Iterative learning control for singularly perturbed systems

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Abstract

The singularly perturbed systems are described by differential equations with a small parameter near higher derivatives. It turns out that most of known iterative learning control algorithms diverge in the presence of singular perturbations. The aim of the present paper is to construct learning control algorithms that are able to overcome this problem. Our approach is based on weak convergence conditions that take into account the whole range of frequency response characteristic in contrast to traditional convergence conditions based on the rough H_∞ norm. The first two proposed methods employ the analog and digital filtering. Whereas the learning operator of the third method approximates the inverse of the regular part of the system and at the same time appears to be a low-pass filter. The simulation of the algorithms for a benchmark model confirms all theoretical considerations.

Key words: iterative learning control, singular perturbations, filtering.

1 Introduction

In this paper we investigate the convergence of learning control algorithms in the presence of singular perturbations. By singular perturbations we mean the perturbations that cause the increase of system dimension. In particular, if a system is described by differential equations, then the singular perturbations are usually represented by some small parameters near higher derivatives. For example, a robotic manipulator with flexible joints which have large stiffness coefficients can be considered as the case of singular perturbations. In this instance the small parameters near higher derivatives are the inverse values of stiffness coefficients. The trajectories of singularly perturbed systems always consist of slow and fast motions [17]. The slow motions correspond to the behaviour of the unperturbed system. Note that in contrast to the singular perturbations, the regular perturbations do not lead to the rise of the system dimension and therefore cannot change dramatically the behaviour of the system [17]. Most learning control algorithms (see extensive surveys [4, 7, 8, 10, 11, 12, 14, 15]) are designed to overcome regular perturbations but only few of them, if any, are able to cope with singular perturbations. The existent learning control algorithms actually magnify the amplitude of high-frequency oscillations which are generated by the singular part of the system. The purpose of the present paper is to develop the learning control algorithms that are able to handle the challenging case of singular perturbations. Our approach is based on weak convergence conditions which take into account the whole range of frequency response characteristic. Another main point of our research is that we consider the convergence of the error to some domain rather than to the zero level. Indeed, as pointed out by most

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authors working in the applied field of iterative learning control [8, 11], we never achieve the convergence to zero in practice. Of course, one can explain this by the presence of stochastic noise in the measurements. The other explanation for this phenomenon can be singular perturbations. It is interesting that both stochastic noise and singular perturbations are represented by high-frequency oscillations. Hence an algorithm which converges in the presence of singular perturbations will also suppress the high-frequency stochastic noise. In the next section we briefly review the standard and weak convergence conditions for the iterative learning control of linear systems.

Finally, we would like to note that the present work is a full paper version of the talk presented at the ILC workshop [1].

2 Convergence conditions

Let us consider a linear cyclically working system described in terms of Laplace transform

$$y_k(s) = H_{yu}(s)u_k(s), \quad k = 0, 1, ..., \tag{1}$$

where $H_{yu}(s)$ is a transfer function of the system, k is a cycle number, and $u_k(s), y_k(s)$ are the Laplace transforms of the control and the output of the system at the k-th cycle, respectively. Without loss of generality we can assume zero initial conditions.

The aim of the iterative learning control is to decrease successively the error

$$z_k(t) = y_k(t) - y_d(t), \quad t \in [0, T], \quad k = 0, 1, ...,$$

where $y^d(t)$ is the desired output of the system and T is the cycle duration. The linear learning control algorithm is the iterative procedure defined as follows:

$$u_{k+1}(s) = u_k(s) - L(s)z_k(s), \quad k = 0, 1, ..., \tag{2}$$

where $L(s)$ is a linear *learning operator*, and $u_0(s)$ is the Laplace transform of an arbitrary control at the first cycle. It is easy to show that the following recurrent equation takes place.

$$z_{k+1}(s) = [I - H_{yu}(s)L(s)]z_k(s), \quad k = 0, 1, ... \tag{3}$$

Let us denote $A(s) := I - H_{yu}(s)L(s)$. By employing $\mathcal{H}_\infty$- and $\mathcal{L}_2$-norms [5], we obtain

$$||z_{k+1}(t)||_{\mathcal{L}_2} \le ||A(s)||_{\mathcal{H}_\infty} ||z_k(t)||_{\mathcal{L}_2}.$$

The above inequality leads to the *standard* convergence conditions for iterative learning procedure (2).

Theorem 1 *Let $A(s) \in \mathcal{H}_\infty$, i.e. $\sup_\omega \sigma_{max}\{A(i\omega)\}$ is bounded ($\sigma_{max}\{M\}$ is a maximal singular eigenvalue of M), and*

$$||A(s)||_{\mathcal{H}_\infty} < 1. \tag{4}$$

Then

$$\lim_{k\to\infty} ||x_k(t)||^2_{\mathcal{L}_2} = \lim_{k\to\infty} \int_0^T ||x_k(t)||^2 dt = 0.$$

In particular, this theorem implies that the learning operator $L(s)$ has to be chosen as close to $H_{yu}^{-1}(s)$ as possible. This fact is referred to as the *inverse operator principle* [14, 15].

Unfortunately, this criterion cannot be satisfied in the case of singular perturbations. For example, in SISO case condition (4) can be written as

$$\sup_{\omega\in[0,\infty)} |1 - H_{yu}(i\omega)L(i\omega)| < 1 \tag{5}$$

This condition can be satisfied only if the difference between the degrees of numerator and denominator of $L(s)$ is precisely equal to the difference between the degrees of denominator and numerator of $H_{yu}(s)$. However, if the system is singularly perturbed, then there are some higher power terms in the denominator of $H_{yu}(i\omega)$ that we did not take into account. Hence $H_{yu}(i\omega)L(i\omega) \to 0$ and $|1 - H_{yu}(i\omega)L(i\omega)| \to 1$ as $\omega \to \infty$, and consequently condition (5) is violated. On the other hand, it is obvious that (4) is only *sufficient* condition, so that the learning procedure can converge even if the condition (4) is not satisfied. This prompts us to obtain weaker sufficient conditions for the convergence of learning procedure (2). A variant of such *weak* convergence conditions is stated in the next theorem [16].

Theorem 2 *Let the following conditions hold:*

(i) $x_0(t) \in \mathcal{L}_2[0,T]$

(ii) $||A(s)||_{\mathcal{H}_\infty} = \sup_\omega \sigma_{max}\{A(i\omega)\} \leq 1,$

(iii) $\sigma_{max}\{A(i\omega)\}$ *has a countable number of local maxima (including the infinite point).*

Then, the iterative learning procedure converges to zero in $\mathcal{L}_2$*-norm, namely*

$$\lim_{k\to\infty} \|x_k(t)\|^2_{\mathcal{L}_2} = 0.$$

Moreover, if condition (iii) is satisfied everywhere with the exception of domain $\overline{\Omega}$*, where* $\sigma_{max}\{A(i\omega)\} = 1$*, then the error of learning control algorithm (2) converges to some non-zero level bounded by*

$$\lim_{k\to\infty} \|x_k(t)\|^2_{\mathcal{L}_2} \leq \frac{1}{2\pi} \int\limits_{\overline{\Omega}} \|X_0(i\omega)\|^2 \, d\omega,$$

where $X_0(s)$ *is the Laplace transform of* $x_0(t)$.

These conditions except the last statement of the theorem were also studied in [2, 13]. Note that in contrast to Theorem 1 the conditions of Theorem 2 do not guarantee the linear rate of convergence.

Let us introduce two Benchmark models that are used throughout the paper to test new algorithms and to demonstrate the problem of singular perturbations in iterative learning control. The first *rigid* benchmark model (see figure 1.a) is a two-mass system with a rigid axle, whereas the second *flexible* benchmark model (see figure 1.b) has an axle with flexibility coefficient c. The rigid benchmark model is described by the next system of equations.

$$\begin{cases} (J_1 + J_2)\ddot{q}_{1k} = u_k \\ u_k = -k_1\dot{z}_{1k} - k_0 z_{1k} + v_k \\ y_k = q_{2k} = q_{1k}, \quad z_{1k,2k} = q_{1k,2k} - q_d \end{cases}$$

And the equations for the flexible model can be written as follows:

$$\begin{cases} J_1\ddot{q}_{1k} + b(\dot{q}_{1k} - \dot{q}_{2k}) + c(q_{1k} - q_{2k}) = u_k \\ J_2\ddot{q}_{2k} + b(\dot{q}_{2k} - \dot{q}_{1k}) + c(q_{2k} - q_{1k}) = 0 \\ u_k = -k_1\dot{z}_{1k} - k_0 z_{1k} + v_k \\ y_k = q_{2k}, \quad z_{1k,2k} = q_{1k,2k} - q_d \end{cases}$$

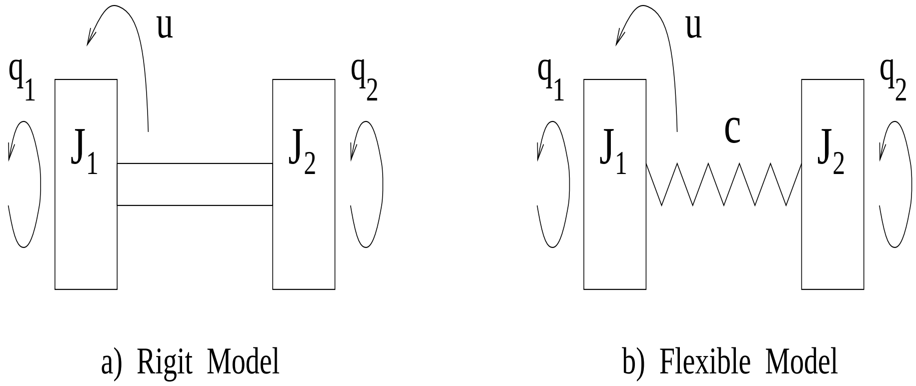

Figure 1: Benchmark models

All simulations in this paper were carried out in MATLAB package with the following values of parameters: $J_1 = 10, J_2 = 10, b = 2, c = 20000$. One can see that the differential equations for the flexible model can be rewritten in the form with the small parameter $1/c$ near the higher derivatives, which is the typical phenomenon in the case of singular perturbations. Next we demonstrate that the singular perturbations destroy the convergence of the classical learning control algorithms. Let us consider the following learning control algorithm [14, 15]:

$$v_{k+1} = v_k - (\bar{J}\ddot{z}_{2k} - k_1\dot{z}_{2k} - k_0 z_{2k}), \tag{6}$$

where $\bar{J} \simeq J_1 + J_2$ is some approximation of the system inertia coefficient. The above algorithm readily satisfies the standard convergence condition (4). The simulation results are shown in figure 2. However, the straightforward application of the algorithm (6) to the flexible

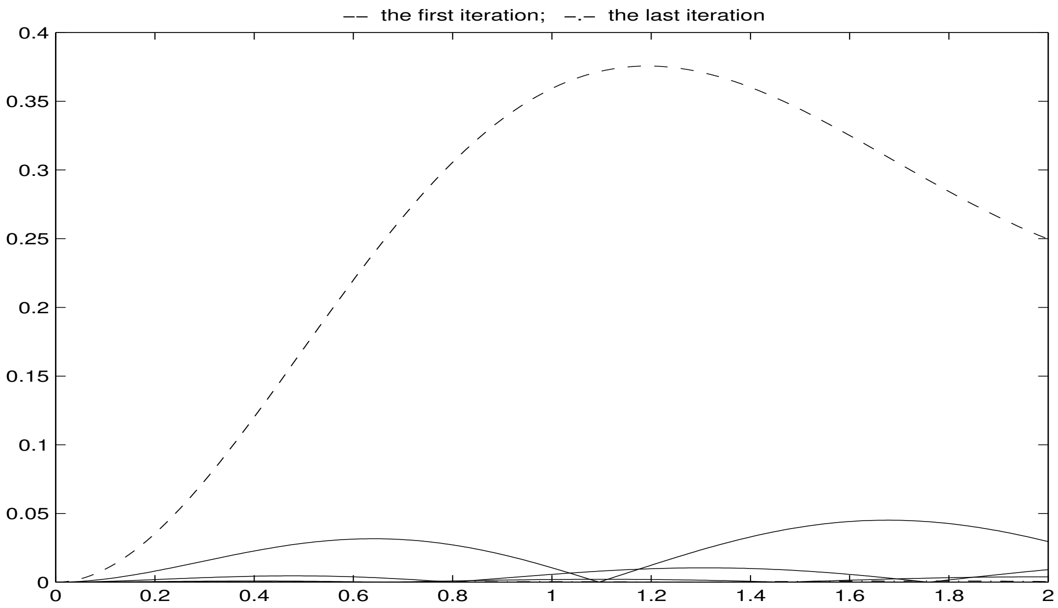

Figure 2: Algorithm (6) applied to the rigid model, $\bar{J} = 15$.

benchmark model results in the rapid divergence of the process after the third iteration (see figure 3). Note that in this case both weak and standard convergence conditions are violated. Next we propose an algorithm which satisfies the conditions of Theorem 2 but violates the restrictive condition (4). Let the learning operator be given in the form

$$L(s) = \frac{k_0}{\omega_*^2}[s^2 + 2\nu s + (\omega_*^2 + \nu^2)], \tag{7}$$

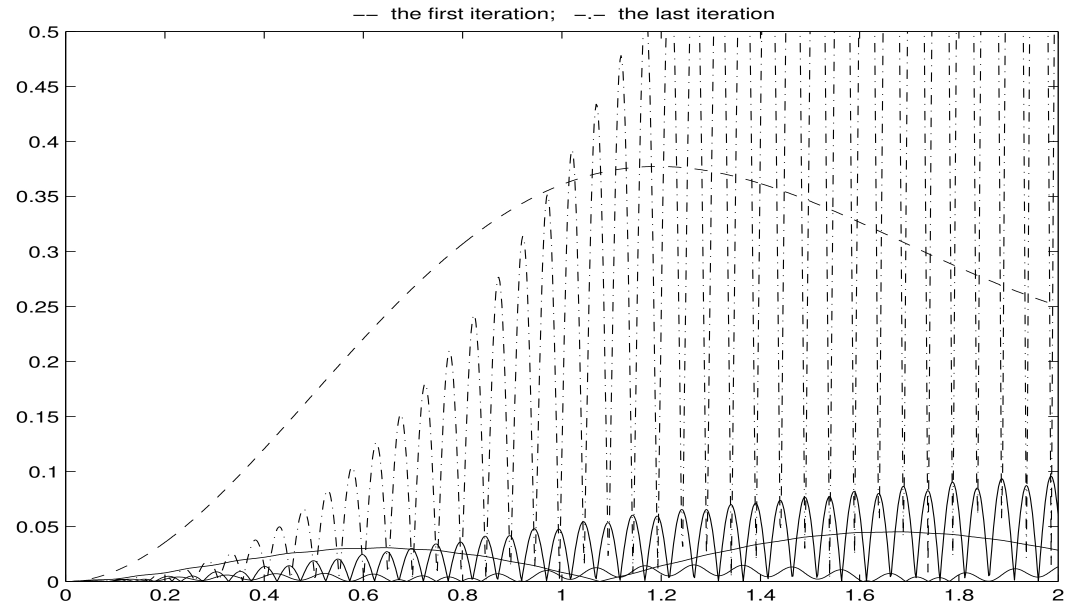

Figure 3: Algorithm (6) applied to the flexible model, $\bar{J} = 15$.

where $\omega_*^2 \approx c/J_2$ and $\nu \approx b$. It is easy to see that $H_{yv}(i\omega)L(i\omega) = O(1/\omega)$, when $\omega \to \infty$, and hence the condition (4) of Theorem 1 is violated. On the other hand one can check (for example, by ploting frequency response for some values of parameters) that the conditions of Theorem 2 are satisfied, if ω_*^2 and ν are sufficiently close to c/J_2 and b, respectively. The numerical simulations also confirm the convergence of the proposed learning control algorithm (see figure 4). Unfortunately, this algorithm is not very robust in respect to the perturbations of parameter c. Moreover, the algorithm will be divergent if we consider the three-mass model instead of the present two-mass model. Namely, it is not robust in respect to higher order singular perturbations. The latter forces us to seek algorithms which will be robust in respect to the singular perturbations of any order.

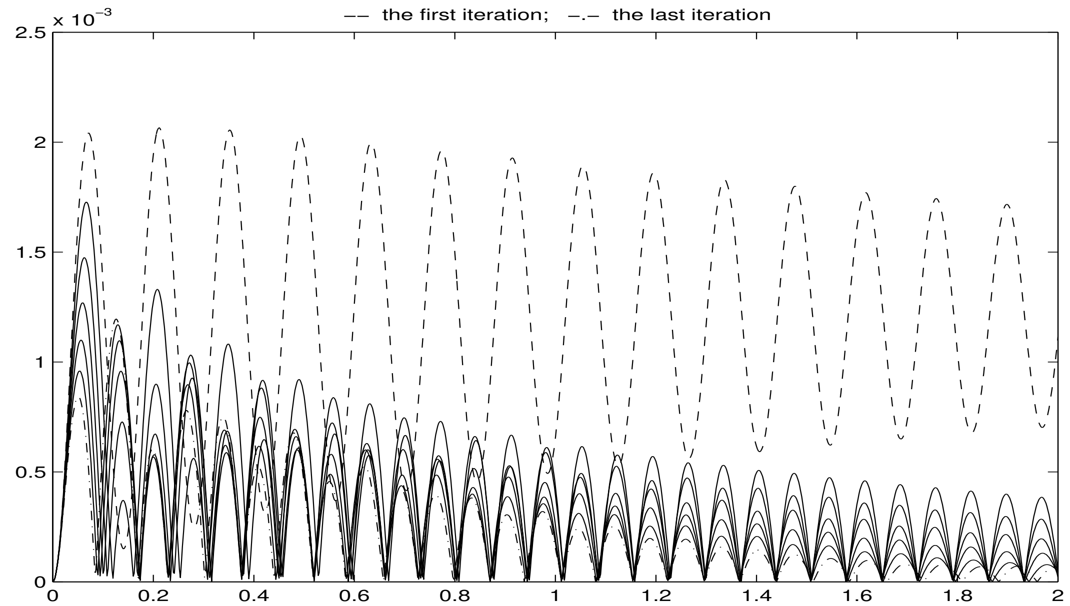

Figure 4: Simulation results for algorithm (7), 5% deviation of the nomianal parameters

3 Learning algorithms with filtration

In this section we construct robust learning control algorithms based on analog and digital filtration. Namely, instead of algorithm (2) let us consider the following modification

$$u_{k+1}(s) = F(s)[u_k(s) - L(s)z_k(s)], \quad k = 0, 1, ..., \tag{8}$$

where $F(s)$ is a transfer function of some filter. Even though the above learning control algorithm has been already analysed in a number of papers [6, 9, 13, 19], only its simplest variant ($F(s) \equiv \gamma$, $0 < \gamma < 1$) [3, 18] was used to construct algorithms that are able to overcome structural uncertainty which leads to the increase of system dimension. One can easily formulate the standard and weak convergence conditions for the learning procedure (8) similar to Theorems 1 and 2. For example, the standard convergence conditions take the form.

Theorem 3 *Let* $F(s), L(s) \in \mathcal{H}_\infty$ *and*

$$||F(s)[I - H_{yu}(s)L(s)]||_{\mathcal{H}_\infty} < 1. \tag{9}$$

Then, the error of the iterative learning procedure (8) converges in $\mathcal{L}_2$*-norm to some non-zero level with transfer function*

$$z_*(s) = (I - F(s)[I - H_{yu}(s)L(s)])^{-1}(F(s) - I)y_d(s)$$

In the presence of singular perturbations the frequency response characteristic of the transfer function $I - H_{yu}(s)L(s)$ has one or more picks in high-frequency domain. See for example figure 5, where $|1-H_{yv}(i\omega)L(i\omega)|$ is ploted for our benchmark model. The magnitude

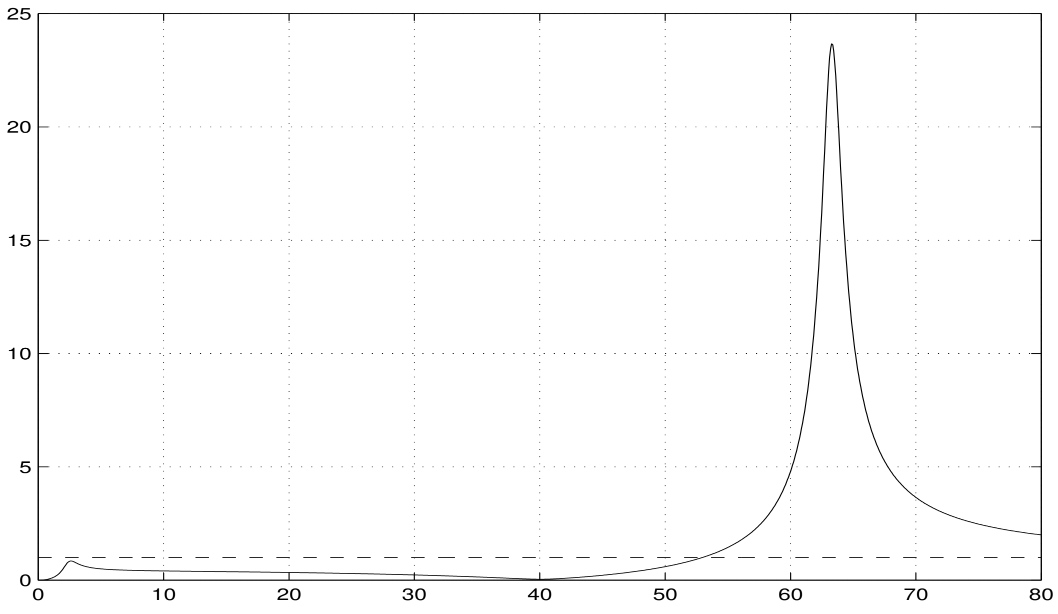

Figure 5: The graph of $|1 - H_{yv}(i\omega)L(i\omega)|$ for the flexible model with learning operator (6)

of these picks usually greater than unity level, and hence the conditions of Theorem 1 and 2 are violated. This means that the high-frequency part of the error will increase after each iteration and the learning procedure will inevitably diverge starting from some iteration. On the other hand, using the modified learning procedure (8) with $F(s)$ as a low-pass filter, we can easily satisfy convergence condition (9). Of course, we have to pay a price for that.

Namely, the error will converge to the non-zero limiting error $z_*(s)$. The limiting error level depends on the choice of $F(s)$ and $L(s)$. Suppose we take $L(s)$ which approximates well $H_{yu}^{-1}(s)$ in the frequency range $[0, \overline{\omega}]$. Then, the cutoff frequency ω_f of the low-pass filter $F(s)$ has to be chosen so that $\omega_f < \overline{\omega}$. For example, in the case of our flexible two-mass model we can take $L(s)$ as in (6) and choose the cutoff frequency ω_f in the interval $(\sqrt{k_0/(J_1+J_2)}, \sqrt{c(J_1+J_2)/J_1J_2})$. We have tested the modified learning control algorithm (8) with two simple low-pass filters. The transfer function of the analog filter has the form

$$F(s) = \frac{1}{s/\omega_f + 1},$$

where w_f is a cutoff frequency of the filter. See simulation results on figure 6. The digital

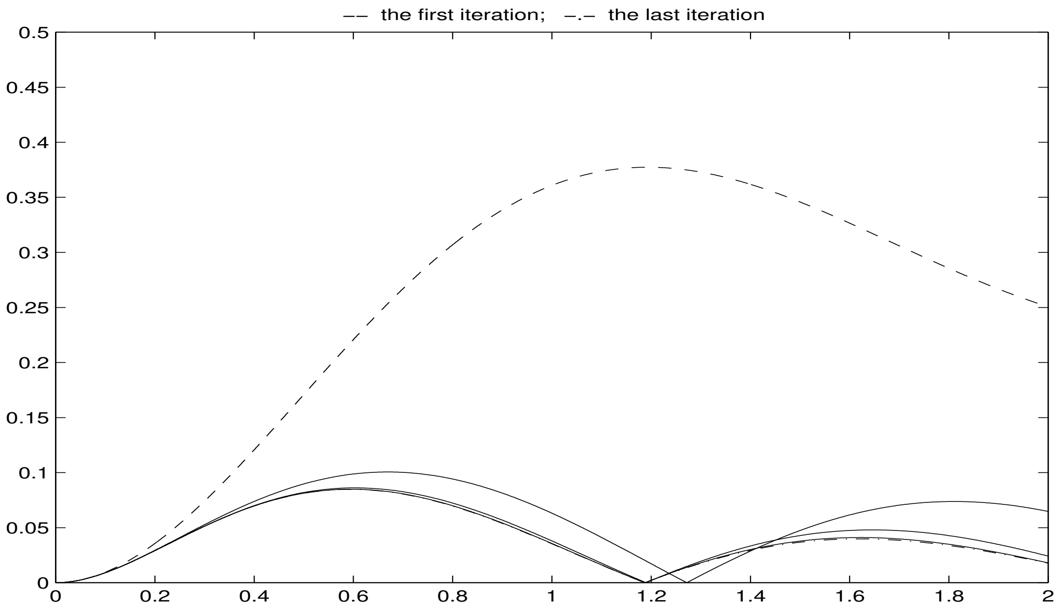

Figure 6: Learning algorithm (8) with analog filtration, 30% deviation of the nomianal parameters

filter that we have studied is given by the convolution

$$y(t) = \frac{1}{\pi} \int_0^T \frac{\sin(\omega_f(t-\tau))}{t-\tau} x(\tau) d\tau, \quad t \in [0, T].$$

Note that the Fourier transform of the function $\sin(\omega_f t)/\pi t$ is the rectangular pulse from $-\omega_f$ to ω_f of height 1. The simulation results are shown on figure 7. As we can see, the modified learning control algorithm based on the digital filtration has much better performance than the algorithm based on the analog filtration.

4 Learning operator with properties of low-pass filter

From the results of the previous section we conclude that it is necessary to use low-pass filtering in the presence of singular perturbations. In this section we propose the learning operator that in the same time appears to be a low-pass filter. This learning operator approximates only the system regular part whose principal structure is known. It can be represented by

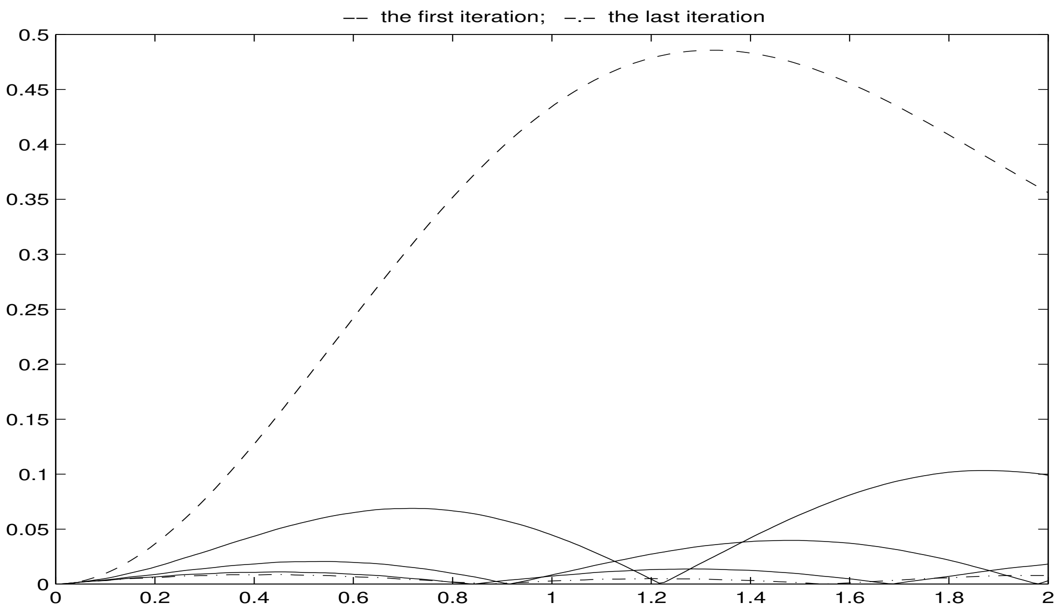

Figure 7: Learning algorithm (8) with digital filtration, 30% deviation of the nomianal parameters

the following convolution

$$u_{k+1}(t) = u_k(t) - \int_0^T f(t-\tau) z_k(\tau) d\tau, \quad t \in [0;T], \tag{10}$$

where the weight function $f(t)$ is given by

$$f(t) = \frac{1}{2\pi} \int_{-\varpi}^{+\varpi} \widehat{H}_{yu}^{-1}(i\omega) e^{i\omega t} d\omega, \quad t \in [-T;+T], \tag{11}$$

The convergence condition of the above algorithm is stated in the next theorem [16].

Theorem 4 *Let $\widehat{H}_{yu}(s)$ be an approximation of the system transfer function $H_{yu}(s)$ in the frequency range $[0, \bar{\omega}]$ and let the following condition hold*

$$\sup_{\omega \in [0,\bar{\omega}]} |1 - H_{yu}(i\omega)\widehat{H}_{yu}(i\omega)| < 1. \tag{12}$$

Then the error of the iterative learning procedure defined by (10) and (11) converges to the limiting error $z_*(t), t \in [0,T]$ bounded by

$$\|z_*(t)\|^2_{\mathcal{L}_2} \leq \frac{1}{\pi} \int_{\varpi}^{\infty} |Z_0(i\omega)|^2 d\omega$$

The proof of this theorem follows from the last statement of Theorem 2. We would like to emphasize that the weight function $f(t)$ can be computed beforehand and the proposed algorithm does not demand any differentiation. Note that in this case only the high-frequency part of the initial error $z_0(t)$ contributes to the level of the limiting error. Hence the limiting error can be made small by choosing the desired output $y_d(t)$ as a smooth function. The choice

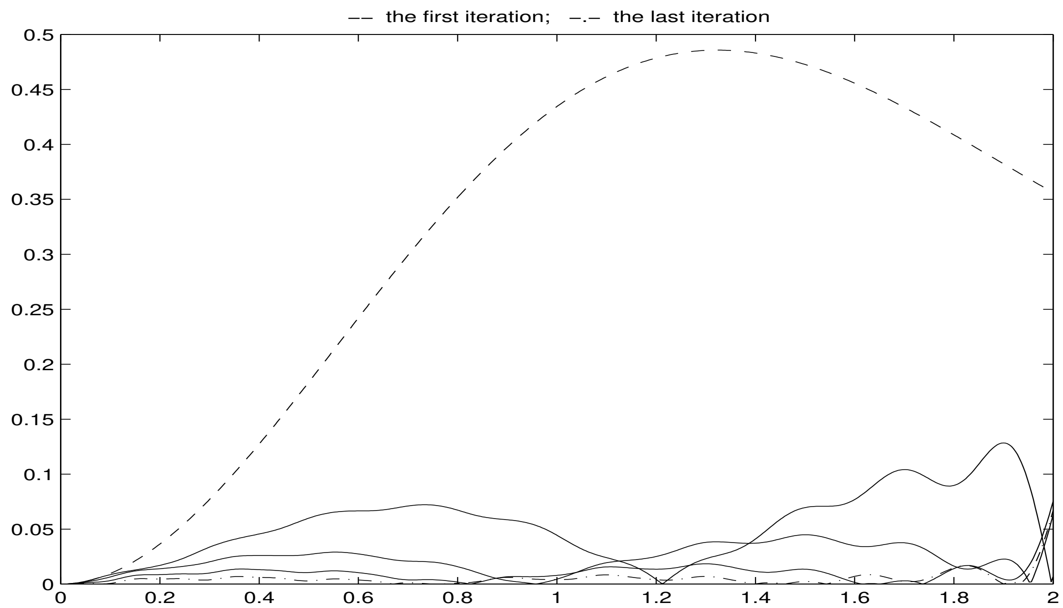

Figure 8: Learning algorithm (10),(11), 30% deviation of the nomianal parameters

of $\bar{\omega}$ is ascertained by condition (12) and the tolerance for the limiting error. Finally we verify the convergence of the proposed algorithm by simulating it on the flexible benchmark model (see figure 8). In (11) we take $\widehat{H}_{yv}(s)$ as follows:

$$\widehat{H}_{yv}(s) = \frac{1}{\bar{J}s^2 + k_1 s + k_0}$$

to approximate the known regular part of the system and we choose parameter $\bar{\omega}$ again from the interval $(\sqrt{k_0/(J_1+J_2)}, \sqrt{c(J_1+J_2)/J_1 J_2})$ to cut out the influence of the unknown singular part of the sytem.

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