# Learning Control Algorithms: Convergence and Robustness

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**Abstract**

The general learning control procedure is considered. It is shown that known convergence conditions for the learning procedure demand operators with unproper transfer functions. Therefore we propose new weaker convergence conditions which admit learning operators with proper transfer functions. A new robust learning algorithm based on these conditions is constructed. The presented algorithm is especially useful for the control of singularly perturbed mechanical systems.

## 1 Introduction

Many industrial systems (for example, robotic manipulators of conveyor lines) work in a cyclic mode. Namely, a system has to repeat the same operation many times returning after each cycle to the same initial conditions. For continuous cyclic dynamic systems and especially for robots the *learning control* was suggested in [1], where such terminology was also introduced. The basic idea of the learning control is very simple and natural: the control scheme should use information about errors from the previous cycles to construct the control for the next cycle. So after receiving a new task, the system has to make several tuning trials, which improve the precision of operations, and afterwards begin to yield operations with the desired accuracy. The first theoretical results for cyclic control of discrete systems were obtained in [7].

A general theory and several applications of the learning control were given in [2, 8, 9]. However it turns out that the sufficient convergence conditions for the learning procedure proposed in [8] are very tight. Namely, according to these conditions to control inertial objects we need to use high derivatives of the error.

In this paper the authors present *weaker* convergence conditions, which allow to construct the learning operators with proper transfer functions. The learning algorithm based on new convergence conditions is proposed. This algorithm is robust in respect to the high-frequency dynamics and does not use derivatives of the error. Such type of algorithms is extremely useful for the control of mechanical systems with weak damping and hidden resonances. Therefore we consider in the last section of the paper the application of the general learning scheme to *singularly perturbed* multi-frequency mechanical systems. Our approach is close to the scheme proposed in [6]. However it is much simpler. Moreover in our paper a regrious proof of convergence is obtained.

## 2 Convergence of learning control algorithms

In this section we present main results of the learning control theory. An interested reader can find detail proofs and explanations in [2, 8, 9].

Let us formulate the general scheme of the learning control. Consider an arbitrary system working cyclically, that can be described in the operator form

$$y_k(t) = G\{u_k(\tau), \tau \in [0,T]\}, \quad t \in [0,T], k = 0, 1, ...,$$

where $G\{\} = H\{\} + \Phi\{\}$ is the system operator consisting of linear part $H$ and nonlinear part $\Phi$, $T$ is the cycle duration, $k$ is the cycle number, and $u_k(t)$, $y_k(t)$, $t \in [0,T]$ are the control and the output of the system on the $k$-th cycle, respectively. The nonlinear part satisfies Lipschitz condition

$$\|\Phi\{u_1\} - \Phi\{u_2\}\| \le \alpha\|u_1 - u_2\|.$$

The aim of the learning control is to decrease successively the error

$$z_k(t) = y_k(t) - y^d(t), \quad k = 0, 1, \dots ,$$

where $y^d(t), t \in [0, T]$ is a desired behavior of the system. The linear learning control algorithm is the iterative procedure

$$u_{k+1}(t) = u_k(t) - F\{z_k(t)\}, \quad k = 0, 1, \dots, \tag{1}$$

where $F\{\}$ is a linear *learning operator*, and $u_0(t)$ is an arbitrary control on the first cycle.

It was shown in [8] that the above learning algorithm is convergent, that is

$$\lim_{k\to\infty} \|z_k(t)\| = 0,$$

if the following condition holds

$$\|I - HF\| + \alpha\|F\| < 1. \tag{2}$$

This condition is quite general, since one can use any functional norms and corresponding operator norms.

In particular, if the system is linear, then the condition (2) can be written in the form

$$\|I - H_{yu}(p)F(p)\|_{\mathcal{H}_\infty} < 1, \tag{3}$$

where $H_{yu}(p)$ is the transfer function of the object, $F(p)$ is the transfer function of the learning operator. Here the operator norm is $\mathcal{H}_\infty$-norm [3, 4, 5] and the functional norm is $\mathcal{L}_2$-norm.

As it was said before, the learning control can be applied in the case, when the operator of the system is not known precisely. Namely, suppose that

$$H_{yu}(p) = \bar{H}_{yu}(p) + \Delta H_{yu}(p),$$

where $\bar{H}_{yu}$ is the known part and $\Delta H_{yu}$ is the unknown perturbation. Then the condition (3) can be modified as follows

$$\|I - \bar{H}_{yu}(p)F(p)\|_{\mathcal{H}_\infty} + \|\Delta H_{yu}(p)F(p)\|_{\mathcal{H}_\infty} < 1 \tag{4}$$

It is convenient to represent the perturbation $\Delta H_{yu}(p)$ in the multiplicative form

$$\Delta H_{yu}(p) = \eta(p)\bar{H}_{yu}(p),$$

where $|\eta(i\omega)| \le C(\omega)$ and $C(\omega)$ characterizes the range of uncertainty.

The simplest method of the construction of learning operator $F(p)$, called in [8, 9] the *inverse operator principle*, is to take

$$F(p) = \bar{H}_{yu}^{-1}(p), \tag{5}$$

of course, if such operator exists. It was demonstrated in [9] how to use this principle in combination with the classical high gain feedback for the control of mechanical systems, in particular, for the robot control.

However, there are some disadvantages of this approach:

- in the case of inertial systems the degree of denominator of $\bar{H}_{yu}(p)$ is greater than the degree of numerator. Thus, if we use the precise model of the object $\bar{H}_{yu}(p) = H_{yu}(p)$ and the inverse operator principle, then the learning operator $F(p) = H_{yu}^{-1}(p)$ appears to be not proper;

- if one uses unprecise model $\bar{H}_{yu}(p) \ne H_{yu}(p)$ and (5), then it follows from the condition (4) that

  $$C(\omega) < 1, \forall\omega.$$

  But the above inequality may be not satisfied, especially, for the objects with hidden resonances.

Moreover, we would like to emphasize that the outlined problems will arise in any other methods of the learning operator construction. Indeed even in the scalar case the condition (3) means that

$$\sup_{\omega} |1 - H_{yu}(i\omega)F(i\omega)| < 1. \tag{6}$$

This condition can be satisfied only if the difference between the degrees of numerator and denominator of $F(p)$ is precisely equal to the difference between the degrees of denominator and numerator of $H_{yu}(p)$. Otherwise, (6) will not hold, since $|1 - H_{yu}(i\omega)F(i\omega)| \to 1$ as $\omega \to \infty$.

On the other hand, it is obvious that (3) is only *sufficient* condition, so that the learning procedure can converge even if the condition (3) is not satisfied. It was stated in [6] that the condition (3) is also necessary. However according to the theorem 1 below, it is not true.

Therefore we are very interested in obtaining weaker sufficient conditions. A variant of such *weak* conditions is proposed in the next section.

## 3 Weak sufficient conditions

Let us consider a sequence of functions $\{x_k(t),\ k = 0, 1, \dots,\ t \in [0, T]\}$ formed by the recurrent equation

$$x_{k+1}(t) = \mathcal{A}x_k(t), \quad k = 0, 1, \dots \tag{7}$$

where $\mathcal{A}$ is a linear operator with the corresponding transfer function $\mathcal{A}(p)$ and $x_0(t) \in \mathcal{L}_{2[0,T]}$ is an initial term of the sequence. We denote the Fourier transform of $x_0(t)$ as $X_0(i\omega)$. Then sufficient convergence conditions of the sequence $\{x_k(t)\}$ can be formulated as follows.

**Theorem 1** *Let $\mathcal{A}(p) \in \mathcal{H}_\infty$, i.e.*

$$\sup_\omega \sigma_{max}\{\mathcal{A}(i\omega)\}$$

*is bounded ( $\sigma_{max}\{M\}$ is a maximal singular eigenvalue of $M$). Let also the following conditions hold:*

*(i)*

$$\sigma_{max}\{\mathcal{A}(i\omega)\} \leq 1, \quad \omega \in [0, \infty),$$

*(ii) $\sigma_{max}\{\mathcal{A}(i\omega)\}$ has the finite number of local maxima,*

*(iii)*

$$\lim_{\omega\to\infty} \sigma_{max}\{\mathcal{A}(i\omega)\} < 1.$$

*Then*

$$\lim_{k\to\infty} \|x_k(t)\|^2_{\mathcal{L}_2} = \lim_{k\to\infty} \int_0^T x_k^2(t)dt = 0.$$

*If $x_0(t)$ is a bounded variation function, then convergence takes place without condition (iii). If (ii) is satisfied everywhere with the exception of the domain $\overline{\Omega}$, where*

$$\sigma_{max}\{\mathcal{A}(i\omega)\} = 1, \quad \omega \in \overline{\Omega},$$

*then*

$$\lim_{k\to\infty} \|x_k(t)\|^2_{\mathcal{L}_2} \leq \frac{1}{2\pi} \int_{\overline{\Omega}} |X_0(i\omega)|^2 d\omega.$$

**Proof:** For simplicity we show the proof only for the scalar case when

$$\sigma_{max}\{\mathcal{A}(i\omega)\} = |\mathcal{A}(i\omega)|.$$

Let us denote $\omega_1, ..., \omega_n$ the coordinates of local maxima where

$$|\mathcal{A}(i\omega_l)| = 1, \quad l = 1, ..., n. \tag{8}$$

From the recurrent formula (7) and Parseval's equality, we get

$$\begin{aligned} \|x_k(t)\|^2_{\mathcal{L}_2} &= \|X_k(p)\|^2_{\mathcal{H}_2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_k(i\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega, \end{aligned}$$

where $X_k(i\omega)$ is the Fourier transform of $x_k(t)$. Denote

$$\Omega = \{\omega / \, |\mathcal{A}(i\omega)|^2 \geq 1 - \varepsilon\}$$

Due to (ii) and (iii), there exists $\varepsilon > 0$, such that

$$\Omega = \bigcup_l \Omega_l, \quad \Omega_l \bigcap \Omega_k = 0, \quad l \neq k$$

$$\Omega_l = [\omega_l - \delta_l, \omega_l + \delta_l'], \quad \delta_l, \delta_l' > 0$$

Moreover, there is only one maximum point $\omega_l$ in each $\Omega_l$. We have

$$\begin{aligned} \|x_k(t)\|^2_{\mathcal{L}_2} &= \frac{1}{2\pi} \int_{\mathcal{R}\setminus\Omega} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega + \\ & \sum_l \frac{1}{2\pi} \int_{\Omega_l} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega \end{aligned} \tag{9}$$

Now let us find bounds for each summand in (9). It is obvious that

$$\frac{1}{2\pi} \int_{\mathcal{R}\setminus\Omega} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega \leq$$

$$\leq (1-\varepsilon)^k \frac{1}{2\pi} \int_{\mathcal{R}\setminus\Omega} |X_0(i\omega)|^2 d\omega \leq (1-\varepsilon)^k \|x_0(t)\|^2_{\mathcal{L}_2} \tag{10}$$

and

$$\frac{1}{2\pi} \int_{\Omega_l} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega \leq$$

$$\leq \max_{\omega\in\Omega_l} |X_0(i\omega)|^2 \frac{1}{2\pi} \int_{\Omega_l} |\mathcal{A}(i\omega)|^{2k} d\omega \; .$$

Note that $|\mathcal{A}(i\omega)| < 1$ if $\omega \in [\omega_l - \delta_l, \omega_l - \frac{\delta_l}{2}] \bigcup [\omega_l + \frac{\delta_l'}{2}, \omega_l + \delta_l']$, hence

$$\int_{\Omega_l} |\mathcal{A}(i\omega)|^{2(k+1)} d\omega < \int_{\Omega_l} |\mathcal{A}(i\omega)|^{2k} d\omega$$

and the sequence of $J_k = \int\limits_{\Omega_l} |\mathcal{A}(i\omega)|^{2k} d\omega$ is decreasing and has limit $J \geq 0$. We prove by contradiction that $J = 0$. So suppose $J > 0$. Then there exist $\bar{\delta}_l < \delta_l$, $\bar{\delta}_l' < \delta_l'$ such that

$$\lim_{k\to\infty} \int_{\omega_l - \bar{\delta}_l}^{\omega_l + \bar{\delta}_l'} |\mathcal{A}(i\omega)|^{2k} d\omega = \frac{J}{2}$$

or

$$\lim_{k\to\infty} [ \int_{\omega_l - \delta_l}^{\omega_l - \bar{\delta}_l} |\mathcal{A}(i\omega)|^{2k} d\omega + \int_{\omega_l + \bar{\delta}_l'}^{\omega_l + \delta_l'} |\mathcal{A}(i\omega)|^{2k} d\omega ] = \frac{J}{2}.$$

However, due to isolating of maxima (ii), there exists $\varepsilon_1 > 0$ such that

$$|\mathcal{A}(i\omega)|^2 \le 1-\varepsilon_1, \quad \omega \in [\omega_l-\delta_l, \omega_l-\bar{\delta}_l] \cup [\omega_l+\bar{\delta}'_l, \omega_l+\delta'_l]$$

The above inequality implies

$$0 < \frac{J}{2} \le \lim_{k\to\infty} (1-\varepsilon_1)^k (\delta_l - \bar{\delta}_l + \delta'_l - \bar{\delta}'_l) = 0$$

We have got contradiction, thus

$$\lim_{k\to\infty} \frac{1}{2\pi} \int\limits_{\Omega_l} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega = 0, \quad l = 1, ..., n$$

Together with (9) and (10) it proves that

$$\lim_{k\to\infty} \|x_k(t)\|^2_{\mathcal{L}_2} = 0.$$

Now we consider the case when the condition (iii) is not satisfied, that is

$$\lim_{\omega\to\infty} |\mathcal{A}(i\omega)| = 1.$$

Since $\mathcal{A}(p) \in \mathcal{H}_\infty$ and (ii), we can find $\omega^* : \omega_n < \omega^* < \infty$ such that $|\mathcal{A}(i\omega)|$ is not decreasing if $\omega \in \Omega^* = \{\omega \ge \omega^*\}$ and

$$|\mathcal{A}(i\omega)| < 1 - \varepsilon, \quad \omega \in (\mathcal{R} \setminus \Omega) \setminus \Omega^*.$$

If $x_0(t)$ is a bounded variation function, then [10]

$$|X_0(i\omega)| \le \frac{c}{|\omega|}$$

and we have

$$J^*_k \triangleq \frac{1}{2\pi} \int\limits_{\Omega^*} |\mathcal{A}(i\omega)|^{2k} |X_0(i\omega)|^2 d\omega \le$$

$$\le \frac{c}{2\pi} \int\limits_{\Omega^*} |\mathcal{A}(i\omega)|^{2k} \frac{1}{\omega^2} d\omega = \frac{c}{2\pi} \int\limits_0^{1/\omega^*} \left|\mathcal{A}(i\xi^{-1})\right|^{2k} d\xi$$

The convergence of the sequence $\{J^*_k\}$ to zero is proved by analogy with $\{J_k\}$.

So the above assertions are correct if $|\mathcal{A}(i\omega)|$ has the finite number of local maxima. On the other hand, if

$$|\mathcal{A}(i\omega)| = 1, \quad \omega \in \overline{\Omega},$$

then we should write in (9) the additional term

$$\frac{1}{2\pi} \int\limits_{\overline{\Omega}} |X_0(i\omega)|^2 d\omega,$$

which determines the convergence domain. ■

Note if we take $\mathcal{A}(p) = I - H_{yu}(p)F(p)$, then we immediately get new *weaker* conditions for convergence of the learning procedure (1).

## 4 A robust learning algorithm

The weak sufficient conditions given in the previous section allow us to construct learning operators with proper transfer functions. Let us consider a *robust* learning algorithm, which is especially useful for the control of mechanical systems with hidden resonances and weak damping. Take the learning operator in the form

$$F(p) = R(p)\bar{H}^{-1}_{yu}(p), \tag{11}$$

where $R(p)$ is a transfer function of the *regularizing* filter

$$R(i\omega) = \begin{cases} 1, & |\omega| \le \overline{\omega} \\ 0, & |\omega| > \overline{\omega} \end{cases} \tag{12}$$

It is obvious that this control algorithm is *robust* in respect to the perturbation of the object frequency characteristic in the range $[\bar{\omega}, \infty)$. The next theorem gives precise statements for SISO systems.

**Theorem 2** *Let the transfer function of the object be*

$$G(p) = [1 + \eta(p)]\bar{H}_{yu}(p),$$

*and the learning operator is chosen according to (11), (12). If*

$$|\eta(i\omega)| < 1, \quad \omega \le \overline{\omega}, \tag{13}$$

*then the learning procedure converges to the domain*

$$\|z(t)\|^2_{\mathcal{L}_2} \le \frac{1}{\pi} \int\limits_{\overline{\omega}}^{\infty} |Z_0(i\omega)|^2 d\omega. \tag{14}$$

**Proof:** That is only enough to check the conditions of the Theorem 1 for the operator $\mathcal{A}(p) = 1 - H_{yu}(p)F(p)$. ■

We would like to emphasize that the learning operator $F(p)$ is not linear-rational transformation in this case. It is convenient to implement the learning algorithm (11), (12) via the convolution

$$u_{k+1}(t) = u_k(t) - \int\limits_0^T f(t-\tau) z_k(\tau) d\tau, \quad t \in [0; T],$$

where

$$\begin{aligned} f(\tau) &= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} F(i\omega) e^{i\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int\limits_{-\overline{\omega}}^{+\overline{\omega}} \bar{H}^{-1}_{yu}(i\omega) e^{i\omega\tau} d\omega ., \tau \in [-T; +T]. \end{aligned}$$

The weight function $f(\tau)$ can be calculated beforehand. Note also that the above algorithm does not demand any differentiation. It is good advantage in comparison with algorithms based on the inverse operator principle. The choice of $\bar{\omega}$ is determined by the robustness criterion as well as the tolerance level of error. Below we shall show the application of this general scheme to the control of mechanical systems.

## 5 Learning control of singularly perturbed mechanical systems

Let us consider a linear mechanical system in the general form

$$\ddot{q} + D\dot{q} + \Omega^2 q = B_u u(t) + B_w w(t), \quad y = Cq, \qquad (15)$$

$$z = y - y^d(t), \qquad (16)$$

where $q$ is the vector of main generalized coordinates, $D, \Omega^2$ are diagonal matrices, $u$ is a control, $w(t)$ is uncontrollable disturbance, and $y,\ y^d(t)$ are an output and its desired behavior. Again we assume that $w(t)$ and $y^d(t)$ are the same for every cycle and $q(0) = 0$. The system of equation (15), (16) can be rewritten in the operator form

$$Z(p) = H_{yu}(p)U(p) + \overline{W}(p),$$

where

$$H_{yu}(p) = C[p^2 I + pD + \Omega^2]^{-1} B_u,$$

$$\overline{W}(p) = C[p^2 I + pD + \Omega^2]^{-1} B_w W(p) - Y^d(p).$$

We can also write the transfer function $H_{yu}(p)$ in the form

$$H_{yu}(p) = \sum_{k=1}^{N} \frac{g_k}{p^2 + d_k p + \Omega_k^2}.$$

Let us enumerate the frequencies $\Omega_i, i = \overline{1,N}$ as follows

$$\Omega_1 \le ... \le \Omega_{\overline{N}} << \Omega_{\overline{N}+1} \le ... \le \Omega_N.$$

One can separate the two groups of frequencies, because in most of real mechanical systems the low-frequency dynamics, corresponding to $\Omega_1, ..., \Omega_{\overline{N}}$, is known well, whereas the higher frequencies $\Omega_{\overline{N}+1}, ..., \Omega_N$ are known unprecisely or even unknown at all. The first group represents the known part of the mechanical object and the second one represents the singular perturbation. Therefore we chose $\bar{\omega}$ and $\bar{H}_{yu}(p)$ such that

$$\Omega_{\overline{N}} < \overline{\omega} < \Omega_{\overline{N}+1}$$

and

$$\bar{H}_{yu}(p) = \sum_{k=1}^{\overline{N}} \frac{\overline{g}_k}{p^2 + \overline{d}_k p + \overline{\Omega}_k^2} + \sum_{k=\overline{N}+1}^{N} \frac{\overline{g}_k}{\overline{\Omega}_k^2}, \qquad (17)$$

where $\overline{g}_k, \overline{d}_k, \overline{\Omega}_k$ are estimates of object parameters. Note that we neglect high-frequency dynamics in (17). Due to use of regularizing filter (12) the learning procedure is convergent, though the limiting error differs from zero. For example, if $u_0(t) = 0$, then

$$Z_0(p) = \overline{W}(p).$$

According to (14) the level of the limiting error is determined by high-frequency parts of disturbance transfer function and desired output. Thus one chooses $\overline{\omega}$ to satisfy a required accuracy of the learning procedure. On the other hand, the robustness condition (13) for low-frequency dynamics has to hold as well. Of course, slight empirical tuning of $\overline{\omega}$ is recommended in every practical realization.

The described above algorithm has been tested on a benchmark model having two degrees of freedom. Simulations show that the algorithm has a small domain of convergence (1-5% of the initial error value) and good robustness in respect to parameter estimations (for example, the convergence takes place when the elasticity parameter is changed in the range -30%,+80%). Moreover the presented learning algorithm has been successfully applied even to nonlinear mechanical models as robot-manipulators.

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