimal policy will be constructed from the solution of the linear program for n-discount optimal policy.

Let us first formally introduce the pair of dual linear programs for the computation of n-discount optimal policy. The primal program can be postulated as follows:

$$\min\ \tilde{g}^{(n)} \tag{3.113}$$

$$\text{s.t.} \qquad \tilde{g}^{(n)} + \sum_j (\delta_{ij} - p(j|i,a))\tilde{h}_j^{(n)} \geq -h_{*i}^{(n-1)},$$
$$i \in \mathbb{X}, \quad a \in \mathbb{A}_n(i).$$

The corresponding dual program is

$$\max \sum_i (-h_{*i}^{(n-1)}) \sum_{a \in \mathbb{A}_n(i)} x_{ia}^{(n)} \tag{3.114}$$

$$\text{s.t.} \quad \sum_{i,a} (\delta_{ij} - p(j|i,a))x_{ia}^{(n)} = 0,\ \sum_{i,a} x_{ia}^{(n)} = 1,\ x_{ia}^{(n)} \geq 0,$$
$$i \in \mathbb{X}, \quad a \in \mathbb{A}_n(i).$$

Note that the pair of the dual linear programs (3.111) and (3.112) for the bias optimality is the base for our induction. Indeed in the case of bias optimality we have $n = 0$, $h_*^{(-1)} := h_*$ and the action set $\mathbb{A}_0(i)$ is defined as in (3.110). From a basic optimal solution of (3.112) we can construct the bias (or in other words 0-discount) optimal policy.

Now let us perform an induction step. Namely, we are going to construct a linear program for the calculation of $(n+1)$-discount optimal policy.

Firstly define a new truncated set of actions. Let $(g_*^{(n)}, h_*^{(n)})$ be an optimal solution to the problem (3.113), then

$$\mathbb{A}_{n+1}(i) \stackrel{def}{=} \{a \in \mathbb{A}_n(i) \mid g_*^{(n)} + \sum_j (\delta_{ij} - p(j|i,a))h_{*j}^{(n)} = -h_{*i}^{(n-1)}\}. \tag{3.115}$$

The induction step is based on the following lemma, which is a generalization of Proposition 6.

Lemma 11 *Under the ergodicity Assumption (E), let f be any deterministic and stationary n-discount optimal policy. Then,*
(i) $f(i) \in \mathbb{A}_{n+1}(i),\ i \in \mathbb{X},$ and
(ii) the corresponding $(n+1)$-discount vector $y_{n+1}(f)$ (see Laurent expansion (3.12)) can be calculated by the formula

$$y_{n+1}(f) = [I - P^*(f)]h_*^{(n)}. \tag{3.116}$$

PROOF: We first prove part (i). Let f be a deterministic n-discount optimal policy. Then, according to the induction assumption, it is average optimal for the truncated MDP model associated with linear programs (3.113) and (3.114). From Subsection 3.5.2 we know that there exists a one-to-one correspondence between the deterministic average optimal policies and the basic optimal solutions of the dual linear program (3.114). Hence there exists a basic optimal solution $x_*^{(n)}$ such that $x_{*if(i)}^{(n)} > 0$. Then from the complementary slackness theorem we conclude that

$$g_*^{(n)} + \sum_j (\delta_{ij} - p_{if(i)j}) h_{*j}^{(n)} = -h_{*i}^{(n-1)},$$

that is $f(i) \in \mathbb{A}_{n+1}(i)$.

As a result of part (i), we have the following equation valid for any n-discount optimal policy

$$g_*^{(n)} + (I - P(f)) h_*^{(n)} = -h_*^{(n-1)}. \tag{3.117}$$

We prove the formula (3.116) by induction. Suppose that for $n = k-1$ it is true. The part (ii) of Proposition 6 provides the induction base. Let us prove that the formula (3.116) holds for $n = k$ as well. Taking into account the structure of coefficients $y_k(f)$ of the Laurent series (3.12), we have

$$y_{k+1}(f) = (-1)^{k+1} H^{k+2}(f) r(f) = -H(f)(-1)^k H^{k+1}(f) r(f) = -H(f) y_k(f).$$

Then, using the induction assumption $y_k(f) = [I - P^*(f)] h_*^{(k-1)}$, formula (3.117) and the properties of the deviation matrix (3.14),(3.15), we obtain

$$\begin{aligned} y_{k+1}(f) &= -H(f) y_k(f) = -H(f)[I - P^*(f)] h_*^{(k-1)} \\ &= H(f)(-h_*^{(k-1)}) = H(f)[g_*^{(k)} + (I - P(f)) h_*^{(k)}] \\ &= H(f)(I - P(f)) h_*^{(k)} = (I - P^*(f)) h_*^{(k)}. \end{aligned}$$

Thus, according to the principle of the mathematical induction, the formula (3.116) holds for any $n \geq -1$.

□

It follows from the Definition 11 that the policy f is $(n+1)$-discount optimal if it is n-discount optimal and in addition the $(n+1)$-discount vector $y_{n+1}(f)$ achieves its maximal value. Now we can apply the considerations which are similar to those of Section 3.5.3. The Lemma 11 part (ii) says that the maximization of the vector $y_{n+1}(f)$ is equivalent to the maximization of the product $-P^*(f) h_*^{(n)}$.

The vector $g^{(n+1)}(f) := P^*(f)(-h_*^{(n)})$ can be considered as the expected average reward vector for the Markov decision process with transition matrix $P(f)$, the immediate reward vector $r^{(n+1)}(i,a) := -h_{*i}^{(n)}$ and the truncated action set $\mathbb{A}_{n+1}(i)$. Note that $\mathbb{A}_{n+1}(i) \neq \emptyset$ for $\forall i \in \mathbb{X}$ due to the part (i) of Lemma 11. We shall refer to this truncated Markov decision process as $(n+1)$-step truncated MDP model. We emphasize that under Assumption (E) the n-step truncated MDP model is completely ergodic for any $n \geq -1$. The optimal value of the expected average reward $g^{(n+1)}(f)$ can be obtained from the next linear program

$$\min \ \tilde{g}^{(n+1)} \tag{3.118}$$

$$\text{s.t.} \qquad \tilde{g}^{(n+1)} + \sum_j (\delta_{ij} - p(j|i,a))\tilde{h}_j^{(n+1)} \geq -h_{*i}^{(n)},$$
$$i \in \mathbb{X}, \quad a \in \mathbb{A}_{n+1}(i),$$

and the average optimal policy for the $(n+1)$-step truncated model can be determined from the corresponding dual linear program

$$\max \sum_i (-h_{*i}^{(n)}) \sum_{a \in \mathbb{A}_{n+1}(i)} x_{ia}^{(n+1)} \tag{3.119}$$

$$\text{s.t.} \quad \sum_{i,a} (\delta_{ij} - p(j|i,a)) x_{ia}^{(n+1)} = 0, \quad \sum_{i,a} x_{ia}^{(n+1)} = 1,$$
$$x_{ia}^{(n+1)} \geq 0, \quad i \in \mathbb{X}, \quad a \in \mathbb{A}_{n+1}(i).$$

Namely, if $x_*^{(n+1)}$ is a basic optimal solution of the dual program (3.119), then the deterministic policy f_* such that

$$f_*(i) = a_i, \quad \text{if} \quad x_{*ia_i}^{(n+1)} > 0, \tag{3.120}$$

is average optimal for the $(n+1)$-step truncated model. The next theorem shows that it is also $(n+1)$-optimal for the original problem.

Theorem 27 *Let f_* be any average optimal policy for the $(n+1)$-step truncated Markov decision process. Then,*

$$y_{n+1}(f_*) = y_{n+1}^{opt},$$

where y_{n+1}^{opt} is the optimal value of the $(n+1)$-discount vector. Namely, the policy f_* is $(n+1)$-discount optimal.

Proof: Let d be any stationary $(n+1)$-discount optimal policy for the original Markov decision problem; such a policy exists, see for example [119, 147]. Of course, d is also an n-discount optimal policy. From Lemma 11 part (ii), we have

$$y_{n+1}^{opt} = y_{n+1}(d) = h_*^{(n)} - P^*(d)h_*^{(n)}. \tag{3.121}$$

Since f_* is an average optimal policy for the $(n+1)$-step truncated model, we can write

$$y^{opt}_{n+1} \geq y_{n+1}(f_*) = h^{(n)}_* - P^*(f_*)h^{(n)}_* \geq h^{(n)}_* - P^*(d)h^{(n)}_*. \qquad (3.122)$$

Of course, here $a \geq b$ means $a_i \geq b_i$, for $\forall i \in \mathbb{X}$. Then, (3.121) and (3.122) imply that

$$y^{opt}_{n+1} = y_{n+1}(f_*),$$

which completes the proof.

□

Thus the system of nested linear programs (3.113),(3.114), $n = 0, 1, ...$ provides an elegant iterative method for determining n-discount optimal policies. Moreover, this approach eventually leads to the computation of the Blackwell optimal policy.

3.5.5 The algorithm for computation of Blackwell optimal policy

Here we present a formal algorithm for the computation of the Blackwell optimal policy. Note that n-discount optimal policies, $n = 0, 1, ...$, are by-products of that algorithm. Moreover if we want to obtain only the n-discount optimal policy for some particular n, then we should perform at most n main iterative steps.

Algorithm:

1. Solve the dual linear program (3.108) for determining the average optimal policy. If it has a unique basic optimal solution, then stop. The policy obtained via (3.109) is Blackwell optimal as well as n-discount optimal for any $n \geq -1$. Otherwise, go to the next step.

2. Compute a solution to the primal program (3.107).

3. Set $n := 0$.

4. Obtain the truncated action set $\mathbb{A}_n$ defined by (3.110) and (3.115).

5. Solve the dual program (3.114) for the n-step truncated MDP model. From a basic optimal solution of (3.114) using (3.120) obtain an n-discount optimal policy. If one needs to satisfy only the n-discount optimality criterion, then stop. Also one should stop, if either the basic optimal solution is unique or $n = |S| - 2$. In this case the Blackwell optimal policy is achieved. (Recall that the Blackwell optimality implies the n-discount optimality for any $n \geq -1$.) Otherwise, proceed to the next step.

6. Solve the primal program (3.113) for the n-step truncated MDP model.

7. Increment the number n by one and return to Step 4.

Note that the Theorem 1.1 guarantees the finite convergence of the above algorithm. The maximal number of repetitions of the main iterative steps 4-7 does not exceed $|S|-2$.

Chapter 4

Applications to Mathematical Programming

4.1 Background and literature review

The general results obtained in Chapter 2 have immediate applications to perturbation analysis of mathematical programs. A perturbed mathematical problem can be written in the following form

$$\min_x f(x, \varepsilon) \tag{4.1}$$

subject to

$$g(x, \varepsilon) \geq 0, \quad h(x, \varepsilon) = 0 \tag{4.2}$$

where $x \in \mathbb{R}^n$, g and h are some vector-valued functions, and ε ia a "small" perturbation parameter. When $\varepsilon = 0$, the above mathematical program is referred to as the *unperturbed* or *original* mathematical program.

The well-established theory of parametric programming (see e.g. [50, 55, 56, 57] and references therein) also deals with the problems of the form (4.1). However, there is an essential difference between our approach and the approach used in the parametric programming. In general, the parametric programming investigates the global properties of (4.1), that is the properties that hold for a large interval of ε. By contrast, we are interested in the asymptotic properties of program (4.1) when the perturbation parameter goes to zero. Since the latter is a more specific problem, more detailed results can be obtained. Moreover, the parametric programming is mainly devoted to the case of a "nice" behaviour of perturbed optimal solutions, that is when the perturbed optimal solution converges to the optimal solution of the original problem if the perturbation vanishes. By contrast, we are also interested in the more difficult case when the perturbed optimal solution is discontinuous with respect to the perturbation parameter. We refer to the former case as a *regular* perturbation and to the latter case as a *singular* perturbation. Note that often singularity arises when the dimension of the feasible set changes with perturbation [116]. Therefore in the ensuing sections we will give more specific definitions of the singular perturbations in terms of constraints.

Many authors investigated the mathematical program (4.1) in the context of regular perturbations. Numerous conditions were given to insure that the perturbed optimal solution converges to the optimal solution of the original problem when ε goes to zero. These conditions are often referred to as *stability conditions* of mathematical programs. An interested reader can find more about the stability theory from the work of the following authors: Bonnans and Shapiro [21], Fiacco [49, 50, 51], Gal and Greenberg [57], Levitin [103], Pervozvanskii and Gaitsgory [116].

On the other hand, the research on the singular perturbation is very limited. The analysis of the singularly perturbed linear programs was originated by Jeroslow [81, 82], though he did not use the term "singular perturbation". His approach, *asymptotic linear programming*, was refined later in several papers [71, 75, 93, 94] (also see more detail review below). In Section 4.2 we extend further this approach. The field of singular perturbations of the nonlinear mathematical programming is still largely uncharted teritority. To the best of our knowledge the contribution of Pervozvanski and Gaitsgory [54, 116] seems to be unique in this area. In particular, they showed that under certain assumptions an optimal solution of a singularly perturbed nonlinear mathematical program can be approximated by an optimal solution of some well-defined auxiliary mathematical program. We will use this principle in Section 4.3. Recently, Coulomb, Filar and Szczechla [31] demonstrated that under some rather general assumptions an optimal solution of the perturbed mathematical program (4.1) can be expanded as a Puiseux series. These are perhaps all results on singular perturbation of mathematical programs that are available at present. Therefore in this part of the thesis we are going to focus on the challenging case of singular perturbations of mathematical programming problems.

Firstly, in Section 4.2 we analyse the perturbed linear program. The perturbed linear program is a mathematical program (4.1)-(4.2) with

$$f(x,\varepsilon) = c(\varepsilon)x, \quad g(x,\varepsilon) = x, \quad \text{and} \quad h(x,\varepsilon) = A(\varepsilon)x - b(\varepsilon).$$

Furthermore, we restrict ourselves to the case when the coefficients of the objective functions $c_i(\varepsilon)$ and the coefficients of constraints $a_{ij}(\varepsilon)$ and $b_i(\varepsilon)$ are linear functions of ε. However, as will be demonstrated in Subsection 4.2.4 our approach can be easily generalized to the case when the coefficients $a_{ij}(\varepsilon), b_i(\varepsilon)$ and $c_i(\varepsilon)$ are polynomials of the perturbation parameter. In Subsection 4.2.3 we construct a method that finds a solution to the perturbed linear program which is optimal for all sufficiently small ε but different from zero. We call this solution *a-optimal*.

Our results are based on the preceding works of [71, 75, 81, 82, 93, 94]. In particular, Jeroslow [81, 82] considered the elements of matrices $A(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ as arbitrary rational functions. To deal with such perturbed LPs, Jeroslow [81, 82] proposed simplex-like method which works directly over the field of rational functions. The main disadvantage of this method is that the polynomials involved in calculation can have a high degree. For instance, the inversion of a basis matrix takes $O(m^4 log(m))$ flops [71]. It can be shown that Jeroslow's method can be viewed as an instance of a more general theory of extending algorithms for parametric problems (see Eaves and Rothblum [46]). Another application of this general theory would be a method which works in the field of series expansions. We will call this method the *asymptotic simplex method*. It uses only initial terms of asymptotic expansions and hence can significantly reduce computational effort. The basic idea of this asymptotic simplex method was given by Pervozvanskii and Gaitsgory [116]. However they imposed some strong regularity restrictions. For example, it was required that the inverse of the perturbed basis matrix has a pole of order one at the singular point $\varepsilon = 0$. Later Lamond [93] proposed a method for the expansion of the inverse of the basis matrix which demands $O(\bar{s}m^3)$ flops, where $\bar{s}$ is the maximal order of poles of the basis matrices. Huang [75] improved further the expansion of the perturbed basis matrix by proposing an algorithm which demands only $O(m^3)$ flops. In another paper Lamond [94] proposed to update the asymptotic expansion for the inverse of the perturbed basis matrix rather then to compute it anew. However, his approach applies only to some particular cases, that is when the inverse of the perturbed basis matrix has the pole of order one. This updating procedure demands $O(m^2)$ operations, which is comparable with the standard simplex method. In Subsection 4.2.3 we propose an updating procedure which deals with the general case and demands only $O(\bar{s}m^2)$ operations. Moreover, our procedure is simpler than the inversion technique of Huang [75] and the updating algorithm of Lamond [94]. It is based on the elegant recurrent formulae of Langenhop [98] and Schweitzer and Stewart [135].

In Subsection 4.2.6 we show how the asymptotic simplex method provides a unified framework for singularly perturbed MDPs, branching Markov decision chains and MDPs with Blackwell optimality criterion.

In Section 4.3 we consider the perturbed mathematical program with linear constrains and nonlinear strictly convex objective function. In particular, we are interested in the singular case when some of the constraints merge. Note that if one attempts to solve this singularly perturbed program by standard gradient projection method [107, 127] for a small fixed ε, one will need to invert the ill-condition matrix $A(\varepsilon)A^T(\varepsilon)$ to compute the projection. This can lead to large errors. On the other hand, we propose a modification of the gradient projection method which avoids ill-conditioned operations by using the asymptotic expansion of the projection with respect to the perturbation parameter. The asymptotic expansion of the projection is efficiently obtained with the help of the results from Section 2.2. Finally, we would like to note that our method of Section 4.3 is in line with the general principle of Pervozvanskii and Gaitsgory [54, 116]: to reduce the ill-posed singularly perturbed mathematical program to some auxuiliary mathematical program, which is well-defined.

4.2 Asymptotic simplex method

4.2.1 Preliminaries

Let us consider the following perturbed linear program.

$$\min_x\{(c^{(0)}+\varepsilon c^{(1)})x\} \tag{4.3}$$

subject to

$$(A^{(0)}+\varepsilon A^{(1)})x = b^{(0)}+\varepsilon b^{(1)}, \quad x \geq 0. \tag{4.4}$$

This is the case of a linear perturbation. Later we show how our method can be generalized to the case of a polynomial perturbation, where the coefficient matrix is of the form

$$A(\varepsilon) = A^{(0)}+\varepsilon A^{(1)}+ ... +\varepsilon^p A^{(p)},$$

and similarly for $b(\varepsilon)$ and $c(\varepsilon)$. As was mentioned in the introductory Section 4.1, we are interested in the determination of an *asymptotically* optimal solution. For linear programming we can define it as follows:

Definition 13 *The set of basic indices B is said to be asymptotically optimal* (*or a-optimal*) *for the perturbed linear program* (4.3), (4.4), *if it is optimal for the linear program* (4.3), (4.4) *with any particular fixed $\varepsilon \in (0,\bar{\varepsilon}]$, where $\bar{\varepsilon} > 0$.*

The effect of perturbations (for small values of ε) can be either small or large. Typically the effect of perturbation is large when the dimension of the perturbed feasible set is different from the dimension of the original feasible set [116]. This underlies the classification of problems into either *regular* or *singular* perturbation problems. More precisely, we have the following definition.

Definition 14 *Let $B = \{j_1, j_2, ..., j_m\}$ be a subset of m indices selected from $\{1, 2, ..., n\}$, and $A_B(\varepsilon)$ be the $m \times m$ submatrix of $A(\varepsilon)$, whose columns correspond to the index set B, for $\varepsilon \geq 0$. In all cases it will be assumed that rank$[A(\varepsilon)] = m$ for $\varepsilon > 0$ and sufficiently small. There are now three cases:*

(i) Regular perturbation: *$A_B^{-1}(0)$ exists whenever $A_B^{-1}(\varepsilon)$ exists for $\varepsilon > 0$ and sufficiently small.*

(ii) Weakly singular perturbation: *rank$[A(0)] = m$ but there exits at least one B such that rank$[A_B(0)] < m$ and rank$[A_B(\varepsilon)] = m$ for $\varepsilon > 0$ and sufficiently small.*

(iii) Strongly singular perturbation: *rank$[A(0)] < m$.*

Note that the above definition is more general then the one proposed in [54, 116]. In particular, the notions of weakly singular and strongly singular perturbations are new. It was shown in [116] that an a-optimal solution of the regularly perturbed LP is always the optimal solution of the original unperturbed LP. However, in the case of singular perturbations it is often not true. Let us demonstrate this phenomenon with the help of the following elegant example [116]

$$\max_{x_1, x_2}\{x_2\}$$

subject to

$$\begin{cases} x_1 + x_2 = 1, \\ (1+\varepsilon)x_1 + (1+2\varepsilon)x_2 = 1 + \varepsilon, \end{cases} \tag{4.5}$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

It is obvious that the system of constraints (4.5) has the unique feasible solution $x_1^*(\varepsilon) = 1, x_2^*(\varepsilon) = 0$, when $\varepsilon > 0$. Of course, this is also an optimal solution if ε is not equal to zero. However, the optimal solution of the original ($\varepsilon = 0$) problem is $x_1^* = 0, x_2^* = 1$, which is not anywhere near the previous solution. Thus we can see that in the singularly perturbed linear programs the gap between the solution of the original problem and $\lim_{\varepsilon\to 0} x^*(\varepsilon)$ may arise.

Since the asymptotic simplex method is based on a generalisation of the revised simplex method to the field of asymptotic expansions, we shall briefly outline the classical revised simplex method in the next subsection.

4.2.2 Revised simplex method

To ensure that this section is self-contained and has consistent notation we briefly outline the now classical, revised simplex method. More detailed explanations and proofs can be found in any standard book on linear programming (e.g., [35, 107]). Consider the following linear program

$$\min_x \{cx\} \tag{4.6}$$

subject to

$$Ax = b, \quad x \geq 0, \tag{4.7}$$

Denote by B the index set of the current basic variables. Let A_B be the corresponding basis and $x_B = A_B^{-1}b$ be the vector of basic variables. Suppose that all elements of this vector are positive (non-degeneracy assumption).

Recall that when applying the revised simplex method the following quantities need to be computed or updated:

(i) Simplex multipliers:

$$\lambda = c_B A_B^{-1}.$$

(ii) Reduced cost coefficients:

$$r_N = \lambda A_N - c_N,$$

where N corresponds to the set of non-basic column indices,
namely $N = \{1, 2, ..., n\} \backslash B$.

(iii) Entering column:

$$y_k = A_B^{-1} a_k,$$

where

$$k = \arg \max_{j \in N} \{r_j | r_j > 0\}.$$

(iv) Exiting column index:

$$p = \arg \min_l \{\frac{[x_B]_l}{[y_k]_l} | y_l > 0\}.$$

Of course, most of the above depend on the knowledge of A_B^{-1} and it is a well known fact that this inverse need not be computed from scratch but can be updated at each iteration.

In particular, if B' is the next set of basic variable indices and we construct the following vector and matrix

$$\xi = [-\frac{y_1}{y_p}, ..., -\frac{y_{p-1}}{y_p}, \frac{1}{y_p}, -\frac{y_{p+1}}{y_p}, ..., -\frac{y_m}{y_p}]^T,$$

$$E = [e_1, ..., e_{p-1}, \xi, e_{p+1}, ..., e_m],$$

then the next inverse basis is given by

$$A_{B'}^{-1} = E A_B^{-1}.$$

4.2.3 Asymptotic simplex method

In this subsection we present the asymptotic simplex method for obtaining an a-optimal solution to the perturbed linear program (4.3),(4.4).

At first let us introduce some rather mild assumptions, which will be relaxed further in Subsection 4.2.4. Let M_ε denote the feasible region of (4.3),(4.4) and M_0 be the feasible region of the unperturbed problem.

Assumption 1 *The region M_0 is bounded.*

The above assumption ensures that basic feasible solutions of the perturbed program (4.3),(4.4) can be expanded as Taylor series (see Lemma 13 for details).

Assumption 2 *The perturbed problem is non-degenerate, namely, every element of the basic feasible vector $x_B(\varepsilon) = A_B^{-1}(\varepsilon)b(\varepsilon), \varepsilon \in (0, \bar{\varepsilon}]$ is positive.*

From the preceding description of the classical revised simplex method we see that everything depends on the inverse basis matrix A_B^{-1}. In the perturbed case it is necessary to realise that the analogous matrix $A_B^{-1}(\varepsilon)$ is a Laurent series whose structure and coefficients determine the asymptotic behaviour, as $\varepsilon \to 0$. In particular,

$$A_B^{-1}(\varepsilon) = \frac{1}{\varepsilon^s} B^{(-s)} + ... + \frac{1}{\varepsilon} B^{(-1)} + B^{(0)} + ... \ . \tag{4.8}$$

If $A_B(\varepsilon)$ becomes singular at $\varepsilon = 0$, then the above series will have a pole of order s at $\varepsilon = 0$ and will contain a nontrivial *singular part*; defined by

$$A_B^S(\varepsilon) = \frac{1}{\varepsilon^s} B^{(-s)} + ... + \frac{1}{\varepsilon} B^{(-1)} \ . \tag{4.9}$$

Similarly, a *regular part* of (4.8) is defined by

$$A_B^R(\varepsilon) = B^{(0)} + \varepsilon B^{(1)} + ... \ . \tag{4.10}$$

Clearly, if $A_B^S(\varepsilon) \neq 0$ and ε is small then standard simplex operations could result in unstable behaviour. The methods developed in this section overcome this difficulty by working with the coefficients of (4.8).

At first sight it might appear that computations involving the series expansion (4.8) would be too difficult. Fortunately, recursive formulae developed by Langenhop [98] and Schweitzer and Stewart [135] provide tools that can be adapted to the revised simplex method. A key observation here is that if $B^{(0)}$ and $B^{(-1)}$ are known, then every coefficient of (4.8) can be obtained according to (see [98], [135] and the Auxiliary Subsection 4.2.5)

$$B^{(k)} = (-B^{(0)} A_B^{(1)})^{k-1} B^{(0)} = B^{(0)} (-A_B^{(1)} B^{(0)})^{k-1}, \quad k = 0, 1, ... \tag{4.11}$$

$$B^{(-k)} = (-B^{(-1)}A_B^{(0)})^{k-1}B^{(-1)} = B^{(-1)}(-A_B^{(0)}B^{(-1)})^{k-1}, \quad k = 1, ..., s, \tag{4.12}$$

where $A_B^{(i)}$ are the coefficients of ε^i in the basis matrix $A_B(\varepsilon)$. Consequently, if $B^{(0)}$ and $B^{(-1)}$ can be efficiently updated, when moving from one basis to another, then any coefficient of (4.8) can also be calculated for the next basis via (4.11),(4.12), if it is needed.

However, we still need to compute $B^{(0)}$ and $B^{(-1)}$ for the initial step of the asymptotic simplex method. There are two solutions to this problem. The first solution is to compute the singular part and the first regular coefficient of asymptotic expansion (4.8) by using methods presented in Section 2.2. The other solution is to start the asymptotic simplex method with an analog of phase 1 method [107]. Note that when we introduce artificial variables in phase 1, the linear program becomes weakly singular, even if it was strongly singular before modification. The later enables us to start from a basis matrix which possesses a Taylor series expansion instead of a Laurent series. This significantly facilitates computations. In addition, if we use the phase 1 method, we need not be concerned about the choice of an initial basic feasible solution.

In asymptotic simplex method we need to compare two functions in some small neighbourhood of zero. This will be done with the help of asymptotic expansion (4.8) and the concept of lexicographical ordering. Before proceeding further, let us recall the definition of *lexicographical ordering* [147, 148].

Definition 15 *A vector a (possibly infinite) is called* lexicographically nonnegative*, written $a \succeq 0$, if the first nonzero element (if any) in a is positive, and a is called* lexicographically positive*, written $a \succ 0$, if $a \succeq 0$ and $a \neq 0$. For two vectors a and b we say that a is* lexicographically greater (strictly) *than b, if $a - b \succeq 0$ ($a - b \succ 0$).*

Suppose that we have an analytic function $g(\varepsilon)$ which can be expanded as a Laurent series at $\varepsilon = 0$ with the finite singular part

$$g(\varepsilon) = \frac{1}{\varepsilon^{-s}}g^{(-s)} + ... + \frac{1}{\varepsilon}g^{(-1)} + g^{(0)} + \varepsilon g^{(1)} + ... \; .$$

We construct from the coefficients of the above series the infinite vector

$$\gamma = [g^{(-s)}, ..., g^{(0)}, g^{(1)}, ...].$$

It is easy to see that $g(\varepsilon) > 0$ for ε sufficiently small and positive $\Leftrightarrow \gamma \succ 0$. Moreover, if $g(\varepsilon)$ is a rational function then only a finite number of elements in γ needs to be checked (see Lemma 12). The comparison (in a neighbourhood of 0) between two functions $g(\varepsilon)$ and $h(\varepsilon)$ possessing Laurent series expansions with finite order poles can be carried out similarly, by considering $g(\varepsilon) - h(\varepsilon)$.

Now let us formally present the *asymptotic simplex method* and comment on its possible implementation. The proof of its finite convergence is given at the end of the subsection.

The asymptotic simplex method:

Let the initial set of basic indices B be given.

Step 1: Obtain or update only the singular and the first regular coefficients of the Laurent series expansion (4.8) for the inverse basis matrix $A_B^{-1}(\varepsilon)$. The implementation of this step is discussed at the end of this section.

Step 2 (idea): In asymptotic simplex method we have to decide which column enters the basis. Namely, among the non-basic elements of the reduced cost vector

$$r_N(\varepsilon) := \lambda(\varepsilon)A_N(\varepsilon) - c_N(\varepsilon), \quad \text{where} \quad \lambda(\varepsilon) := c_B(\varepsilon)A_B^{-1}(\varepsilon), \tag{4.13}$$

we need to find such k that

$$k \in \arg\max_{j \in N}\{r_j(\varepsilon) | r_j(\varepsilon) > 0, \varepsilon \in (0, \bar{\varepsilon}]\}.$$

Substituting (4.8) into (4.13), we obtain the next asymptotic expansion

$$r_N(\varepsilon) = \frac{1}{\varepsilon^s} r_N^{(-s)} + \frac{1}{\varepsilon^{s-1}} r_N^{(-s+1)} + \ldots \quad ,$$

where

$$r_N^{(i)} := \lambda^{(i)} A_N^{(0)} + \lambda^{(i-1)} A_N^{(1)} - \delta_{0i} c_N^{(0)} - \delta_{1i} c_N^{(1)} \quad \text{and} \quad \lambda^{(i)} := c_B^{(0)} B^{(i)} + c_B^{(1)} B^{(i-1)}.$$

Let us construct the following (semi-infinite) matrix

$$\rho = \begin{bmatrix} r_N^{(-s)} \\ r_N^{(-s+1)} \\ \ldots \end{bmatrix}$$

and denote its i-th column by ρ_i.

As mentioned above, the lexicographical ordering can be used to compare functions in the "small" neibourhood near zero. In particular, it is easy to see that

$$\arg\max_{j \in N}\{r_j(\varepsilon) | r_j(\varepsilon) > 0, \varepsilon \in (0, \bar{\varepsilon}]\} = \arg \text{lex-}\max_{j \in N}\{\rho_j | \rho_j \succ 0\},$$

where "lex-max" is a maximum with respect to the lexicographical ordering and "arg lex-max" is an index at which "lex-max" is attained. Note that to compare two reduced cost coefficients $r_i(\varepsilon)$ and $r_j(\varepsilon)$ for sufficiently small ε we need only to check a *finite* number of elements of the vectors ρ_i and ρ_j. This follows from the fact that $r_i(\varepsilon)$ and $r_j(\varepsilon)$ are rational functions (see Lemma 12). A practical implementation of the lexicographical entering rule is as follows:

Step 2 (implementation): Set $i := -s$ and $\lambda^{(-s-1)} := 0$.

Step 2a) Calculate the i-th term of the Laurent expansion for the vector of simplex multipliers

$$\lambda^{(i)} := c_B^{(0)} B^{(i)} + c_B^{(1)} B^{(i-1)}.$$

Step 2b) Calculate the i-th term of the Laurent expansion for the vector of non-basic reduced cost coefficients

$$r_N^{(i)} := \lambda^{(i)} A_N^{(0)} + \lambda^{(i-1)} A_N^{(1)} - \delta_{0i} c_N^{(0)} - \delta_{1i} c_N^{(1)}$$

where δ_{0i} and δ_{1i} are the Kronecker deltas. Let $N^{(-s-1)} := N$ and $N^{(i)} = \{j \in N^{(i-1)} | r_j^{(i)} = 0\}$. If $r_j^{(i)} < 0$ for $j \in N^{(i-1)}$, then STOP; the current solution is a-optimal. If there is an index k such that

$$k := \arg \max_{j \in N^{(i-1)}} \{r_j^{(i)} | r_j^{(i)} > 0\},$$

then k identifies the entering non-basic variable; go to Step 3.

Step 2c) If the algorithm does not stop and k has not been identified in Step 2b, and $i < m+1$, then $N^{(i)}$ is not empty. Increment the index i and return to Step 2a to consider the higher order approximation of $r_N(\varepsilon)$.

Remark 26 *Lemma 12, proved in sequel, ensures that if index $i = m+1$ is reached in Step 2c and $N^{(m+1)}$ is still non-empty, then $r_j(\varepsilon) \equiv 0$ for $j \in N^{(m+1)}$ and ε sufficiently small. The latter implies that $r_N(\varepsilon) \leq 0$ for all ε sufficiently small, that is, the current solution is a-optimal.*

Step 3 (idea): Now, as in the revised simplex method, we have to find out which elements of the vector $y_k(\varepsilon) = A_B^{-1}(\varepsilon) a_k(\varepsilon)$ are positive for $\varepsilon > 0$ and sufficiently small. Namely, we have to identify the set of indices $P := \{l | [y_k(\varepsilon)]_l > 0, \varepsilon \in (0, \bar{\varepsilon}]\}$. Towards this, as in Step 2, we first expand $y_k(\varepsilon)$ as a Laurent series

$$y_k(\varepsilon) = \frac{1}{\varepsilon^s} y_k^{(-s)} + \frac{1}{\varepsilon^{s-1}} y_k^{(-s+1)} + \ldots$$

and then define an auxiliary semi-infinite matrix

$$\upsilon = [y_k^{(-s)}, y_k^{(-s+1)}, \ldots].$$

Let υ_l denote the l-th row of matrix υ. Then the set P can be described as $P = \{l | \upsilon_l \succ 0\}$ and the practical procedure for its determination is

Step 3 (implementation): Set $i := -s$, $B^{(-s-1)} := 0$ and $P = \emptyset$.

Step 3a) Calculate the i-th term of the Laurent expansion for $y_k(\varepsilon)$

$$y_k^{(i)} = B^{(i)} a_k^{(0)} + B^{(i-1)} a_k^{(1)}.$$

Step 3b) Let $Q^{(-s-1)} := \{1, ..., m\}$ and $Q^{(i)} := \{j \in Q^{(i-1)} | [y_k^{(i)}]_j = 0\}$. Add the index $j \in Q^{(i-1)}$ to the set P if $[y_k^{(i)}]_j > 0$. If $Q^{(i)} = \emptyset$, then go to Step 3d.

Step 3c) If $Q^{(i)} \neq \emptyset$ and $i < m$, then increment i by one and return to Step 3a. However, if the index i reached m, then go to the next Step 3d. Lemma 12 guarantees that $[y_k(\varepsilon)]_j \equiv 0, j \in Q^{(m)}$.

Step 3d) Stop. At this point the set P is determined.

Remark 27 *From Lemma 13 one can easily deduce that the feasible region for the perturbed problem is bounded. The latter implies that in our setting the set P is always non-empty.*

Step 4 (idea): Now we have to choose a basic variable which exits the basis, namely we have to find

$$p \in \arg\min_l \left\{ \frac{[x_B(\varepsilon)]_l}{[y_k(\varepsilon)]_l} \;\mid\; l \in P, \varepsilon \in (0, \bar{\varepsilon}] \right\}.$$

According to the previous step the functions $[y_k(\varepsilon)]_l, l \in P$ can be expressed as a Laurent series

$$[y_k(\varepsilon)]_l = \varepsilon^{q_l} [y_k^{(q_l)}]_l + \varepsilon^{q_l+1} [y_k^{(q_l+1)}]_l + ..., \tag{4.14}$$

with $y_l^{(q_l)} > 0$. Due to the non-degeneracy Assumption 2 and Lemma 13 $[x_B(\varepsilon)]_l$ can be expressed as power series with a positive leading coefficient

$$[x_B(\varepsilon)]_l = \varepsilon^{t_l} x_{Bl}^{(t_l)} + \varepsilon^{t_l+1} x_{Bl}^{(t_l+1)} + ..., \qquad x_{Bl}^{(t_l)} > 0. \tag{4.15}$$

Then by Lemma 14 the quotient $\Delta_l(\varepsilon) := [x_B(\varepsilon)]_l / [y_k(\varepsilon)]_l$ can also be written in terms of Laurent series

$$\Delta_l(\varepsilon) = \varepsilon^{t_l - q_l} (\Delta_l^{(0)} + \varepsilon \Delta_l^{(1)} + \varepsilon^2 \Delta_l^{(2)} + ...), \tag{4.16}$$

where the coefficients $\Delta_l^{(i)}$ are calculated by simple recurrent formulae (see Lemma 14). Thus, similarly to the previous steps the lexicographical ordering can be implemented as follows:

Step 4 (implementation): Set $i := 0$.

Step 4a) Form the set of indices corresponding to the maximal powers of the leading coefficients in (4.16).

$$R^{(-1)} := \{j | j = \arg\max_l \{t_l - q_l | l \in P\}\}$$

Step 4b) Calculate the (q_l+i)-th and (t_l+i)-th terms of the expansions (4.14),(4.15) respectively

$$[y_k^{(q_l+i)}]_l = [B^{(q_l+i)}]_l a_k^{(0)} + [B^{(q_l+i+1)}]_l a_k^{(1)}, \quad l \in R^{(i-1)},$$

and

$$x_{Bl}^{(t_l+i)} = [B^{(t_l+i)}]_l b^{(0)} + [B^{(t_l+i+1)}]_l b^{(1)}, \quad l \in R^{(i-1)},$$

Step 4c) Calculate the i-th coefficient of expansion (4.16)

$$\Delta_l^{(i)} = (x_{Bl}^{(t_l+i)} - \sum_{j=0}^{i-1} [y_k^{(q_l+i-j)}]_l \Delta_l^{(j)})/[y_k^{(q_l)}]_l, \quad l \in R^{(i-1)}.$$

Step 4d) Form the following set of indices

$$R^{(i)} := \{j | j = \arg\min_l \{\Delta_l^{(i)} | l \in R^{(i-1)}\}\}.$$

If $R^{(i)}$ consists of a unique index p, then go to Step 5. If $R^{(i)}$ is not a singleton and $i < 2m+1$, then we should take into account the higher order approximation of $\Delta_l(\varepsilon)$. Namely, increment i by one and return to Step 4b. However, if $i = 2m+1$ then choose any $p \in R^{(i)}$ and go to Step 5.

Remark 28 *Again Lemma 12 guarantees that $\Delta_p(\varepsilon) \equiv \Delta_q(\varepsilon)$, if $p, q \in R^{(2m+1)}$.*

Step 5: Construct a new basis $A_{B'}(\varepsilon)$ obtained from $A_B(\varepsilon)$ by replacing $a_p(\varepsilon)$ with $a_k(\varepsilon)$. Go to Step 1.

This completes the algorithm.

Remark 29 *Note that if we know the first regular and the first singular terms of the Laurent expansion (4.8), then the computation of Laurent series coefficients for simplex quantities $\lambda(\varepsilon), x_B(\varepsilon)$ and $y(\varepsilon)$ can be easily performed by the following recurrent formulae*

$$\lambda^{(t)} = \lambda^{(t-1)} D_1, \quad y_k^{(t)} = D_2 y_k^{(t-1)},$$

$$x_B^{(t)} = D_2 x_B^{(t-1)}, \quad t \geq 2,$$

and

$$\lambda^{(-t)} = \lambda^{(-t+1)} F_1, \quad y_k^{(-t)} = F_2 y_k^{(-t+1)},$$

$$x_B^{(-t)} = F_2 x_B^{(-t+1)}, \quad t \geq 3,$$

where $D_1 := -A_B^{(1)} B^{(0)}, D_2 := -B^{(0)} A_B^{(1)}$ and $F_1 := -A_B^{(0)} B^{(-1)}, F_2 := -B^{(-1)} A_B^{(0)}$.

As in the revised simplex method, it is possible to update the expansion (4.8) for the inverse of the new basis matrix $A_{B'}(\varepsilon)$ via the multiplication of the series by $E(\varepsilon) = [e_1, ..., e_{p-1}, \xi(\varepsilon), e_{p+1}, ..., e_m]$, where $\xi(\varepsilon) = [-\frac{y_1(\varepsilon)}{y_p(\varepsilon)}, ..., -\frac{y_{p-1}(\varepsilon)}{y_p(\varepsilon)}, \frac{1}{y_p(\varepsilon)}, -\frac{y_{p+1}(\varepsilon)}{y_p(\varepsilon)}, ..., -\frac{y_m(\varepsilon)}{y_p(\varepsilon)}]^T$. Since the division of two Laurent series in the scalar case is not a problem (see Lemma 14), one can easily obtain the Laurent series for $E(\varepsilon)$

$$E(\varepsilon) = \frac{1}{\varepsilon^t}E^{(-t)} + ... + \frac{1}{\varepsilon}E^{(-1)} + E^{(0)} + ... \tag{4.17}$$

Consequently, the coefficients $B'^{(k)}, k = -s', -s'+1, ...$ of the Laurent series for $A_{B'}^{(-1)}(\varepsilon)$ can be calculated by the following formula

$$B'^{(k)} = \sum_{i+j=k} E^{(i)}B^{(j)}, \quad k = -s', -s'+1, ... \ . \tag{4.18}$$

However, we would like to emphasize that we need to update only the coefficients $B'^{(-1)}, B'^{(0)}$ by the above formula. The other coefficients, if needed, can be restored by iterative formulae (4.11),(4.12) in a more efficient way. The computational complexity for this updating procedure is given in the next proposition.

Proposition 7 *The updating procedure for terms $B^{(-1)}$ and $B^{(0)}$ of the Laurent series expansion (4.8) requires $O(\bar{s}m^2)$ operations, where $\bar{s}$ is the maximal order of poles of the Laurent expansions for basis matrices.*

PROOF: Note that for our updating procedure we need to compute $\bar{s}$ terms of the Laurent series (4.17). To calculate the Laurent series for $E(\varepsilon)$, we need to calculate m scalar Laurent expansions for elements of $\xi(\varepsilon)$. This can be done by applying the recursive formulae from Lemma 14. Since the computation of each scalar expansions takes $O(\bar{s}^2)$ flops, the computation of first $\bar{s}$ terms of Laurent series (4.17) requires $O(\bar{s}^2m)$ operations. Note that since matrix $E^{(i)}$ has a special structure, the matrix multiplication $E^{(i)}B^{(j)}$ takes only $O(m^2)$ operations. Then the calculation of $B^{(-1)}$ and $B^{(0)}$ by formula (4.18) demands $O(\bar{s}m^2)$ flops. Consequently, the whole updating procedure requires $O(\bar{s}^2m + \bar{s}m^2)$ operations, which is equivalent to $O(\bar{s}m^2)$, since $\bar{s} \leq m$.

□

Remark 30 *If $\bar{s} << m$ (as can be expected in practice), then the estimated number of operations $O(\bar{s}m^2)$ needed in our updating procedure could be significiantly less than $O(m^3)$, which is required in Huang's method [75].*

Now we prove the finite convergence of the asymptotic simplex method.

Theorem 28 *Let the assumptions 1 and 2 hold. Then the asymptotic simplex method finds the a-optimal basic index set for perturbed LP* (4.3), (4.4)*,* $0 < \varepsilon < \bar{\varepsilon}$*, in a finite number of steps. Furthermore, let* B^* *denote this a-optimal basic index set, then the basic variables of the a-optimal solution can be expressed by the power series*

$$x_{B^*}(\varepsilon) = x^{(0)}_{B^*} + \varepsilon x^{(1)}_{B^*} + \ldots \quad , \qquad \varepsilon < \min\{\bar{\varepsilon}, 1/||D_2||\} \tag{4.19}$$

where $x^{(0)}_{B^*} = B^{(0)}b^{(0)} + B^{(-1)}b^{(1)}, x^{(1)}_{B^*} = B^{(1)}b^{(0)} + B^{(0)}b^{(1)}$ *and the subsequent coefficients are calculated by the recurrent formula* $x^{(k)}_{B^*} = D_2 x^{(k-1)}_{B^*}, \quad k \geq 2$*, with* D_2 *as in Remark 29. The matrices* $B^{(0)}$ *and* $B^{(-1)}$ *are coefficients of the Laurent series (4.8) corresponding to the basic set* B^**.*

Proof: At each iteration of the asymptotic simplex method it is necessary to determine the column that enters the basis and the column that exits it. Namely, it is necessary to determine the following numbers

$$k \in \arg\max_{j \in N}\{r_j(\varepsilon) | r_j(\varepsilon) > 0\}, \quad p \in \arg\min_{l}\{\frac{x_{Bl}(\varepsilon)}{y_l(\varepsilon)} | y_l(\varepsilon) > 0\}.$$

According to Lemma 12, in the asymptotic simplex method k and p are determined in a finite number of steps by a recursive procedure analogous to lexicographic ordering of the coefficients of Laurent/power series expansions.

Next let us show that the asymptotic simplex method has a finite number of iterations. Note that after each iteration the objective function $c(\varepsilon)x$ is “decreased” by the subtraction of the function

$$r_k(\varepsilon)\frac{x_{Bp}(\varepsilon)}{y_p(\varepsilon)}$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$. Since all quantities in the above expression are positive for small $\varepsilon > 0$ (in particular, $x_{Bp}(\varepsilon) > 0$ for $\varepsilon \in (0, \bar{\varepsilon}]$ due to the nondegeneracy assumption), after each iteration the objective function is strictly decreased for all $\varepsilon \in (0, \bar{\varepsilon}]$. Hence, cycling is imposible and the asymptotic simplex method converges in a finite number of iterations.

The series expansion (4.19) is obtained by substituting the Laurent expansion (4.8) into $x_{B^*}(\varepsilon) = A^{-1}_{B^*}(\varepsilon)(b^{(0)} + \varepsilon b^{(1)})$ and noticing that $x_{B^*}(\varepsilon)$ cannot have a singular part because of Lemma 13. The inequality $\varepsilon < 1/||D_2||$ is the standard convergence condition for Neumann series [87, p. 32].

□

Once an a-optimal basis is found by the asymptotic simplex method, one can exactly calculate the optimal solution for any sufficiently small value of the perturbation parameter. Namely,

Corollary 8 *There is a precise updating formula for the optimal solution of the perturbed linear program*

$$x_{B^*}(\varepsilon) = x_{B^*}^{(0)} + \varepsilon[I - \varepsilon D_2]^{-1} x_{B^*}^{(1)}, \quad \varepsilon < \min\{\overline{\varepsilon}, \frac{1}{||D_2||}\} \tag{4.20}$$

where $x_{B^*}^{(0)}$, *and* $x_{B^*}^{(1)}$ *are as in Theorem 28.*

Note that this updating formula is computationally stable even in the case of singular perturbations, since one needs to invert the matrix which is close to the identity.

Computational aid:

Finally, we would like to present a practical modification of the asymptotic simplex method. Note that the main computational difficulties can arise in Step 2 and Step 3, if the functions $r_j(\varepsilon)$ and $y_l(\varepsilon)$ are identically zero, namely, if $r_j(\varepsilon) = 0, y_l(\varepsilon) = 0$ for any ε. In this case we are forced to calculate all terms in corresponding expansions up to the $(m+1)$-st term. Of course, we are interested in identifying such identically zero elements by an efficient method.

One simple heuristic solution for the above problem is proposed here. Note that if $r_j(\varepsilon) = 0$ or $y_l(\varepsilon) = 0$ for $\varepsilon \in (0, \overline{\varepsilon}]$ then these equalities hold for any $\varepsilon \in \mathbb{R}$. This fact follows from the observation that $r_j(\varepsilon)$ and $y_l(\varepsilon)$ are rational functions of ε and every rational function has either no zero or isolated zeros or it is identically zero. Therefore we can detect elements that are identically zero not only in the neighbourhood of $\varepsilon = 0$ but also at any point $\varepsilon \in \mathbb{R}$.

For instance, choose an ε^* such that the basis matrix $A_B(\varepsilon^*)$ is well-conditioned. Then calculate $r_j(\varepsilon^*)$ and $y_l(\varepsilon^*)$ (now we can use directly formulas $r_N(\varepsilon^*) = \lambda(\varepsilon^*)A_N(\varepsilon^*) - c_N(\varepsilon^*)$ and $y(\varepsilon^*) = A_B^{-1}(\varepsilon^*)a_k(\varepsilon^*)$ instead of expansions). If $r_j(\varepsilon^*) \neq 0$ and $y_l(\varepsilon^*) \neq 0$, then the functions are certainly not identically zero. If we obtain some zero elements, then we should add to ε^* arbitrary small variation and check whether this is a local zero or an identical zero. Of course, ε^* and its small variation (if necessary) are chosen according to the features of the specific problem.

4.2.4 Generalizations

In this subsection we show how the asymptotic simplex method can be modified to solve problems more general than LP (4.3), (4.4) under Assumptions 1 and 2.

We first point out that our method can be readily generalized to an LP with polynomial perturbations. Namely, suppose that the coefficients $A(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ in the perturbed LP are polynomials of ε. In particular, a basis matrix has the form

$$A_B(\varepsilon) = A_B^{(0)} + \varepsilon A_B^{(1)} + ... + \varepsilon^p A_B^{(p)}. \tag{4.21}$$

Clearly, the lexicographical entry and exit rules will need only some "technical" changes. Note that in the case of polynomial perturbations one needs at worst to check $(m+1)p$ terms of Laurent expansions for the entry rule and $2mp$ terms for the exit rule. The main difficulties that we face now are in the inversion of the polynomial matrix (4.21). It turns out that the methods of Lamond [93, 94] and Huang [75] for calculating the Laurent series

$$A_B^{-1}(\varepsilon) = \frac{1}{\varepsilon^s} B^{(-s)} + \frac{1}{\varepsilon^{s-1}} B^{(-s+1)} + ... \tag{4.22}$$

cannot be applied (at least not directly) to the polynomial perturbation problem. This is because their methods are heavily dependent on the linearity of the perturbation. Note that the iterative formulae (4.11),(4.12) that we use in our analysis were also derived for the case of linear perturbations. However, they can be generalized for the case of polynomial perturbations. The next generalized recursive formula is due to Korolyuk and Turbin [91]

$$B^{(k+1)} = -\sum_{i=0}^{p-1} \left(\sum_{j=0}^{p-1-i} B^{(-j)} A^{(j+i+1)} \right) B^{(k-i)}, \quad k = 0, 1, ... \quad . \tag{4.23}$$

Note that the above formula is a particular case of the more general formula (2.96) from Section 2.5. Thus, only some singular and the first regular terms of Laurent expansion (4.22) have to be obtained or updated. The other terms, if needed, can be computed in an efficient way by the recursive formula (4.23). On the first iteration of the asymptotic simplex method one may use the algorithm from Section 2.2 or 2.3 to get the initial Laurent expansion. For the next iterations one can use the generalized version of the updating algorithm that we introduced in the previous section.

Note that Assumption 1 guarantees that an a-optimal solution can be expanded in Taylor series. We have introduced this assumption in order to restrict ourselves to the most common and interesting case where the a-optimal solution differs from the optimal solution of the unperturbed problem but both solutions are finite. In this case there exists a computationally stable updating formula (4.20). Of course, one can consider a perturbed LP without this restriction. Then one will need to deal with the basic solutions in the form of general Laurent series with singular terms. Again, the asymptotic algorithm for that case would not be much different from the one presented in Subsection 4.2.3.

4.2.5 Auxiliary results

Here we present several auxiliary theoretical results that help to develop the asymptotic simplex method.

The next elegant property of the Laurent expansion for the singularly perturbed inverse matrix was originally established by Langenhop [98]. Later Schweitzer [133] greatly simplified the proof. Here we give yet another modification of Schweitzer's proof.

Theorem 29 (Langenhop,Schweitzer) *Consider a linearly perturbed matrix* $A(\varepsilon) = A_0 + \varepsilon A_1 \in \mathrm{IR}^{m \times m}$ *and suppose that* $detA(\varepsilon) \neq 0, 0 < \varepsilon < \overline{\varepsilon}$ *and* $det(A_0) = 0$. *Then the inverse operator* $B(\varepsilon) = A^{(-1)}(\varepsilon)$ *can be expanded as a Laurent series*

$$B(\varepsilon) = \frac{1}{\varepsilon^s} B^{(-s)} + ... + \frac{1}{\varepsilon} B^{(-1)} + B^{(0)} + \varepsilon B^{(1)} + ..., \quad s \leq m,$$

The regular part $B^R(\varepsilon) = B^{(0)} + \varepsilon B^{(1)} + ...$ *and the singular part* $B^S(\varepsilon) = \frac{1}{\varepsilon^s} B^{(-s)} + ... + \frac{1}{\varepsilon} B^{(-1)}$ *of this series can be represented in the close analytic forms*

$$B^R(\varepsilon) = B^{(0)}(I + \varepsilon A_1 B^{(0)})^{-1} = (I + \varepsilon B^{(0)} A_1)^{-1} B^{(0)} \tag{4.24}$$

$$B^S(\varepsilon) = \varepsilon^{-1} B^{(-1)}(I + \varepsilon^{-1} A_0 B^{(-1)})^{-1} = (I + \varepsilon^{-1} B^{(-1)} A_0)^{-1} \varepsilon^{-1} B^{(-1)} \tag{4.25}$$

Moreover if the coefficients $B^{(0)}, B^{(-1)}$ *are given, then the other coefficients of the Laurent expansion are calculated by simple iterative formulae*

$$B^{(k)} = (-B^{(0)} A_1)^{k-1} B^{(0)} = B^{(0)}(-A_1 B^{(0)})^{k-1}, \quad k = 0, 1, ... \tag{4.26}$$

$$B^{(-k)} = (-B^{(-1)} A_0)^{k-1} B^{(-1)} = B^{(-1)}(-A_0 B^{(-1)})^{k-1}, \quad k = 1, ..., s \tag{4.27}$$

Proof: It is easy to check that the matrix $B(\varepsilon)$ satisfies the following resolvent-type identity

$$B(\varepsilon_2) - B(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2) B(\varepsilon_2) A_1 B(\varepsilon_1)$$

Suppose ε_2 is fixed and take regular parts of the above identity with respect to ε_1

$$B(\varepsilon_2) - B^R(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2) B(\varepsilon_2) A_1 B^R(\varepsilon_1) + B(\varepsilon_2) A_1 B^{(-1)}, \tag{4.28}$$

and let $\varepsilon_1 = \varepsilon_2$,

$$B(\varepsilon_2) - B^R(\varepsilon_2) = B(\varepsilon_2) A_1 B^{(-1)},$$

and again take regular parts (with respect to ε_2)

$$0 = B^R(\varepsilon_2) A_1 B^{(-1)} \tag{4.29}$$

Now in (4.28) take regular parts with respect to ε_2 (with ε_1 fixed).

$$B^R(\varepsilon_2) - B^R(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2) B^R(\varepsilon_2) A_1 B^R(\varepsilon_1) - B^{(-1)} A_1 B^R(\varepsilon_1) + B(\varepsilon_2) A_1 B^{(-1)}$$

The last summand in the above equation is zero due to (4.29). Let $\varepsilon_2 = \varepsilon_1$,

$$0 = 0 - B^{(-1)} A_1 B^R(\varepsilon_1).$$

So, we have obtained

$$B^R(\varepsilon_2) - B^R(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2)B^R(\varepsilon_2)A_1B^R(\varepsilon_1). \tag{4.30}$$

By setting $\varepsilon_2 = \varepsilon, \varepsilon_1 = 0$, we get

$$B^R(\varepsilon) - B^{(0)} = -\varepsilon B^R(\varepsilon)A_1B^{(0)}$$

$$B^R(\varepsilon)[I + \varepsilon A_1 B^{(0)}] = B^{(0)}$$

or $B^R(\varepsilon) = B^{(0)}[I+\varepsilon A_1B^{(0)}]^{-1}$. To obtain the other formula $B^R(\varepsilon) = [I+\varepsilon B^{(0)}A_1]^{-1}B^{(0)}$ just set in (4.30) $\varepsilon_2 = 0, \varepsilon_1 = \varepsilon$.

The similar considerations can be applied to obtain the resolvent-type identity for the singular part

$$B^S(\varepsilon_2) - B^S(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2)B^S(\varepsilon_2)A_1B^S(\varepsilon_1)$$

and then, via equating coefficients of $\varepsilon_{1,2}^{-1}$ in this identity, we get the formulae (4.25).

Finally, expanding the analytic formulas (4.24),(4.25) in the Neumann power series, we immediately obtain (4.26),(4.27).

□

Lemma 12 (Lamond,Huang) *Suppose $c(\varepsilon) = a(\varepsilon)/b(\varepsilon)$ is a rational function with the degrees of the polynomials $a(\varepsilon)$ and $b(\varepsilon)$ being m and n, respectively. Then the function $c(\varepsilon)$ can be expanded as a Laurent series*

$$c(\varepsilon) = \frac{1}{\varepsilon^s}c^{(-s)} + \frac{1}{\varepsilon^{s-1}}c^{(-s+1)} + ...$$

in some punctured neighbourhood of zero with the order of pole s that is at most n. Moreover, if $c^{(-s)} = c^{(-s+1)} = ... = c^{(m)} = 0$, then $c(\varepsilon) \equiv 0$.

Proof: See papers of Lamond [93, 94] and Huang [75].

□

Lemma 13 *Let Assumption 1 hold. Then, any basic feasible solution of the perturbed program (4.3), (4.4) can be expanded as the Taylor series*

$$x_B(\varepsilon) = x_B^{(0)} + \varepsilon x_B^{(1)} + ...$$

for $\varepsilon \in (0, \overline{\varepsilon}]$, where $\overline{\varepsilon} > 0$ is sufficiently small.

Proof: Recall that any basic feasible solution can be given by the formula

$$x_B(\varepsilon) = A_B^{-1}(\varepsilon)b(\varepsilon). \tag{4.31}$$

According to Theorem 29, the inverse basis matrix $A_B^{-1}(\varepsilon)$ possesses, in general, a Laurent series expansion. Thus one can see from the formula (4.31), that $x_B(\varepsilon)$ possesses a Laurent series expansion in some punctured neibourhood of $\varepsilon = 0$ as well.

Now we shall show that the Laurent series for $x_B(\varepsilon)$ does not have a singular part. Suppose this is not the case and some basic feasible solution has a Laurent series with a non-trivial singular part. The latter implies that there exists a sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ such that $\varepsilon_k \to 0$ and $||x_B(\varepsilon_k)|| \to \infty$ as $k \to \infty$.

Next we define the following auxiliary sequence

$$y_k := \frac{x_B(\varepsilon_k)}{||x_B(\varepsilon_k)||}.$$

Note that $||y_k|| = 1,\ k = 1, 2, \dots$. According to the Bolzano-Weierstrass principle on the bounded sequences in the finite dimensional spaces, there exists a convergent subsequence $\{y_{k_l}\}$ with the limit $\overline{y}$ as $k_l \to \infty$.

From the definition of the basic feasible solution one gets

$$A(\varepsilon_{k_l})y_{k_l} = \frac{b(\varepsilon_{k_l})}{||x(\varepsilon_{k_l}||},$$

for each k_l. Now let k_l go to the infinity to obtain

$$A^{(0)}\overline{y} = 0 \quad \text{and} \quad \overline{y} \geq 0.$$

Take any feasible solution of the unperturbed problem x_f such that

$$A^{(0)}x_f = b^{(0)}, \quad x_f \geq 0.$$

It is easy to see that $x_f + \lambda\overline{y}$ is also feasible solution for any $\lambda \geq 0$. The latter means that the original feasible region M_0 is unbounded, which is a contradiction to Assumption 1.

Thus, every basic feasible solution of the perturbed program can be expanded as a Taylor series.

□

Remark 31 *We would like to note that the first term $x_B^{(0)}$ of the Taylor expansion for $x_B(\varepsilon)$ might not be a* basic *feasible solution for the original program. This may occur in the case of singular perturbations.*

Lemma 14 (Markushevich) *Let analytic functions $a(\varepsilon)$ and $b(\varepsilon)$ be represented by the power series $a(\varepsilon) = \varepsilon^t a^{(t)} + \varepsilon^{t+1}a^{(t+1)} + \dots, a^{(t)} \neq 0$ and $b(\varepsilon) = \varepsilon^q b^{(q)} + \varepsilon^{q+1}b^{(q+1)} + \dots, b^{(q)} \neq 0$ respectively. Then the division of these analytic functions can be also expressed as a power series (for sufficiently small ε)*

$$c(\varepsilon) = \frac{a(\varepsilon)}{b(\varepsilon)} = \varepsilon^{t-q}c^{(0)} + \varepsilon^{t-q+1}c^{(1)} + \dots, \quad c^{(0)} \neq 0, \tag{4.32}$$

whose coefficients are calculated by the recurrent formula

$$c^{(k)} = [a^{(t+k)} - \sum_{i=0}^{k-1} b^{(q+k-i)} c^{(i)}]/b^{(q)}, \quad k = 0, 1, 2, \dots . \tag{4.33}$$

PROOF: It is well known [111] that the division of two analytic functions with poles with finite orders is also an analytic function with a pole of finite order. Then $c(\varepsilon)$ can be expanded as a Laurent series near $\varepsilon = 0$ that converges in a punctured disc of the origin with non-zero radius. Moreover, the coefficients of expansion (4.32) are unique. The recursive formula (4.33) is easily derived from the system of equations obtained by equating the terms with the same power of ε in the identity $b(\varepsilon)c(\varepsilon) = a(\varepsilon)$.

□

4.2.6 Application of the asymptotic simplex method to MDPs

The asymptotic simplex method from Subsection 4.2.3 (or its extension to the case of polynomial perturbation) can be applied to the next four MDP models.

Model I: MDP with Blackwell optimality criterion.

The Blackwell optimal policy in MDP is a discount optimal policy for all discount factor sufficiently close to one, or, equivalently, the policy which is optimal for all interest rate sufficiently close to zero. The Blackwell optimal policy can be determined by using the asymptotic linear programming approach [6, 71, 93, 94]. Namely, one needs to find a solution to the following linear program which is optimal for all interest rates $\rho > 0$ and sufficiently small.

$$\max \sum_{i,a} (1+\rho) r(i,a) x_{ia}$$

$$\sum_{i,a} [(1+\rho)\delta_{ij} - p(j|i,a)] x_{ia} = 1, \quad x_{ia} \geq 0. \tag{4.34}$$

Note that the above linear program can be immediately written in the form (4.3),(4.4) with $\varepsilon = \rho$. Hordijk *et al* [71] developed a refined version of Jeroslow's asymptotic linear programming method [81, 82] to solve (4.34). Later Lamond [94] applied his version of the asymtotic simplex method to the above parametric LP. Of course, our algorithm can be applied to this model as well. Lamond [94] pointed out to the fact that in this particular case the Laurent series (4.8) has always a simple pole. The latter implies that the complexity of the basis updating is $O(n^2)$, which is comparable with the complexity of the basis updating for the ordinary simplex method.

Model II: Markov branching decision chains.

The Markov branching decision chains were introduced by Huang and Veinott in [74]. These are MDPs with immediate rewards dependent on the interest rate. Namely, it is assumed that $r(i,a) = r^{\rho}(i,a)$ is a polynomial of the interest rate ρ. To find a policy which is optimal for all sufficiently small ρ (Strong-value policy [74]), we need to solve only slightly modified version of (4.34), that is

$$\max \sum_{i,a} (1+\rho) r^{\rho}(i,a) x_{ia}$$

$$\sum_{i,a} [(1+\rho)\delta_{ij} - p(j|i,a)] x_{ia} = 1, \quad x_{ia} \geq 0. \tag{4.35}$$

Note that after a simple modification of Step 2 of our algorithm, the asymptotic simplex method can be applied to solve this problem. The method of finding the optimal policy proposed by Huang and Veinott [74] also employs the asymptotic linear programming approach. Their method is based on the subsequent solution of the augmented LPs, whose objective functions and the right hand sides but not the constraint coefficient matrix depend on the parameter. Of course, this facilitates the implementation of the asymptotic linear programming, however they have to deal with large matrices. Moreover, since, as in the Model I, the Laurent series (4.8) has only a simple pole, the dependance of the constraint coefficient matrix on the parameter does not make much harm.

Model III: Singularly perturbed MDP with average criterion.

In Section 3.4 we applied the asymptotic linear programming [81, 82, 71], which is based on the ordering in the field of rational functions, to singularly perturbed MDPs with average criterion. Since one needs to solve the following parametric program

$$\max \sum_{i,a} r(i,a) x_{ia}$$

$$\sum_{i,a} [\delta_{ij} - p(j|i,a) - \varepsilon d(j|i,a)] x_{ia} = 0,$$

$$\sum_{a} x_{ja} + \sum_{i,a} [(\delta_{ij} - p(j|i,a) - \varepsilon d(j|i,a)] y_{ia} = \beta_j,$$

$$x_{ia} \geq 0, \qquad y_{ia} \geq 0, \tag{4.36}$$

the asymptotic simplex method, based on the Laurent series expansions, can be applied as well. Moreover, since the basis updating in the asymptotic linear programming takes $O(m^4 log(m))$ flops, and the asymptotic simplex method takes $O(\bar{s}m^2)$ flops; obviously, the latter method appears to be much faster.

Model IV: Singularly perturbed MDP with killing interest rate

Quadrat and Delebecque [39, 120] considered singularly perturbed MDPs with "killing interest rate" $\rho(\varepsilon) = \mu\varepsilon^l$, where l is the order of a time scale. This model exhibits the necessity of different control regimes for the different time scales. Quadrat and Delebecque [39, 120] provided a lexicographical policy improvement algorithm for the solution of the perturbed dynamic programming equation. Alternatively, the extension of our asymptotic simplex method for the polynomial perturbation can be used in this problem. Here the parametric linear program takes the following form

$$\max \sum_{i,a} (1+\mu\varepsilon^l) r(i,a) x_{ia}$$

$$\sum_{i,a} [(1+\mu\varepsilon^l)\delta_{ij} - p(j|i,a) - \varepsilon d(j|i,a)] x_{ia} = 1, \quad x_{ia} \geq 0. \tag{4.37}$$

Generalized Model:

Finally, we would like to note that the Models I,II and IV can be viewed as particular cases of the next unified scheme. Let transition probabilities $p^\varepsilon(j|i,a)$, immediate rewards $r^\varepsilon(i,a)$ and interest rate $\rho(\varepsilon)$ of an MDP model be polynomials of the parameter ε. Then a policy which is optimal for all sufficiently small values of parameter ε can be found from the next perturbed linear program.

$$\max \sum_{i,a} (1+\rho(\varepsilon)) r^\varepsilon(i,a) x_{ia}$$

$$\sum_{i,a} [(1+\rho(\varepsilon))\delta_{ij} - p^\varepsilon(j|i,a)] x_{ia} = 1, \quad x_{ia} \geq 0. \tag{4.38}$$

The above perturbed LP can be efficiently solved by the generalization of the asymptotic simplex method proposed in Subsection 4.2.4. Note that we retrieve Model I with $\rho(\varepsilon) = \varepsilon$, $r^\varepsilon(i,a) = r(i,a)$, $p^\varepsilon(j|i,a) = p(j|i,a)$; Model II with $\rho(\varepsilon) = \varepsilon$, $r^\varepsilon(i,a) = \sum_{k=0}^p \varepsilon^k r_k(i,a)$, $p^\varepsilon(j|i,a) = p(j|i,a)$; and Model IV with $\rho(\varepsilon) = \mu\varepsilon^l$, $r^\varepsilon(i,a) = r(i,a)$, $p^\varepsilon(j|i,a) = p(j|i,a) + \varepsilon d(j|i,a)$.

4.3 Gradient projection method for singularly perturbed mathematical programs

4.3.1 Preliminaries

In this section we analyse a mathematical program with non-linear strictly convex objective function and linear constraints which depend on a small perturbation parameter ε. Namely, consider the program (Pr_ε)

$$\min_x f(x)$$

subject to

$$A(\varepsilon)x = b(\varepsilon), \tag{4.39}$$

where $x \in \mathrm{IR}^n$, $f \in C^1$, $A(\varepsilon) \in \mathrm{IR}^{m\times n}$ is an analytic matrix valued function, $b(\varepsilon) \in \mathrm{IR}^m$ is analytic vector valued function and the level sets $L_c = \{x|f(x) \leq c\}$ are assumed to be compact (or empty) for every $c \in \mathrm{IR}$. The corresponding *unperturbed* program (Pr_0) is given by

$$\min_x f(x)$$

subject to

$$A(0)x = b(0) \tag{4.40}$$

Suppose that $x^{opt}(\varepsilon)$ and $x^{opt}(0)$ are optimal solutions of the perturbed and unperturbed problems, respectively. We are mainly interested in the *singular perturbation* case, when $x^{opt}(\varepsilon) \not\to x^{opt}(0)$ as $\varepsilon \to 0$. This situation takes place when some of the perturbed constraints (4.39) merge as the perturbation parameter goes to zero [116]. Here we provide a modification of the gradient projection method [107, 127] to solve the auxiliary well-defined mathematical program. The key element of our method is the use of an asymptotic expansion of the projection operator onto the perturbed feasible set. As will be shown in the next section, this expansion can be readily calculated with the help of the results from Section 2.2.

4.3.2 Perturbations of linear manifolds and projections

Singular perturbations occur when the rank of the perturbed constraint matrix $A(\varepsilon)$ is strictly greater than the rank of the unperturbed matrix $A(0)$. To investigate the behaviour of the linear manifold described by (4.39) with respect to the perturbation parameter, let us consider the auxiliary quadratic program (AP)

$$\min (x - x_0)^T (x - x_0)$$

subject to (4.39), where x_0 is some arbitrary point in IR^n. The above (convex) quadratic program admits simple solution $x(\varepsilon)$ in the folowing closed form:

$$x(\varepsilon) = d(\varepsilon) + P(\varepsilon)x_0 \tag{4.41}$$

where

$$d(\varepsilon) = A^T(\varepsilon)(A(\varepsilon)A^T(\varepsilon))^{-1}b(\varepsilon) \tag{4.42}$$

is a perpendicular from the origin to the perturbed linear manifold described by (4.39) and

$$P(\varepsilon) = I - A^T(\varepsilon)(A(\varepsilon)A^T(\varepsilon))^{-1}A(\varepsilon) \tag{4.43}$$

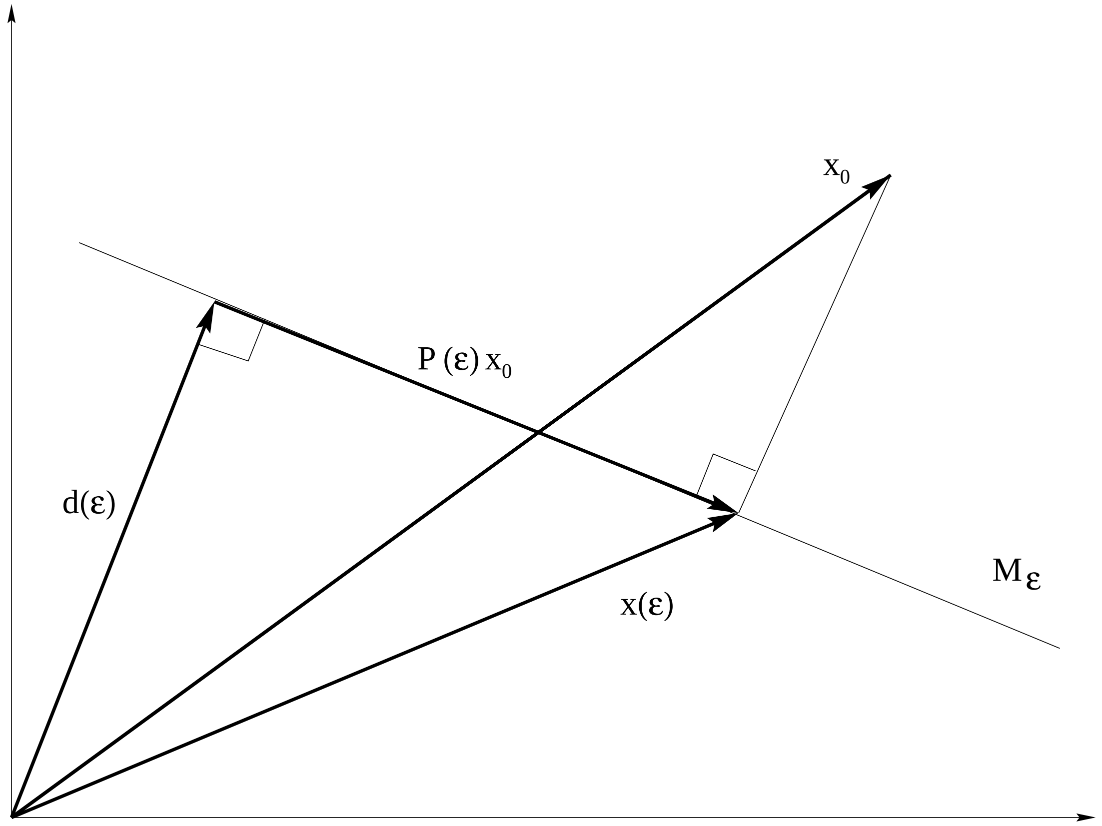

Figure 4.1: Projection onto the perturbed manifold

is a projection operator onto this manifold (see Figure 4.1). In sequel we will show that the perturbed projection always possesses a Maclaurin series expansion around $\varepsilon = 0$, even though the matrix $(A(\varepsilon)A^T(\varepsilon))^{-1}$ may have a singularity at this point. The perpendicular $d(\varepsilon)$ can be either finite or infinite when the perturbation parameter tends to zero. This underlies the further classification. Note that the unperturbed problem can be infeasible, even if the perturbed problem is feasible. In this case, the perturbed linear manifold defined by (4.39) moves to infinity (that is, $||d(\varepsilon)|| \to \infty$) when the perturbation parameter vanishes. However, in the case of feasible unperturbed problem it is possible that the perturbed manifold does not move away when $\varepsilon \to 0$, that is $||d(\varepsilon)||$ is bounded when $\varepsilon \to 0$. Therefore, it is sufficient for our purposes to demand the boundedness of this quantity in some small neighbourhood of $\varepsilon = 0$.

Now let us analyse the dependance of the projection matrix $P(\varepsilon)$ on a small parameter ε. First we need the following auxiliary lemma.

Lemma 15 *Let an orthogonal projection $Q(\varepsilon)$ depend on a parameter ε. Then its norm and its elements are uniformly bounded with respect to the parameter ε.*

PROOF: The proof follows immediately from the fact that $||Q(\varepsilon)x|| \leq ||x||$ if $Q(\varepsilon)$ is an orthogonal projection matrix.

□

Remark 32 *Note that the above statement is not true in general for a non-orthogonal projection.*

Clearly, $P(\varepsilon)$ defined above is a projection [107, 127]. If $P(\varepsilon)$ is symmetric, then it is an orthogonal projection [28]. Indeed, we have

$$(A^T(\varepsilon)(A(\varepsilon)A^T(\varepsilon))^{-1}A(\varepsilon))^T =$$

$$= A^T(\varepsilon)(A(\varepsilon)A^T(\varepsilon))^{-T}(A^T(\varepsilon))^T = A^T(\varepsilon)(A(\varepsilon)A^T(\varepsilon))^{-1}A(\varepsilon).$$

Hence $P(\varepsilon)$ is orthogonal. Now we are able to formulate and prove the main result of this section.

Theorem 30 *The projection matrix $P(\varepsilon)$ possesses Maclaurin series expansion at $\varepsilon = 0$.*

PROOF: The proof for the regular case follows immediately from the Neumann expansion [87] of $[A(\varepsilon)A^T(\varepsilon)]^{-1}$. Consequently we consider the singular case.

Since in the singular case the rows of matrix $A(\varepsilon)$ become linearly dependent when the perturbation parameter tends to zero, the matrix $A(\varepsilon)A^T(\varepsilon)$ does not have a full rank when $\varepsilon = 0$ [141]. Using the results of Section 2.2 we conclude that the inverse of $A(\varepsilon)A^T(\varepsilon)$ possesses a Laurent series expansion in some neighbourhood of $\varepsilon = 0$. Namely,

$$[A(\varepsilon)A^T(\varepsilon)]^{-1} = \frac{1}{\varepsilon^s}C_{-s} + \frac{1}{\varepsilon^{s-1}}C_{-s+1} + ... + \frac{1}{\varepsilon}C_{-1} + C_0 + \varepsilon C_1 + ... \tag{4.44}$$

for $0 < |\varepsilon| < \varepsilon_*$. This implies that the projection $P(\varepsilon) = I - A^T(\varepsilon)(A(\varepsilon)A^T(\varepsilon))^{-1}A(\varepsilon)$ can be expanded as a Laurent series as well. However, $P(\varepsilon)$ is an orthogonal projection and hence it is uniformly bounded for $0 < |\varepsilon| < \varepsilon_*$. Consequently, the Laurent expansion for $P(\varepsilon)$ cannot have any terms with negative powers of ε, that is, $P(\varepsilon)$ possesses a Maclaurin series at $\varepsilon = 0$.

□

Even though by using the general results from Section 2.2 one can obtain Maclaurin series for $P(\varepsilon)$ at $\varepsilon = 0$ in the general case of analytic perturbation $A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + ...$, $b(\varepsilon) = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + ...$ the calculations become much more easy and transparent in the case of linear perturbations. Namely, let us assume that $A(\varepsilon) = A_0 + \varepsilon A_1$ and $b(\varepsilon) = b_0 + \varepsilon b_1$. This implies that we need to obtain the Laurent series (4.44) for the inverse of quadratic perturbation $A_0A_0^T + \varepsilon(A_0A_1^T + A_1A_0^T) + \varepsilon^2 A_1A_1^T$. Note that if we have in hand the coefficients C_{-1}, C_0 and C_1, then the other coefficients of the regular part of (4.44) can efficiently be computed by the recursive formula (4.23) from Subsection 4.2.4. In this particular setting, this recursive formula has the following form

$$C_{k+1} = -[C_0(A_0A_1^T + A_1A_0^T) + C_{-1}A_1A_1^T]C_k - [C_0A_1A_1^T]C_{k-1}, \ k = 1, 2, ... \qquad (4.45)$$

The coefficients $C_{-s}, ..., C_1$ can be computed by the methods of Section 2.2. Next, upon substituting the Laurent series (2.2) into (4.43) and equating coefficients of like powers of ε, we obtain the power series for the projection matrix

$$P(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k P_k, \qquad (4.46)$$

where $P_k = \delta_{0k}I - A_0^TC_kA_0 - A_1^TC_{k-1}A_0 - A_0^TC_{k-1}A_1$, $k = 0, 1, ...$ and δ_{0k} is Kroneker delta. In the sequel we will also need the expansion for $d(\varepsilon)$. Again, upon the substitution of (4.44) into (4.42) and equating coefficients with with the same powers of ε, we obtain the next Laurent series

$$d(\varepsilon) = \sum_{k=-s}^{\infty} \varepsilon^k d_k, \qquad (4.47)$$

where $d_{-s} = A_0^TC_{-s}b_0$ and $d_k = A_0^TC_kb_0 + A_0^TC_{k-1}b_1 + A_1^TC_{k-1}b_0$, $k = -s+1, -s+2, ...$.

4.3.3 Application of the gradient projection method to the limiting mathematical program

Let M_ε denote the perturbed linear manifold described by the system (4.39), M_0 denote the linear manifold corresponding to unperturbed constraints (4.40) and finally let M_* denote (when it exists) a *limiting linear manifold*, that is the limit of set M_ε as ε goes to zero (see Figure 4.2). Strictly speaking, by this we mean that $d(\varepsilon) \to d_0$ and $P(\varepsilon) \to P_0$ as $\varepsilon \to 0$. As mentioned before, here we are interested in the case where the distance from the origin to the limiting manifold M_* is finite. Equivalently, the power series expansion (4.47) for $d(\varepsilon)$ does not have any terms with negative powers of ε

$$d(\varepsilon) = d_0 + \varepsilon d_1 + \varepsilon^2 d_2 + ... \ .$$

Note that in this case $||d_0|| = \text{dist}\{0, M_*\}$ and P_0 (the first term of (4.46)) is an orthogonal projection operator onto M_*. Observe that, in general, $P_0 \neq P(0)$. Actually, M_* is uniquely characterized by d_0 and P_0, that is, any vector y from M_* can be written in the form

$$y = d_0 + P_0x$$

for some $x \in \mathbb{R}^n$.

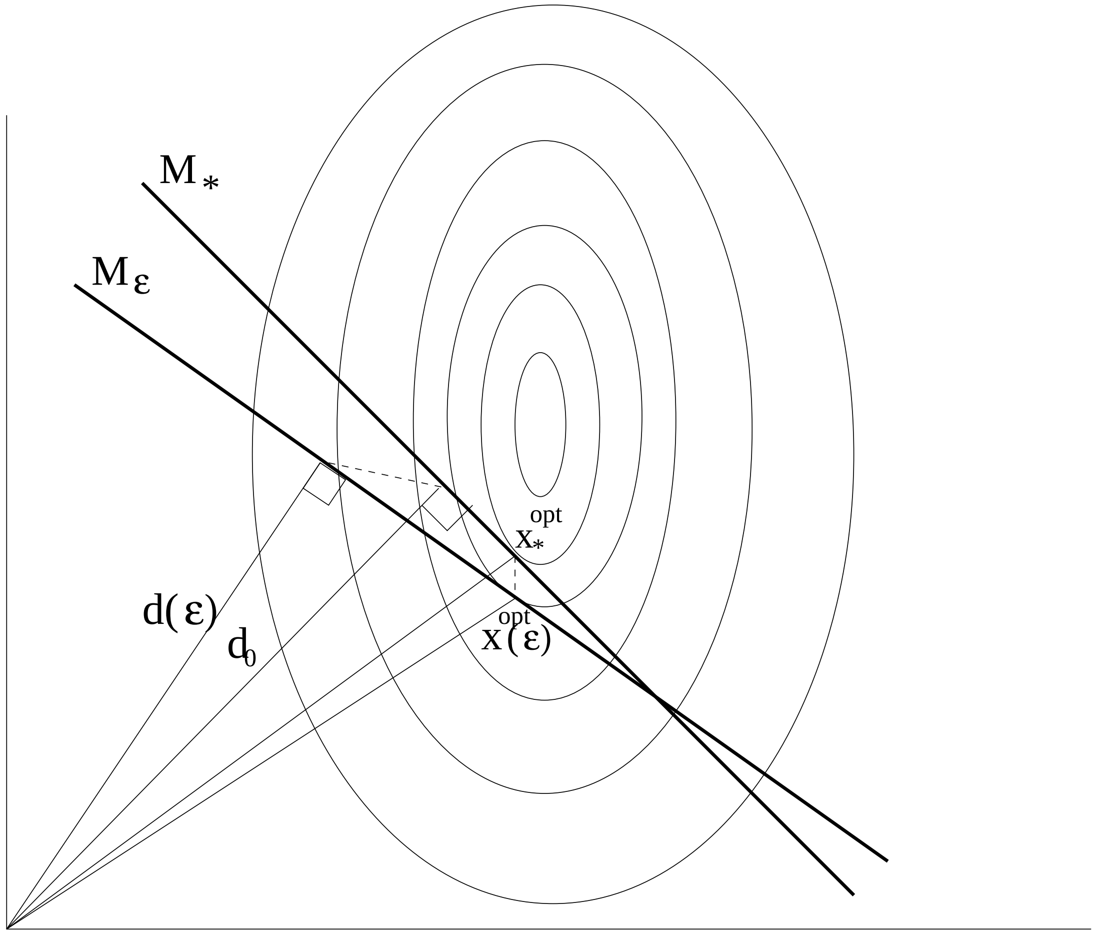

Figure 4.2: The limiting manifold and the auxiliary mathematical program

Now let us review briefly the gradient projection method [107, 127]. Suppose we want to find an optimal solution to the following mathematical program

$$\min_x f(x)$$

subject to

$$Ax = b.$$

Here we assume that f(x) is strictly convex with compact level sets. Then, it is known that a unique optimal solution exists and it can be found by the iterative *gradient projection method* [107, 127]. First we construct the projection matrix $P = I - A^T(AA^T)^{-1}A$ onto the feasible region and find any feasible solution x_0. Then, the gradient projection method is performed according to:

$$x_{k+1} = x_k - \alpha_k P g_k, \tag{4.48}$$

where $g_k := \nabla f(x_k)$ and $\alpha_k := \arg\min_\alpha \{f(x_k - \alpha P g_k)\}$. Since $f(x)$ is a strictly convex function with compact level sets, there exists a simple necessary and sufficient condition for a feasible x to be optimal, namely, $P\nabla f(x) = 0$.

Now suppose we need to solve the perturbed mathematical program (Pr_ε) when the perturbation is singular. If one tries to apply the above gradient projection method directly to (Pr_ε), one will face at least two problems. The first problem is that the matrix $A(\varepsilon)A^T(\varepsilon)$ becomes ill-conditioned when ε is close to zero. This will lead to an incorrect computation of the projection $P(\varepsilon)$. The latter in turn implies that the sequence $\{x_k\}$ generated by the gradient projection method may leave the feasible set. The second problem is that, since ε may not be known exactly or even be unknown, it is very difficult to find the first feasible solution $x_0 \in M_\varepsilon$. In other words, the perturbed feasible set described by (4.39) is very sensitive to changes in ε in the case of singular perturbations.

As a solution to the outlined problems, we propose to substitute the ill-posed program (Pr_ε) with a specially constructed auxiliary *limiting mathematical program*. This is in line with the general approach to singular perturbation problems [3, 15, 54, 116]. Namely, the limiting mathematical program (Pr_*) can be defined as follows:

$$\min_x \{f(x) | x \in M_*\},$$

where M_* is a limiting feasible set of (4.39). Now we can solve the above limiting program by the gradient projection method. As an initial feasible solution one can take $x_0 = d_0$. Then, the iterative procedure takes the form.

$$x_{k+1} = x_k - \alpha_k P_0 g_k\ , \tag{4.49}$$

where P_0 is the first term in (4.46). As a result, we obtain an approximation to optimal solution x_*^{opt} of the auxiliary limiting program (Pr_*), which we shall show is close to the optimal solution of the perturbed problem for small values of the perturbation parameter. Namely, we can state the following result. Its geometric interpretation is given in Figure 4.2.

Theorem 31 *Suppose the distance from the origin to the limiting manifold M_* is finite and f is strictly convex with compact level sets. Then, the optimal solution $x^{opt}(\varepsilon)$ of the perturbed mathematical program (Pr_ε) converges to the optimal solution x_*^{opt} of the limiting program (Pr_*) as ε goes to zero.*

PROOF: Let us consider the optimality equations for the perturbed program

$$\nabla f(x^{opt}) + \lambda^T A(\varepsilon) = 0, \tag{4.50}$$

$$A(\varepsilon)x^{opt} = b(\varepsilon). \tag{4.51}$$

In the case of regular perturbations, that is, when $A(\varepsilon)$ does not change the rank at $\varepsilon = 0$, we can apply the implicit function theorem to show that $x^{opt}(\varepsilon)$ is continuous at $\varepsilon = 0$ and converges to the optimal solution of (Pr_0) (and hence to the optimal solution of (Pr_*), which in this case coincides with (Pr_0)) as $\varepsilon \to 0$. The case of singular perturbations requires a more detailed argument.

Let us choose $\bar{\varepsilon} > 0$ such that $A(\varepsilon)$ has a constant rank on the interval $(0, \bar{\varepsilon}]$. It is always possible to do so in the finite dimensional case. For any $0 < \underline{\varepsilon} < \bar{\varepsilon}$ the optimal solution of the perturbed problem $x^{opt}(\varepsilon)$ is continuous on the closed interval $[\underline{\varepsilon}, \bar{\varepsilon}]$. The justification of that fact is the same as in the above case of regular perturbations.

Now let us prove, by contradiction, that $x^{opt}(\varepsilon)$ is bounded at $\varepsilon = 0$. Since $f(x)$ is strictly convex with compact level sets and $d(\varepsilon) \to d_0$ as $\varepsilon \to 0$, there exists a constant c such that $d(\varepsilon)$ belongs to the set $L_c = \{x \in \mathrm{I\!R^n} | \mathrm{f(x)} \leq \mathrm{c}\}$ for $\varepsilon \in [0, \bar{\varepsilon}]$. Suppose on contrary that $\|x^{opt}(\varepsilon)\| \to \infty$ as $\varepsilon \to 0$. Then, there exists some ε' for which $f(x^{opt}(\varepsilon')) > c$. On the other hand $f(d(\varepsilon')) \leq c$, since $d(\varepsilon') \in L_c$. Consequently, $f(x^{opt}(\varepsilon')) > f(d(\varepsilon'))$. We came to the contradiction and hence $x^{opt}(\varepsilon)$ is bounded on $[0, \bar{\varepsilon}]$.

Next we show that in fact $x^{opt}(\varepsilon)$ has a finite limit, say x_*, as ε goes to zero. Note that if this limit exists it belongs to M_*. Suppose on contrary that there is no limit, then, since $x^{opt}(\varepsilon)$ is bounded on $[0, \bar{\varepsilon}]$, there exist at least two sequences $\{\varepsilon'_k, k = 0, 1, ... | \varepsilon'_k \to 0\}$ and $\{\varepsilon''_k, k = 0, 1, ... | \varepsilon''_k \to 0\}$ such that $x^{opt}(\varepsilon'_k) \to x'_* \in M_*$, $x^{opt}(\varepsilon''_k) \to x''_* \in M_*$ as $k \to \infty$, and $x' \neq x''$. Since $x^{opt}(\varepsilon'_k)$ and $x^{opt}(\varepsilon''_k)$ are optimal solutions of the perturbed problems $(P_{\varepsilon'_k})$ and $(P_{\varepsilon''_k})$, respectively, we can write the following optimality conditions

$$P(\varepsilon'_k)\nabla f(x^{opt}(\varepsilon'_k)) = 0, \quad P(\varepsilon''_k)\nabla f(x^{opt}(\varepsilon''_k)) = 0.$$

Recall that $P(\varepsilon) \to P_0$ as $\varepsilon \to 0$ and $f \in C^1$. Hence, we can let k go to infinity. The latter results in

$$P_0 \nabla f(x'_*) = 0, \quad P_0 \nabla f(x''_*) = 0,$$

with $x'_*, x''_* \in M_*$ and $x'_* \neq x''_*$. This contradicts the fact that the limiting mathematical program has a unique solution, as the objective function is strictly convex with compact level sets and the limiting constraints form a linear manifold [107]. Consequently, the limit $x_* = \lim_{\varepsilon \to 0} x^{opt}(\varepsilon)$ exists. Moreover, from the optimality condition $P_0 \nabla f(x_*) = 0$ we conclude that it is an optimal solution to (P_*). This completes the proof.

$\square$

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