Perturbation analysis of reduced resolvents and generalized inverses

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Abstract

We investigate analytic perturbations of the reduced resolvent of a finite-dimensional linear operator (also known as Drazin inverse in the linear algebra literature). Our approach is based on spectral theory of linear operators as well as on a new notion of group reduced resolvent. It allows to treat regular and singular perturbations in a unified framework. We provide an algorithm for computing the coefficients of the Laurent series of the perturbed reduced resolvent. In particular, the regular part coefficients can be calculated by simple recursive formulae. Finally, we apply these results to the perturbation analysis of Moore-Penrose generalized inverses.

1 Introduction

This paper is concerned with the analysis of the generalized inverses of a finite-dimensional perturbed linear operator. If the analysis of the eigenvalues and eigenvectors of a perturbed linear operator has been carried out in detail (see e.g. [5, 22]), few results are available for generalized inverses.

We consider the *reduced resolvent* of the perturbed operator, which is also known as *Drazin generalized inverse* in linear algebra [7, 10]. However, as we use spectral theory and the theory of analytic functions we would like to refer to this mathematical object as reduced resolvent rather than Drazin inverse [22].

Note that in this paper we distinguish between regular and singular perturbations. The perturbation is said to be *regular* if it does not change the dimension of null space. Otherwise, the perturbation is said to be *singular*. One of the main advantages of the present approach is that it allows us to analyse both regular and singular perturbations in a unified framework.

It turns out that the perturbation analysis of the reduced resolvent can be applied to the perturbation analysis of the group inverse and the *Moore-Penrose generalized inverse*. Stewart [34] has given a comprehensive analysis of the regular perturbation case and pointed out that the error bounds grow infinitely as the perturbation vanishes in the singular case. He has provided error bounds and a first order approximation for the general additive perturbation of the generalized inverse. The analytic perturbation is more restrictive than the general additive perturbation. However, it allows to perform a deeper analysis of the singular case. The case of singular perturbations is interesting for it provides information on the lack of continuity of solutions to perturbed linear systems. The work on singular perturbation of the generalized inverse is very limited. Deif [11] proposed an algorithm for computating the Laurent series for the generalized inverse of an analytically perturbed matrix. However, to

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implement his algorithm one has to obtain power series expansions for *all* the quantities in the singular value decomposition, namely for all the eigenvectors and eigenvalues. This can be very costly, especially for matrices of high dimension. In contrast, we propose an algorithm, which works in some *reduced* subspaces of small dimension.

In this paper, we prove the existence and obtain the Laurent series expansion of the reduced resolvent of a perturbed (finite-dimensional) linear operator. As a by-product, we also obtain the Laurent series of the Moore-Penrose generalized inverse of an analytically perturbed operator.

We provide recursive formulae for the (matrix) coefficients of the regular part of the Laurent series expansion for the reduced resolvent of the perturbed operator. Besides their theoretical interest, the recursive formulae are particularly useful when one needs to compute a significant number of terms in the Laurent series expansion. These formulae require knowledge of the singular part coefficients and the latter are obtained via a simple and new reduction technique. In particular, this reduction technique uses a novel *group reduced resolvent* different from the one considered in Chatelin [10]. Furthermore, our approach does not require *a priori* knowledge of the order of the pole which is obtained as a by-product.

Note that if an operator is invertible, then its reduced resolvent is just the ordinary inverse. Therefore, our results are applicable to the classical problem of the inversion of nearly singular operators [1, 3, 8, 13, 15, 16, 21, 23, 24, 25, 28, 29, 33, 35, 36], a problem with many important practical applications.

Last but not least, the limit matrix in the Taylor series expansion of the 0-eigenprojection matrix, has a simple exression in terms of (a) the original unperturbed 0-eigenprojection, (b) the perturbation matrices and (c), the singular part of the reduced resolvent. This provides some insight on how the perturbed 0-eigenvectors relate to the original 0-eigenvectors.

Let us also mention an important issue. In the course of the procedure that we propose, one has to determine the multiplicity of zero eigenvalues and verify whether they are semi-simple. Of course, neither task is easy in the presence of rounding errors. (Note that for the perturbation of the Moore-Penrose generalized inverse the semi-simplicity issue is not relevant, as we transform the problem into an equivalent problem with symmetric matrices). Even though these issues should not be underestimated, the primary focus of this paper is to develop a specific reduction process for computing the Laurent series expansion of perturbed reduced resolvents, that we think is novel in several aspects. The details of the practical implementation are beyond the scope of this paper and represent a topic for future research.

The results of the present paper can be immediately applied to singular perturbations of Markov chains [2, 19, 20, 32]. For example, the deviation matrix of a Markov chain is just (with minus sign) the reduced resolvent of the Markov chain generator. To demonstrate this application and verify the validity of our theoretical results, a simple illustrative example is provided at the end of the paper. Another application is concerned with perturbations of spectral inverses arising in the chemical networks, as treated in the work of Bohl and Lancaster [6]. Yet another interesting potential application of our results is the perturbation analysis of least square problems [11, 34].

The paper is organized as follows. In Section 2, we present the notation and introduce the basic notions of generalized inverses and reduced resolvents. In Section 3, for a better understanding of the paper, we briefly outline the methodology and underline some important computational issues. In Section 4, we prove the existence of a Laurent series expansion for the reduced resolvent of a perturbed matrix. In Section 5, we provide recursive formula for the coefficients of the regular part of the Laurent series. For clarity of exposition, the proof of the main result is postponed to an Appendix. In Section 6, we propose a specific reduction process that permits to compute the coefficients of the singular part and finally, we derive a simple expression for the limiting 0-eigenprojection matrix as the perturbation vanishes.

2 Preliminaries

In this section we recall some basic definitions and properties of generalized inverses [7] as well as several useful facts from spectral theory and from the theory of analytic functions [5, 22, 26].

We start with the classical definition of Moore-Penrose generalized inverse, that is:

Definition 1 *If $A \in \mathbb{R}^{m\times n}$, then the Moore-Penrose generalized inverse (or pseudo-inverse) $A^\dagger \in \mathbb{R}^{m\times n}$ is defined by*

$$AA^\dagger A = A \tag{1}$$

$$A^\dagger AA^\dagger = A^\dagger \tag{2}$$

$$(AA^\dagger)^T = AA^\dagger \tag{3}$$

$$(A^\dagger A)^T = A^\dagger A \tag{4}$$

The next properties of the pseudo-inverse [7] will be used in the sequel.

$$(A^T)^\dagger = (A^\dagger)^T \tag{5}$$

$$A^T = A^T AA^\dagger = A^\dagger AA^T \tag{6}$$

$$(A^T A)^\dagger = A^\dagger A^{T\dagger} \tag{7}$$

$$A^\dagger = (A^T A)^\dagger A^T = A^T(AA^T)^\dagger \tag{8}$$

An important particular case of the Moore-Penrose generalized inverse is the so-called *group inverse*, defined as follows:

Definition 2 *Suppose that A is a square matrix. The group inverse A^g, if it exists, is characterized as the unique matrix satisfying the next three equations.*

$$AA^g A = A \tag{9}$$

$$A^g AA^g = A^g \tag{10}$$

$$AA^g = A^g A. \tag{11}$$

Existence of the group inverse of $A \in \mathbb{R}^{n\times n}$ is equivalent to the possibility to decompose the space $\mathbb{R}^n$ into a *direct* sum of the null space and the range of A (see e.g. [7]).

We now show that computing the Moore-Penrose generalized inverse reduces to computing the group inverse of a square symmetric matrix.

Lemma 1 *The Moore-Penrose generalized inverse of A satisfies the identity*

$$A^\dagger = (A^T A)^g A^T = A^T(AA^T)^g. \tag{12}$$

PROOF: To prove the above formula, we only need to verify that the Moore-Penrose generalized inverse $(A^T A)^\dagger$ is also the group inverse of $A^T A$. Indeed, $(A^T A)^\dagger$ satisfies (9)–(11). It is obvious that (9) and (10) hold, since by definition, the generalized inverse satisfies (1), (2). The last identity (11) is obtained via

$$(A^T A)(A^T A)^\dagger = A^T AA^\dagger A^{T\dagger} = A^T AA^\dagger A^{\dagger T} = A^T A^{\dagger T} = (A^\dagger A)^T = A^\dagger A = (A^T A)^\dagger (A^T A),$$

using (7), (5), (6), (4) and (8) respectively. Thus, the matrix $(A^T A)^\dagger$ satisfies (11), and, therefore, $(A^T A)^g = (A^T A)^\dagger$, which immediately yields (12).

□

Any (finite-dimensional) linear operator $A \in \mathbb{R}^{n\times n}$ possesses the following spectral representation

$$A = \sum_{i=0}^{p} (\lambda_i P_i + D_i), \tag{13}$$

where $p+1$ is the number of distinct eigenvalues of A, P_i is the eigenprojection and D_i is the nilpotent operator corresponding to the eigenvalue λ_i. By convention, λ_0 is the zero eigenvalue of A, that is, $\lambda_0 = 0$. In the case, where there is no zero eigenvalue, the eigenvalues are enumerated from $i = 1$. The *resolvent* is another very important operator in spectral theory.

Definition 3 *The following operator-valued function of the complex parameter ζ is called the resolvent of the operator $A \in \mathbb{R}^{n\times n}$.*

$$R(\zeta) = (A - \zeta I)^{-1}.$$

The resolvent satisfies the *resolvent identity*:

$$R(\zeta_1) - R(\zeta_2) = (\zeta_1 - \zeta_2) R(\zeta_1) R(\zeta_2), \tag{14}$$

for all $\zeta_1, \zeta_2 \in \mathbb{C}$. The resolvent has singularities at the points $\zeta = \lambda_k$, where λ_k are the eigenvalues of A. In a neighbourhood of each singular point λ_k the resolvent can be expanded as a Laurent series

$$R(\zeta) = -\sum_{n=1}^{m_k-1} \frac{1}{(\zeta-\lambda_k)^{n+1}} D_k^n - \frac{1}{\zeta-\lambda_k} P_k + \sum_{n=0}^{\infty} (\zeta-\lambda_k)^n S_k^{n+1}, \tag{15}$$

where S_k is the *reduced resolvent* corresponding to the eigenvalue λ_k with geometric multiplicity m_k.

In particular, the reduced resolvent S_0 corresponding to the zero eigenvalue is denoted by $A^{\#}$ and we simply refer to it as the *reduced resolvent*. Note that the reduced resolvent is also known in linear algebra as the *Drazin generalized inverse*. The reduced resolvent has the following basic properties [22]

$$AA^{\#} = I - P_0, \tag{16}$$

$$P_0 A^{\#} = 0. \tag{17}$$

This characterization shows that $A^{\#}$ is the "inverse" of A in the complementary subspace to the generalized null space of A. Here by *generalized* null space we mean a subspace which is spanned by all eigenvectors and generalized (Jordan) eigenvectors corresponding to the zero eigenvalue. Note that P_0 is a projection onto this generalized null space.

Moreover, if the underlying space admits a decomposition into the *direct* sum of the null space and the range of the operator A (recall that this is a necessary and sufficient condition for the existence of the group inverse [7]), then the reduced resolvent and the group inverse coincide, and the following Laurent expansion holds

$$R(\zeta) = -\frac{1}{\zeta} P_0 + \sum_{n=0}^{\infty} \zeta^n (A^{\#})^{n+1}. \tag{18}$$

Since the reduced resolvent is the constant term in the Laurent series (15) and (18), it can be calculated via the following Cauchy integral formula [22, 26]

$$A^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta)\, d\zeta, \tag{19}$$

where Γ_0 is a closed positively-oriented contour in the complex plane, enclosing 0 but no other eigenvalue of A. The above formula will play a crucial role in the sequel.

The reduced resolvent also has a simple expression in terms of eigenprojections, eigenvalues and nilpotent operators of the original operator A. Namely,

$$A^{\#} = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i} P_i + \sum_{j=1}^{m_i - 1} (-1)^j \frac{1}{\lambda_i^{j+1}} D_i^j \right\}. \tag{20}$$

The above expression again demonstrates that the reduced resolvent is the inverse operator in the complementary subspace to the generalized null space. Moreover, this expression exactly represents the inverse operator A^{-1} whenever A has no zero eigenvalue.

3 Outline of the methodology

Let $A_0 \in \mathbb{R}^{m \times n}$ be a finite-dimensional linear operator and let $A(z)$ be an analytic perturbation of A_0, given by:

$$A(z) = A_0 + zA_1 + z^2 A_2 + \ldots , \tag{21}$$

where $|z| < z_{max}$ is a small complex perturbation parameter. [1] The main goal of the present paper is to compute the Laurent series expansion for the reduced resolvent $A^{\#}(z)$ of the perturbed operator (21), that is,

$$A^{\#}(z) \; = \; \sum_{j=-s}^{+\infty} z^j \, H_j \tag{22}$$

for $0 < |z| < z_{max}$. From a practical point of view, only the terms of the singular part (and, perhaps, one or two regular terms) are important to analyse the impact of the perturbation on $A^{\#}(z)$.

We next outline the main points of the methodology developed in the ensuing sections:

- We will show that the series (22) can be written as

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z} B^{\#\Omega}(z) + \frac{1}{z^2} C^{\#\Omega}(z) + \ldots + \frac{1}{z^s} Z^{\#}(z), \tag{23}$$

  where each of the terms $A^{\#\Omega}(z)$, $B^{\#\Omega}(z), \ldots$ is the *group reduced resolvent* (a novel notion to be introduced later) of some operator acting on a space of smaller dimension. All these group reduced resolvents possess Taylor series expansions, a crucial property that we exploit in our computational procedure. For instance,

$$B^{\#\Omega}(z) = B_0^{\#\Omega} + \sum_{j=1}^{\infty} z^j B_j^{\#\Omega}.$$

  Each term of the Taylor series of $A^{\#\Omega}(z)$, $B^{\#\Omega}(z), \ldots, Z^{\#}(z)$ contribute to (22). For example, if the order of the pole is $s = 2$, then in (22),

$$\begin{array}{rcl} H_{-2} & = & C_0^{\#}, \\ H_{-1} & = & B_0^{\#} + C_1^{\#}, \\ H_0 & = & A_0^{\#} + B_1^{\#} + C_2^{\#}, \\ H_1 & = & A_1^{\#} + B_2^{\#} + C_3^{\#}, \\ & \cdots & \end{array} \tag{24}$$

[1]If we analyse the reduced resolvent or the group inverse of the perturbed operator, we always assume that $m = n$.

Note in passing that the singular terms plus the first regular term of (22) require only $s(s+1)/2$ coefficients of the Taylor series $A^{\#\Omega}(z)$, $B^{\#\Omega}(z), \ldots, Z^{\#}(z)$, a small number whenever s is small. And indeed, this is the case in many practical applications, as already mentioned in [6, 2, 19, 20].

- For computing the regular part terms $H_0, H_1, \ldots$, two approaches are possible. The first one is just to compute as many terms of $A^{\#\Omega}(z)$, $B^{\#\Omega}(z), \ldots$ as needed in (24). Since each term of $A^{\#\Omega}(z)$, $B^{\#\Omega}(z), \ldots$ is computationally demanding, this approach is reasonable for a few terms only, and as already pointed out, in most practical applications this will be the case. However, if one wishes to compute a significant number of regular terms, then there is an alternative approach which uses the expansion of the eigenprojection $P_0(z)$ for the perturbed operator, that is, when the coefficients of the series
$$P_0(z) = P_0 + \sum_{j=1}^{\infty} z^j P_j.$$
are available. Several methods for the computation of these coefficients can be found in [5, 10, 22]. In this case, we propose a *simple* recursive formula for the calculation of the regular part coefficients $H_0, H_1, \ldots$ in (22).

4 Existence of a Laurent series expansion

In this section we prove the existence of a Laurent series expansion for the reduced resolvent $A^{\#}(z)$ of the analytically perturbed operator $A(z)$. The latter will imply the existence of a Laurent series expansion for both the group inverse and the Moore-Penrose generalized inverse.

First consider the resolvent $R(\zeta, z) := (A(z) - \zeta I)^{-1}$ of the perturbed operator $A(z)$. One can expand $R(\zeta, z)$ in a power series with respect to the complex variable z near $z = z_0$, as follows (see e.g. [22])

$$R(\zeta, z) \;=\; R(\zeta, z_0) + \sum_{n=1}^{\infty} (z - z_0)^n R^{(n)}(\zeta, z_0), \tag{25}$$

where

$$R^{(n)}(\zeta, z_0) \;:=\; \sum_{\nu_1 + \ldots + \nu_p = n} (-1)^p R(\zeta, z_0) A_{\nu_1} R(\zeta, z_0) A_{\nu_2} \cdots R(\zeta, z_0) A_{\nu_p} R(\zeta, z_0),$$

where A_{ν_k} are the coefficients of $A(z)$ and $\nu_k \geq 1$. The above expansion is called the second Neumann series for the resolvent. It is *uniformly* convergent for z sufficiently close to z_0 and $\zeta \in D$, where D is a compact subset of the complex plane which does not contain the eigenvalues of $A(z_0)$ [22].

Theorem 1 *Let $A(z)$ be the analytic perturbation of the matrix A_0 given by* (21). *Then, the reduced resolvent $A^{\#}(z)$ of the perturbed operator $A(z)$ can be expanded as a Laurent series*

$$A^{\#}(z) \;=\; \sum_{n=-s}^{+\infty} z^n H_n \tag{26}$$

for $0 < |z| < z_{max}$.

Proof: We first show that there exists a domain $0 < |z| < z_{max}$ such that $A^{\#}(z)$ can be expanded in Taylor series at any point z_0 in this domain. For a fixed, arbitrary $z > 0$, (19) becomes

$$A^{\#}(z) = \frac{1}{2\pi i} \int_{\Gamma_0(z)} \frac{1}{\zeta} R(\zeta, z) d\zeta, \tag{27}$$

where $\Gamma_0(z)$ is a closed positively-oriented curve enclosing the origin but no other eigenvalue of $A(z)$.

With z_{max} less than the modulus of any non-zero eigenvalue of A_0, expand the perturbed resolvent in the power series (25), around the point z_0 (with $0 < |z_0| < z_{max}$). Then, substituting that series in the integral formula (27), yields

$$A^{\#}(z) = \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} \left[R(\zeta, z_0) + \sum_{n=1}^{\infty} (z - z_0)^n R^{(n)}(\zeta, z_0) \right] d\zeta.$$

Since the power series for $R(\zeta, z)$ is uniformly convergent for z sufficiently close to z_0, we can integrate the above series term by term,

$$\begin{aligned} A^{\#}(z) &= \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} R(\zeta, z_0)\, d\zeta + \sum_{n=1}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} [R^{(n)}(\zeta, z_0)]\, d\zeta \\ &= A^{\#}(z_0) + \sum_{n=1}^{\infty} (z - z_0)^n H_n(z_0), \end{aligned} \tag{28}$$

where the coefficients are defined by

$$H_n(z_0) := \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} [R^{(n)}(\zeta, z_0)]\, d\zeta.$$

The convergence of power series (28) in some non-empty domain $0 < |z| < z_{max}$ can be shown by using the bounds for the contour integrals (for very similar development see for example [22, Ch.2, Sec.3]). From the power series (28), we can see that $A^{\#}(z)$ is holomorphic in the domain $0 < |z| < z_{max}$. Consequently, by Laurent's theorem [26, v. II, p. 7], we conclude that $A^{\#}(z)$ possesses a Laurent series expansion at $z = 0$ (with radius of convergence z_{max}), i.e.,

$$A^{\#}(z) = \sum_{n=-\infty}^{+\infty} z^n H_n. \tag{29}$$

We next show that the pole at $z = 0$ can be at most of finite order. Consider the spectral representation (20) for the reduced resolvent of the perturbed operator $A(z)$

$$A^{\#}(z) = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i(z)} P_i(z) + \sum_{j=1}^{m_i - 1} (-1)^j \frac{1}{\lambda_i^{j+1}(z)} D_i^j(z) \right\}.$$

From Kato [22], we know that the perturbed eigenvalues $\lambda_i(z)$ are bounded in $|z| \leq z_{max}$ and they have at most algebraic singularities. Furthermore, the eigenprojections $P_i(z)$ and nilpotents $D_i(z)$ can also have only algebraic singularities and poles of finite order. Therefore, none of the functions $\lambda_i(z)$, $P_i(z)$ and $D_i(z)$ can have an essential singularity. This latter fact implies that their finite sums, products or divisions as in $A^{\#}(z)$, do not have an essential singularity as well and, consequently, the order of pole in (29) is finite. This completes the proof.

□

Note that the Laurent series for the perturbed group inverse does not always exist. Indeed, the existence of the group inverse of the unperturbed operator does not imply the existence of the group inverse of the perturbed operator. However, existence results for the Laurent series for the group inverse $A^g(z)$ and the Moore-Penrose generalized inverse $A^{\dagger}(z)$ are direct consequences of the above theorem. The following is a general sufficient condition for the existence of the Laurent series for the perturbed group inverse.

Corollary 1 *Let the group inverse $A^g(z)$ of the analytically perturbed operator $A(z)$ exist in some non-empty (possibly punctured) neighbourhood of $z = 0$. Then the group inverse $A^g(z)$ can be expanded as a Laurent series around $z = 0$ with a non-zero radius of convergence.*

PROOF: Whenever the group inverse exists, it coincides with the reduced resolvent. Therefore, the existence of a Laurent series follows from Theorem 1.

□

By contrast, the Laurent series for the Moore-Penrose generalized inverse $A^{\dagger}(z)$ always exists.

Corollary 2 *Let $A^{\dagger}(z)$ be the Moore-Penrose generalized inverse of the analytically perturbed matrix $A(z)$. Then, $A^{\dagger}(z)$ possesses a Laurent series expansion*

$$A^{\dagger}(z) = \frac{1}{z^s} B_{-s} + \ldots + \frac{1}{z} B_{-1} + B_0 + z B_1 + \ldots \tag{30}$$

in some non-empty region around the origin.

PROOF: Rewriting (12) for the perturbed operator $A(z)$ yields

$$A^{\dagger}(z) = (A^T(z) A(z))^g A^T(z).$$

Note that the group inverse of a symmetric matrix always exists. Since the group inverse is a particular case of the reduced resolvent and the product or sum of two analytic functions is an analytic function again, we can write

$$A^{\dagger}(z) = \mathcal{A}^{\#}(z) A^T(z), \tag{31}$$

where $\mathcal{A}(z) = \sum_{k=0}^{\infty} z^k \mathcal{A}_k$ with $\mathcal{A}_k = \sum_{i=0}^{k} A_i^T A_{k-i}$. Note that $\mathcal{A}(z)$ is an analytically perturbed operator. According to Theorem 1, $\mathcal{A}^{\#}(z)$ has a Laurent series expansion, and from (31), we may conclude that the Laurent series for the perturbed pseudo-inverse $A^{\dagger}(z)$ exists as well.

□

Remark 1 *We would like to note that the statement of Corollary 2 can be considered as a particular case of more general existence results in [3, 4]. However, the reduced resolvent does not belong to the family of relative inverses and hence one cannot apply (at least, not immediately) the results of [3, 4] to the proof of Theorem 1.*

Remark 2 *We would like to emphasize that according to (31), computing the Laurent series of the perturbed pseudo-inverse $A^{\dagger}(z)$ reduces to computing the Laurent series of a reduced resolvent. Moreover, $\mathcal{A}(z)$ is a symmetric perturbation, that is, $\mathcal{A}_k^T = \mathcal{A}_k, k = 0, 1, \ldots$ This guarantees that the reduction process defined in Section 6 is indeed applicable in this case.*

5 Recursive formulae for the regular part coefficients

In this subsection, we derive recursive formulae for the coefficients of the regular part of the Laurent series (26). We use the same analytic technique as in [22, 23], based on Cauchy contour integrals and resolvent-like identities.

First, observe that the structure of the reduced resolvent $(A_0+zA_1+z^2A_2+...)^{\#}$ is similar to the structure of the classical resolvent $(A_0-zI)^{-1}$. Moreover, $A^{\#}(z)$ becomes precisely the resolvent if $A_1=-I$ and $A_k=0$ for $k\geq 2$. Therefore, one can expect that these two mathematical objects have some similar features. It turns out that the reduced resolvent of an analytically perturbed operator $A(z)$ satisfies an identity similar to the resolvent identity (14).

Lemma 2 *The reduced resolvent $A^{\#}(z)$ of the analytically perturbed operator $A(z)=\sum_{k=0}^{\infty} z^k A_k$ satisfies the resolvent-like identity:*

$$A^{\#}(z_1)-A^{\#}(z_2)=\sum_{k=1}^{\infty}(z_2^k-z_1^k)A^{\#}(z_1)A_kA^{\#}(z_2)+A^{\#}(z_1)P_0(z_2)-P_0(z_1)A^{\#}(z_2), \quad (32)$$

where $P_0(z)$ is the eigenprojection matrix corresponding to the zero eigenvalue.

PROOF: Consider the following expression

$$A(z_2)-A(z_1) = \sum_{k=1}^{\infty}(z_2^k-z_1^k)A_k.$$

Premultiplying by $A^{\#}(z_1)$ and postmultiplying by $A^{\#}(z_2)$, yields

$$A^{\#}(z_1)A(z_2)A^{\#}(z_2)-A^{\#}(z_1)A(z_1)A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k-z_1^k)A^{\#}(z_1)A_kA^{\#}(z_2).$$

Then, using (16), we get

$$A^{\#}(z_1)[I-P_0(z_2)]-[I-P_0(z_1)]A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k-z_1^k)A^{\#}(z_1)A_kA^{\#}(z_2).$$

Equivalently,

$$A^{\#}(z_1)-A^{\#}(z_2)=\sum_{k=1}^{\infty}(z_2^k-z_1^k)A^{\#}(z_1)A_kA^{\#}(z_2)+A^{\#}(z_1)P_0(z_2)-P_0(z_1)A^{\#}(z_2),$$

which is the desired identity (32).

□

In the next theorem, we obtain a general relation between the coefficients of the Laurent series (26).

Theorem 2 *Let $H_k, k=-s,-s+1,...$ be the coefficients of the Laurent series (26) and $P_0(z)=\sum_{k=0}^{\infty} z^k P_{0k}$ be a power series for the eigenprojection corresponding to the zero eigenvalue of the perturbed operator. Then the coefficients $H_k, k=-s,-s+1,...$ satisfy the following relation*

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_kH_{m+i-k+1}=-(\eta_n+\eta_m-1)H_{n+m+1}$$

$$
-\left\{\begin{array}{ll} 0, & m<0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1}+z_1P_{0m+2}+...]dz_1, & m\geq 0 \end{array}\right.
$$

$$
-\left\{\begin{array}{ll} 0, & n<0 \\ \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+...]A^{\#}(z_2)dz_2, & n\geq 0, \end{array}\right. \tag{33}
$$

where

$$
\eta_m := \left\{\begin{array}{ll} 1, & m\geq 0, \\ 0, & m<0. \end{array}\right.
$$

For the sake of clarity of presentation, the detailed proof is postponed to the Appendix. Now the recursive formula for the coefficients of the regular part of the Laurent series (26) becomes a corollary of the above general result.

Corollary 3 *Suppose that the coefficients $H_k, k=-s,...,-1,0$ and $P_{0k}, k=0,1,...$ are given. Then, the coefficients of the regular part of the Laurent expansion (26) can be computed by the following recursive formula:*

$$
\begin{aligned}
H_{m+1} \;=\;& -\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s}H_{-j}A_{i+j+1}\right)H_{m-i}-\sum_{i=1}^{m}P_{0m+1-i}H_i \\
& -(P_{0m+1}H_0+...+P_{0m+1+s}H_{-s})-(H_{-s}P_{0m+1+s}+...+H_0P_{0m+1})
\end{aligned} \tag{34}
$$

for $m=0,1,\ldots$.

Proof: Let us take $n=0$, $m>0$ and then simplify the last two terms in (33), so that

$$
\begin{aligned}
& \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1}+z_1P_{0m+2}+...]dz_1 \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{1}{z_1}[\frac{1}{z_1^s}H_{-s}+...+\frac{1}{z_1}H_{-1}+H_0+...][P_{0m+1}+z_1P_{0m+2}+...]dz_1 \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{1}{z_1}\frac{1}{z_1^{m+1}}\frac{1}{z_1^s}[H_{-s}P_{0m+1+s}+...+H_0P_{0m+1}]dz_1 \\
&= H_{-s}P_{0m+1+s}+...+H_0P_{0m+1}
\end{aligned} \tag{35}
$$

The last term can be dealt with in a similar fashion.

$$
\begin{aligned}
& \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+...]A^{\#}(z_2)dz_2 \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{01}+z_2P_{02}+...]A^{\#}(z_2)dz_2 \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{1}{z_2}\frac{1}{Z_2^m}[P_{01}+z_2P_{02}+...][\frac{1}{z_1^s}H_{-s}+...+\frac{1}{z_1}H_{-1}+H_0+...]dz_2 \\
&= \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{1}{z_2}[P_{01}H_m+P_{02}H_{m-1}+...+P_{0m}H_1+P_{0m+1}H_0+...+P_{0m+1+s}H_{-s}]dz_2 \\
&= \sum_{i=1}^{m}P_{0m+1-i}H_i+(H_{-s}P_{0m+1+s}+...+H_0P_{0m+1})
\end{aligned} \tag{36}
$$

Substituting (35) and (36) into (33) with $n = 0$ and $m > 0$, we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = -H_{m+1} - (H_{-s}P_{0m+1+s} + ... + H_0 P_{0m+1})$$

$$-(P_{0m+1}H_0 + ... + P_{0m+1+s}H_{-s}) - \sum_{i=1}^{m} P_{0m+1-i}H_i.$$

Rearranging terms in the above expression, we get

$$\begin{aligned} H_{m+1} &= -\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s} H_{-j}A_{i+j+1}\right) H_{m-i} - \sum_{i=1}^{m} P_{0m+1-i}H_i \\ &\quad -(P_{0m+1}H_0 + ... + P_{0m+1+s}H_{-s}) - (H_{-s}P_{0m+1+s} + ... + H_0 P_{0m+1}) \end{aligned}$$

which is the recursive formula (34).

□

Remark 3 *If the perturbed operator $A(z)$ is invertible for $0 < |z| < z_{max}$, then the inverse $A^{-1}(z)$ can be expanded as a Laurent series*

$$A^{-1}(z) = \frac{1}{z^{-s}}H_{-s} + ... + \frac{1}{z}H_{-1} + H_0 + zH_1 + ...$$

and the formula (34) becomes

$$H_{m+1} = -\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s} H_{-j}A_{i+j+1}\right) H_{m-i}, \quad m = 0, 1, ...$$

which is the formula given by Korolyuk and Turbin [23]. Furthermore, if the perturbed operator is invertible and the perturbation is linear $A(z) = A_0 + zA_1$ we retrieve the recurrent formula obtained by Langenhop [25], and Schweitzer and Stewart [33]

$$H_{m+1} = (-H_0 A_1)H_m, \quad m = 0, 1, ...$$

Note that, using (34) together with (31), one can easily derive the recursive formulae for the regular part of the Laurent series expansion (30) of the Moore-Penrose generalized inverse.

6 Reduction process

We have seen that the regular terms in the Laurent series expansion (26) of $A^{\#}(z)$ can be computed recursively by (34). However, to apply (34), one first needs to compute the terms $H_{-s}, \ldots, H_{-1}, H_0$, that is, the terms of the singular part. One way to achieve this was discussed in [1, 2, 18]. That approach is based on the recursive solution of the fundamental equations, which are obtained by substitution of the Laurent series (26) into (16-17) and the identification of terms with the same powers of z. One may show (see [18]) that a *finite* subsytem of the fundamental equations provides a *unique* solution for $H_{-s}, \ldots, H_{-1}, H_0$. This subsystem can be solved by an algebraic reduction process similar to the one defined in [1, 2]. However, here we would like to pursue an alternative approach based on spectral theory and analytic function theory. In particular, we define a *group reduced resolvent* which permits us to treat *singular* and *regular* perturbations in a unified framework.

6.1 Regular case

In this subsection we apply analytic function techniques to express the power series for the reduced resolvent of the perturbed operator in the case of *regular* perturbations, that is, when the dimension of the null space of the operator does not change with perturbations. In other words, when there is no splitting of the zero eigenvalue of the perturbed operator at $z = 0$. The latter implies that the expansion (28) is valid in some neighbourhood of $z_0 = 0$, and any contour $\Gamma_0 := \Gamma_0(0)$ chosen so that it does not enclose eigenvalues other than zero. Namely, the expansion (28) takes the form

$$A^{\#}(z) = A_0^{\#} + \sum_{n=1}^{\infty} z^n A_n^{\#}, \tag{37}$$

where

$$A_0^{\#} = A^{\#}(0), \quad A_n^{\#} = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{1}{\zeta}[R^{(n)}(\zeta)]\, d\zeta$$

and

$$R^{(n)}(\zeta) = \sum_{\nu_1+\ldots+\nu_p=n} (-1)^p R(\zeta)A_{\nu_1}R(\zeta)A_{\nu_2}\cdots R(\zeta)A_{\nu_p}R(\zeta).$$

It turns out that it is possible to express the coefficients $A_n^{\#}$ in terms of

- the unperturbed reduced resolvent $A^{\#}(0)$,
- the eigenprojection P_0 corresponding to the zero eigenvalue of A_0,
- the perturbation matrices $A_n, n = 1, 2, \ldots$.

The next theorem gives the precise statement.

Theorem 3 *Suppose that the operator A_0 is perturbed analytically as in (21) and assume that the zero eigenvalue of A_0 is semi-simple and the perturbation is regular. Then, the matrices $A_n^{\#}, n = 1, 2, \ldots$ in the expansion (37) are given by the following formula*

$$A_n^{\#} = \sum_{p=1}^{n} (-1)^p \sum_{\substack{\nu_1+\ldots+\nu_p=n \\ \mu_1+\ldots+\mu_{p+1}=p+1 \\ \nu_j\ge 1,\, \mu_j \ge 0}} S_{\mu_1}A_{\nu_1}S_{\mu_2}\ldots A_{\nu_p}S_{\mu_{p+1}}, \tag{38}$$

where $S_0 := -P_0$ and $S_k := (A_0^{\#})^k, k = 1, 2, \ldots$.

Proof: Since Γ_0 encloses only the zero eigenvalue, we have by (37)

$$A_n^{\#} = \frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{\zeta}R^{(n)}(\zeta)d\zeta = \sum_{\nu_1+\ldots\nu_p=n}(-1)^p\frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{\zeta}R(\zeta)A_{\nu_1}R(\zeta)\ldots A_{\nu_p}R(\zeta)d\zeta.$$

$$= \sum_{\nu_1+\ldots\nu_p=n}(-1)^p \mathrm{Res}_{\zeta=0}\{\frac{1}{\zeta}R(\zeta)A_{\nu_1}R(\zeta)\ldots A_{\nu_p}R(\zeta)\}$$

In order to compute the above residue, we replace $R(\zeta)$ by its Laurent series (18) in the expression

$$\frac{1}{\zeta}R(\zeta)A_{\nu_1}R(\zeta)\ldots A_{\nu_p}R(\zeta),$$

and collect the terms with $1/\zeta$, that is, the terms

$$\sum_{\sigma_1+\ldots+\sigma_{p+1}=0} S_{\sigma_1+1}A_{\nu_1}S_{\sigma_2+1}\ldots A_{\nu_p}S_{\sigma_p+1}.$$

Next, we change indicies $\mu_k := \sigma_k + 1, k = 1, \ldots, p+1$, and rewrite the above sum as:

$$\sum_{\mu_1+\ldots+\mu_{p+1}=p+1} S_{\mu_1}A_{\nu_1}S_{\mu_2}\ldots A_{\nu_p}S_{\mu_p},$$

which yields (38).

□

Remark 4 *Of course, formula (38) is computationally demanding due to the combinatorial explosion. However, only few terms will be computed by this formula (see the arguments developed in Section 3).*

6.2 Singular case

We now show that by using a *reduction process*, we can transform the original *singular* problem into a *regular* one. We would like to emphasize that the reduction process of this section is different from the algebraic reduction technique proposed in [1, 2, 18]. This reduction process can be viewed as complimentary to the existing reduction process based on spectral theory which was applied by many authors [5, 9, 12, 22] to the eigenvalue problem. Moreover, to the best of our knowledge, applying the reduction technique to analytical perturbations of generalized inverses is new.

To develop the reduction technique in the context of generalized inverses, we will need to introduce a new notion of *group reduced resolvent*. A definition based on spectral representation is as follows:

Definition 4 *Let $A : R^n \to R^n$ be a linear operator with the spectral representation (13). Then, the* group reduced resolvent *$A^{\#\Lambda}$ relative to the group of eigenvalues $\Lambda := \{\lambda_i\}_{i=0}^k$ is defined as follows:*

$$A^{\#\Lambda} \stackrel{def}{=} \sum_{i=k+1}^{p} \left\{ \frac{1}{\lambda_i}P_i + \sum_{j=1}^{m_i-1}(-1)^j \frac{1}{\lambda_i^{j+1}}D_i^j \right\}$$

Remark 5 *Note that the reduced resolvent (see formula (20)) is a particular case of the group reduced resolvent. In this case, the group of eigenvalues consists only of the zero eigenvalue. In addition, we would like to mention the book of Chatelin [10], where a different group reduced resolvent is defined algebraically via a Sylvester equation.*

From our definition, the properties of a group reduced resolvent follow easily. In particular, in the next theorem, we will obtain an alternative analytic expression of the group reduced resolvent that will play a crucial role in perturbation analysis.

Theorem 4 *Let A be a linear operator with representation (13). Then, the group reduced resolvent relative to the eigenvalues $\Lambda = \{\lambda_i\}_{i=0}^k$ is given by*

$$A^{\#\Lambda} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta, \tag{39}$$

where Γ is a contour in the complex plane which encloses the set of eigenvalues $\{\lambda_i\}_{i=0}^k$, but none of the other eigenvalues $\{\lambda_i\}_{i=k+1}^p$.

PROOF: It is a well known fact [22] that the resolvent can be represented by

$$-\sum_{i=0}^{p}\left[\frac{1}{\zeta-\lambda_i}P_i+\sum_{j=1}^{m_i}\frac{1}{(\zeta-\lambda_i)^{j+1}}D_i^j\right].$$

Substituting the above expression into the integral of (39) yields

$$\frac{1}{2\pi i}\int_\Gamma\frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta \;=\; -\frac{1}{2\pi i}\int_\Gamma\sum_{i=0}^{p}\left[\frac{1}{\zeta(\zeta-\lambda_i)}P_i+\sum_{j=1}^{m_i}\frac{1}{\zeta(\zeta-\lambda_i)^{j+1}}D_i^j\right]d\zeta.$$

Using the fact that for every positive integer l

$$\mathrm{Res}_{\zeta=0}\frac{1}{\zeta(\zeta-\lambda)^l}=\frac{1}{(-\lambda)^l}\quad\text{and}\quad\mathrm{Res}_{\zeta=\lambda}\frac{1}{\zeta(\zeta-\lambda)^l}=-\frac{1}{(-\lambda)^l},$$

we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma\frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta &= \sum_{i=0}^{k}\mathrm{Res}_{\zeta=\lambda_i}\left\{-\sum_{i=0}^{p}\left[\frac{1}{\zeta(\zeta-\lambda_i)}P_i+\sum_{j=1}^{m_i}\frac{1}{\zeta(\zeta-\lambda_i)^{j+1}}D_i^j\right]\right\}\\ &= \sum_{i=k+1}^{p}\left\{\frac{1}{\lambda_i}P_i+\sum_{j=1}^{m_i-1}(-1)^j\frac{1}{\lambda_i^{j+1}}D_i^j\right\}.\end{aligned}$$

According to Definition 4, the latter expression is equal to the group reduced resolvent, so that the proof is complete.

□

Lemma 3 *Let $P=\sum_{i=0}^{k}P_i$ be the projection corresponding to the group of eigenvalues $\Lambda=\{\lambda_i\}_{i=0}^{k}$, then*

$$A^{\#\Lambda}=(A[I-P])^{\#}.$$

PROOF: Since $P_iP_j=\delta_{ij}P_i$ and $D_iP_j=\delta_{ij}D_i$, we have $P_i[I-P]=0$, $D_i[I-P]=0$ for $i=0,...,k$ and $P_i[I-P]=P_i$, $D_i[I-P]=D_i$ for $i=k+1,...,p$. Then, by (13) and above

$$(A[I-P])^{\#}=\left(\sum_{i=0}^{p}(\lambda_iP_i+D_i)[I-P]\right)^{\#}=\left(\sum_{i=k+1}^{p}(\lambda_iP_i+D_i)\right)^{\#}.$$

Using formula (20), we obtain

$$(A[I-P])^{\#}=\sum_{i=k+1}^{p}\left\{\frac{1}{\lambda_i}P_i+\sum_{j=1}^{m_i-1}(-1)^j\frac{1}{\lambda_i^{j+1}}D_i^j\right\}.$$

The latter is equal to the group reduced resolvent $A^{\#\Lambda}$ by Definition 4.

□

Now equipped with this new notion of *group reduced resolvent*, we go back to perturbation analysis. The group of the perturbed eigenvalues $\lambda_i(z)$ such that $\lambda_i(z)\to 0$ as $z\to 0$ is called the 0-group (see e.g. Kato [22]). We denote the 0-group of eigenvalues by Ω. The eigenvalues of the 0-group split from zero when the perturbation parameter differs from zero. Since the eigenvalues of the perturbed operator are analytic functions except at some points with algebraic singularities [22], each eigenvalue of the 0-group (other than 0) can be written as

$$\lambda_i(z) = z^\nu \lambda_{i\nu} + o(z^\nu), \tag{40}$$

with $\lambda_{ik} \neq 0$ and ν is a positive rational number. The reduction technique is essentially based on the semi-simplicity assumption of reduced operators [22], which will be introduced below. Under that assumption, the power ν in (40) must be an integer. The latter implies that we can partition the 0-group into subsets that we call z^l-groups. Namely, we say that the eigenvalue $\lambda_i(z)$ belongs to the z^l-group if $\lambda_i(z) = z^l \lambda_{il} + o(z^l)$, with $\lambda_{il} \neq 0$. We denote z^l-group by Λ_l.

Consider now the spectral representation of the perturbed reduced resolvent.

$$A^{\#}(z) = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i(z)} P_i(z) + \sum_{j=1}^{m_i-1} (-1)^j \frac{1}{\lambda_i^{j+1}(z)} D_i^j(z) \right\},$$

where $\{\lambda_i(z)\}_{i=1}^k$ is the 0-group. From the above formula one can see that in this case, the Laurent expansion for the reduced resolvent $A^{\#}(z)$ will possess terms with negative powers of z. Moreover, it turns out that under our assumptions, the z^k-group eigenvalues contribute to the terms of the Laurent expansion for $A^{\#}(z)$ with negative powers $-k, -k+1, ..., -1$ as well as to the regular part of the Laurent expansion.

The basic idea is to first treat the part of the perturbed operator corresponding to the eigenvalues which do not tend to zero as $z \to 0$. Then we treat subsequently the parts of the perturbed operator corresponding to the eigenvalues which belong to the z-group, z^2-group and so on.

What is helpful is that to treat the part of $A(z)$ corresponding to z^{k+1}-group we have to perform the same algorithm as for the part of the perturbed operator corresponding to the z^k-group. These steps constitute the (finite) reduction process.

Now we implement the general idea we have just briefly outlined above. Consider a fixed contour Γ_0 that encloses only the zero eigenvalue of the unperturbed operator A_0. Note that by continuity of eigenvalues the 0-group of eigenvalues of the perturbed operator $A(z)$ lies inside Γ_0 for z sufficiently small. Therefore, we may define the group reduced resolvent relative to the 0-group of eigenvalues as follows:

$$A^{\#\Omega}(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta, z) d\zeta \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} (A(z) - \zeta I)^{-1} d\zeta.$$

Since $A^{\#\Omega}(z)$ is an analytic function in some neighbourhood of the origin, it can be expanded as a power series

$$A^{\#\Omega}(z) \;=\; A_0^{\#\Omega} + \sum_{i=1}^{\infty} z^i A_i^{\#\Omega}. \tag{41}$$

Note that $A_0^{\#\Omega} = (A_0)^{\#}$ and from Theorem 3 it follows that the other coefficients $A_i^{\#\Omega}, i = 1, 2, ...$ can be calculated by the formula (38). We would like to emphasize that in general the group reduced resolvent $A^{\#\Omega}(z)$ is different from the reduced resolvent $A^{\#}(z)$. The example of Subsection 6.3 demonstrates this phenomenon. However, we note that $A^{\#\Omega}(z)$ does coincide with $A^{\#}(z)$ in the case of regular perturbations.

Another operator that is used extensively in the reduction process is the *group projection*

$$P(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} R(\zeta, z)\, d\zeta,$$

which describes the subspace corresponding to the eigenvalues which split from zero. The group projection is an analytic function in some small neighbourhood of the origin (see e.g. [22]).

Next, as in the classical reduction process [22, 5], we define the restriction $B(z)$ of the operator $A(z)$ to the subspace determined by the group projection $P(z)$, that is,

$$B(z) \;:=\; \frac{1}{z} A(z) P(z) \;=\; \frac{1}{2\pi i z} \int_{\Gamma_0} \zeta R(\zeta, z)\, d\zeta,$$

where Γ_0 is some fixed contour enclosing only the zero eigenvalue of the unperturbed operator A_0. For the operator $B(z)$ to be analytic at zero, we need the following assumption.

Assumption S1 *The zero eigenvalue of the operator A_0 is semi-simple, that is the nilpotent operator D_0 corresponding to $\lambda_0 = 0$ is equal to zero.*

Note that this assumption is not too restrictive. For example, in the case of a self-adjoint perturbation operator, the zero eigenvalue of A_0 is semi-simple. This is in particular the case when one studies the generalized inverse of an analytically perturbed operator since it reduces to study a symmetric perturbation of the reduced resolvent (see Remark 2). Whenever Assumption S1 is satisfied, the operator $B(z)$ can be expressed as a power series [22]

$$B(z) \;=\; B_0 + \sum_{i=1}^{\infty} z^i B_i,$$

with $B_0 = P(0) A_1 P(0)$, and

$$B_n \;=\; -\sum_{p=1}^{n+1} (-1)^p \sum_{\substack{\nu_1 + \ldots + \nu_p = n+1 \\ \mu_1 + \ldots + \mu_{p+1} = p-1 \\ \nu_j \geq 1, \mu_j \geq 0}} S_{\mu_1} A_{\nu_1} S_{\mu_2} \ldots A_{\nu_p} S_{\mu_{p+1}}, \tag{42}$$

where $S_0 := -P(0)$ and $S_k := ((A_0)^{\#})^k$.

Since the operator $B(z)$ is analytic in some neighbourhood of the origin, we can again construct the expansion for its group reduced resolvent

$$B^{\#\Omega}(z) \;=\; (B_0)^{\#} + \sum_{i=1}^{\infty} z^i B_i^{\#\Omega}. \tag{43}$$

The coefficients $B_i^{\#\Omega}, i = 1, 2, \ldots$ are calculated by the formula given in Theorem 3. This is the first reduction step. To continue, we must distinguish between two cases:

(i) If the splitting of the zero eigenvalue terminates, and consequently $B(z)$ is a *regular* perturbation of B_0, then $B^{\#\Omega}(z) = B^{\#}(z)$ and the reduced resolvent of the perturbed operator $A(z)$ is given by

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z} B^{\#}(z). \tag{44}$$

By substituting the series expansions (41) and (43) for $A^{\#\Omega}(z)$ and $B^{\#}(z)$ into (44), we obtain the Laurent series expansion for $A^{\#}(z)$, which has a simple pole at zero.

(ii) If the zero eigenvalue splits further, the expression

$$A^{\#\Omega\setminus\Lambda_1}(z) = A^{\#\Omega}(z) + \frac{1}{z} B^{\#\Omega}(z)$$

represents *only* the *group reduced resolvent* relative to the eigenvalues constituting the 0-group but not the z-group, and we have to continue the reduction process. In fact, we now consider $B(z)$ as a singular perturbation of B_0, and repeat the procedure with $B(z)$. The 0-group of eigenvalues of B_0 contains all the z^k-groups of $A(0)$ (with $k \geq 2$), but *not* the z-group. Specifically, we construct the next-step reduced operator

$$C(z) = z^{-1}B(z)Q(z),$$

where $Q(z)$ is the eigenprojection corresponding to the 0-group of the eigenvalues of $B(z)$. Again, to ensure that $C(z)$ is an analytic function of z, we assume

Assumption S2 *The zero eigenvalue of B_0 is semi-simple.*

We would like to emphasize that the subsequent reduction steps are totally identical with the first one. At each reduction step, we make the assumption Sk that the analogue of B_0 at step k has a semi-simple 0-eigenvalue. The final result is stated in the next theorem.

Theorem 5 *Let Assumptions Sk hold. Then, the reduction process terminates after a finite number of steps, say s, and the perturbed reduced resolvent $A^{\#}(z)$ has the following expression:*

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#\Omega}(z) + \frac{1}{z^2}C^{\#\Omega}(z) + ... + \frac{1}{z^s}Z^{\#}(z). \tag{45}$$

PROOF: Consider the first reduction step. Since $R(P(z))$ and $R(I - P(z))$ represent a direct decomposition of $\mathbb{R}^n$ and the subspace $R(P(z))$ is invariant under the operator $A(z)$, we can write

$$\begin{aligned} A^{\#}(z) &= (A(z)[I - P(z)] + A(z)P(z))^{\#} = (A(z)[I - P(z)])^{\#} + (A(z)P(z))^{\#} \\ &= A^{\#\Omega}(z) + z^{-1}(z^{-1}A(z)P(z))^{\#}, \end{aligned}$$

where Lemma 3 was used to get the first term of the right-hand-side. In view of Assumption $S1$, the operator $B(z) = z^{-1}A(z)P(z)$ is analytic in z and hence, one can apply the next reduction step. Similarly, Assumptions $Sk, k = 1, 2, \ldots$ guarantee that the reduction process can be carried out. Since the splitting of the zero eigenvalue has to terminate after a finite number of steps [22], we conclude that the reduction process has to terminate after a finite number of steps as well. Indeed, we successively eliminate the eigenvalues of the z-group, the z^2-group, etc ... Let $\lambda_i(z) = z^s\lambda_{is} + ...$ be the last eigenvalue which splits from zero. Then the corresponding reduced operator $Z(z)$ is regularly perturbed and the associated reduced resolvent $Z^{\#}(z)$ has the power series defined by Theorem 3. This completes the proof.

□

Summarizing, to obtain the Laurent series for $A^{\#}(z)$, and back to the discussion in Section 3, there are two cases to distinguish. First, if one needs only few regular terms of $A^{\#}(z)$, then it suffices to replace $A^{\#\Omega}(z), B^{\#\Omega}(z), ...$ in (45) by their respective power series (41) computed during the reduction process. Note that only few terms of the power series $A^{\#\Omega}(z), B^{\#\Omega}(z), ...$ are needed. Otherwise, if one wishes to compute a significant number of regular terms, then compute only $H_{-s}, \ldots, H_{-1}, H_0$ as above (in which case, again, only a few terms of $A^{\#\Omega}(z), B^{\#\Omega}(z), ...$ are needed) and then use the recursive formula (34). Of course, one needs first to compute the power series expansion of the eigenprojection $P_0(z)$, which can be obtained by several methods [5, 10, 22].

Moreover, applying formula (31), one can easily obtain the Laurent series for the Moore-Penrose generalized inverse of a perturbed operator. Since $\mathcal{A}_k, k = 0, 1, ...$ are symmetric (cf. Remark 2) the reduced operators are symmetric as well, and consequently, Assumptions Sk are satisfied.

Remark 6 *If the operator $A(z)$ has an inverse for $z \neq 0$, then the above algorithm can be used to calculate its Laurent expansion. Note that this problem, which was studied quite extensively [3, 8, 13, 15, 16, 21, 23, 24, 25, 28, 29, 33, 35, 36] is a particular case of our general approach.*

6.3 Example

As was mentioned in the introduction, the perturbation analysis of the reduced resolvent can be applied directly to the theory of singularly perturbed Markov chains. Namely, the reduced resolvent of the generator of a Markov chain is just with minus sign the deviation matrix of this chain. The deviation matrix plays a crucial role in the Markov chain theory. For example, it is used to obtain mean first-passage times. Taking into acount the above remark, we consider an example of a perturbed Markov chain. We have taken the example from Schweitzer [32] to demonstrate the validaty of our approach. Let us consider the following perturbed operator.

$$A(z) = A_0 + zA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} + z \begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ -4 & 3 & 1 \end{bmatrix}$$

Note that $-A(z)$ is the generator of a Markov chain. The zero eigenprojection and the reduced resolvent of the unperturbed matrix A_0 is given by

$$P(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \quad A_0^{\#} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

In this instance, the Laurent expansion for $A^{\#}(z)$ has a simple pole. Using Hassin and Haviv's method [17] for the determination of the singularity order of the perturbed Markov chains, one can check that

$$A^{\#}(z) = \frac{1}{z} H_{-1} + H_0 + zH_1 + \ldots$$

By applying the reduction process, we compute the singular coefficient H_{-1} and the first regular coefficient H_0. Since the zero eigenvalues of the reduced operators are always semi-simple in the case of perturbed Markov chains [12], we conclude from Theorem 5 that

$$H_{-1} = B_0^{\#}, \quad \text{and} \quad H_0 = A_0^{\#} + B_1^{\#\Omega}.$$

To compute $B_0^{\#}$ and $B_1^{\#\Omega}$, we need to calculate the first two terms of the expansion for the reduced operator $B(z)$. In particular,

$$B_0 = P(0)A_1P(0) = \begin{bmatrix} 2 & -1 & -1 \\ -3.5 & 1.75 & 1.75 \\ -3.5 & 1.75 & 1.75 \end{bmatrix}$$

$$B_1 = -(A_0^{\#}A_1P(0)A_1P(0) + P(0)A_1A_0^{\#}A_1P(0) + P(0)A_1P(0)A_1A_0^{\#}) = \frac{1}{8} \begin{bmatrix} 0 & 4 & -4 \\ -24 & 5 & 19 \\ 20 & -17 & -3 \end{bmatrix}$$

calculated with the help of (42). Next, we calculate the eigenprojection corresponding to the zero eigenvalue of the operator B_0, that is,

$$Q(0) = \frac{1}{22} \begin{bmatrix} 14 & 4 & 4 \\ 14 & 15 & -7 \\ 14 & -7 & 15 \end{bmatrix}.$$

Now using formula (38) from Theorem 3, we obtain

$$B_1^{\#\Omega} = Q(0)B_1(B_0^{\#})^2 - B_0^{\#}B_1B_0^{\#} + (B_0^{\#})^2B_1Q(0) = \frac{1}{2662} \begin{bmatrix} -16 & 52 & -36 \\ -236 & 41 & 195 \\ 248 & -201 & -47 \end{bmatrix}.$$

Thus, we finally obtain

$$H_{-1} = B_0^{\#} = \frac{1}{5.5^2}\begin{bmatrix} 1 \\ -1.75 \\ -1.75 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 8 & -4 & -4 \\ -14 & 7 & 7 \\ -14 & 7 & 7 \end{bmatrix}$$

and

$$H_0 = B_1^{\#\Omega} + A_0^{\#} = \frac{1}{1331}\begin{bmatrix} -8 & 26 & -18 \\ -118 & 686 & -568 \\ 124 & -766 & 642 \end{bmatrix}.$$

Note that the above H_{-1} coincides with the one given in [32]. Independently of the above calculations and using MATLAB symbolic toolbox, we have also expanded $A^{\#}(z)$ as a Laurent series and checked that H_0 is indeed correct.

The above method for the computation of the perturbed Markov chain deviation matrix has at least two advantages in comparison with the algebraic methods proposed in [2, 19, 31]. Fistly, this method can easily cope with the case where the perturbed Markov chain has several ergodic classes. Secondly, our approach does not require the knowledge of the series expansion for the ergodic projection, that is, the eigenprojection corresponding to the zero eigenvalue of the Markov chain generator. However, if we do have in hand the expansion for the ergodic projection, we can use the recursive formula (34) to compute the regular coefficients. Let us compute by the recursive formula the coefficient H_1 for our example. Using any method as in [9, 12, 19, 20, 23, 27, 31, 32], one can compute the coefficients for the expansion of the ergodic projection associated with z and z^2.

$$P_{01} = \frac{1}{121}\begin{bmatrix} 2 & -12 & 10 \\ 2 & -12 & 10 \\ 2 & -12 & 10 \end{bmatrix} \quad P_{02} = \frac{1}{1331}\begin{bmatrix} 32 & -192 & 160 \\ 32 & -192 & 160 \\ 32 & -192 & 160 \end{bmatrix}$$

Then, according to formula (34), we have

$$\begin{aligned} H_1 &= -(H_0A_1)H_0 - (P_{01}H_0 + P_{02}H_{-1}) - (H_{-1}P_{02} + H_0P_{01}) \\ &= \frac{1}{14641}\begin{bmatrix} -368 & 1856 & -1488 \\ -2128 & 12416 & -10288 \\ 1744 & -10816 & 9072 \end{bmatrix}. \end{aligned}$$

The latter coefficient also coincides with the one obtained with the help of MATLAB symbolic toolbox.

6.4 Asymptotics for $P_0(z)$

We know that the eigenprojection $P_0(z)$ of the perturbed operator corresponding to the identically zero eigenvalue is analytic in some (punctured) neighbourhood of $z = 0$ [4, 14], that is,

$$P_0(z) \;=\; P_{00} + \sum_{k=1}^{\infty} z^k P_{0k}, \quad \text{for} \quad z \neq 0. \tag{46}$$

For regular perturbations, P_{00} is just $P_0(0)$, and the group projection coincides with the eigenprojection. This is *not* the case for singular perturbations.

Therefore, an interesting question is how P_{00} in (46) relates to the original matrix $P_0(0)$ in the general case and how the power series (46) can be computed. The answers to these questions are provided below.

Proposition 1 *The coefficients of the power series (46) for the perturbed eigenprojection are given by*

$$P_{0k} = -\sum_{i=0}^{s+k} A_i H_{k-i}, \quad k = 1, 2, \ldots \; .$$

PROOF: The above formula is obtained by equating the terms with the same power of z in the identity (16) for the perturbed operators.

□

Corollary 4 *As $z \to 0$, the limit matrix P_{00} satisfies*

$$P_{00} \; = \; P_0(0)\,[I - \sum_{i=1}^{s} A_i H_{-i}]. \tag{47}$$

where $P_0(0)$ is the 0-eigenprojection of the unperturbed matrix A_0.

PROOF: When substituting the Laurent series expansion (29) into (16)-(17), and equating the terms of same power, one obtains in particular,

$$I - P_{00} \; = \; A_0 H_0 + A_1 H_{-1} + \ldots + A_s H_{-s} \; . \tag{48}$$

In addition, from $A(z)P_0(z) = 0$, we immediately obtain:

$$A_0 P_{00} \; = \; 0 \; \text{ so that } \; P_{00} \; = \; P_0(0)V, \tag{49}$$

for some matrix V. Moreover, as $P_0(0)^2 = P_0(0)$, we also have $P_0(0)P_{00} = P_0(0)^2 V = P_0(0)V = P_{00}$. Therefore, pre-multiplying both sides of (48) by $P_0(0)$, and using $P_0(0)A_0 = 0$, one obtains (47), the desired result.

□

Hence, (47) relates in a simple manner, the limit matrix P_{00} to the original 0-group $P_0(0)$, in terms of the perturbation matrices $A_k, k = 1, ..., s$, the original matrix $P_0(0)$, and the coefficients $H_{-k}, k = 1, ..., s$ of the singular part of $A(z)^{\#}$. This shows how the perturbed 0-eigenvectors compare to the unperturbed ones, for small z. Observe that in the case of a linear (or first-order) perturbation, only the singular term H_{-1} is involved. Finally, the regular case is obtained as a particular case, since then, $H_{-k}, k = 1, \ldots, s$ vanish so that $P_{00} = P_0(0)$.

Appendix

To prove Theorem 2 we use the Cauchy contour integration and the residue technique [26, 22, 23]. First we present some auxiliary results.

Lemma 4 *Let Γ_1 and Γ_2 be two closed positively-oriented contours in the complex plane around zero and let $z_1 \in \Gamma_1, z_2 \in \Gamma_2$. Furthermore, assume that the contour Γ_2 lies inside the contour Γ_1. Then the following formulae hold:*

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{z_2^{-m-1}}{z_2 - z_1} dz_2 = -\eta_m z_1^{-m-1}, \tag{50}$$

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{z_1^{-m-1}}{z_2 - z_1} dz_1 = -(1 - \eta_n) z_2^{-m-1}, \tag{51}$$

with

$$\eta_m := \begin{cases} 0, & m < 0, \\ 1, & m \geq 0. \end{cases}$$

and

$$\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{z_2^{-m-1}P_0(z_2)}{z_2 - z_1}dz_2 = \begin{cases} 0, & m < 0, \\ -z_1^{-m-1}[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}], & m \geq 0. \end{cases} \tag{52}$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{z_1^{-n-1}P_0(z_1)}{z_2 - z_1}dz_1 = \begin{cases} -z_2^{-n-1}P_0(z_2), & n < 0, \\ -[P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...], & n \geq 0. \end{cases} \tag{53}$$

Proof: The proof of formulae (50),(51) is given in the book of Kato [22] and in the book of Korolyuk and Turbin [23].

Let us establish the auxiliary integral (52). If $m < 0$, then the function $\frac{z_2^{-m-1}P_0(z_2)}{z_2 - z_1}$ is analytic inside the area enclosed by the contour Γ_2 and hence the auxiliary integral (52) is equal to zero by the Cauchy Integral Theorem. To deal with the case $m \geq 0$, we first expand the function $\frac{z_2^{-m-1}P_0(z_2)}{z_2 - z_1}$ as a Laurent series.

$$\frac{z_2^{-m-1}P_0(z_2)}{z_2 - z_1} = -\frac{z_2^{-m-1}P_0(z_2)}{z_1(1 - z_2/z_1)} = -z_1^{-1}z_2^{-m-1}[P_{00} + z_2 P_{01} + z_2^2 P_{02} + ...][1 + \frac{z_2}{z_1} + \frac{z_2^2}{z_1^2} + ...] =$$

$$= z_2^{-m-1}(-z_1^{-1})P_{00} + ... + z_2^{-1}(-z_1^{-1})[\frac{1}{z_1^m}P_{00} + \frac{1}{z_1^{m-1}}P_{01} + ... + P_{0m}] + ...$$

Then, according to the Residue Theorem, we have

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{z_2^{-m-1}P_0(z_2)}{z_2 - z_1}dz_2 &= (-z_1^{-1})[\frac{1}{z_1^m}P_{00} + \frac{1}{z_1^{m-1}}P_{01} + ... + P_{0m}] \\ &= -z_1^{-m-1}[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}]. \end{aligned}$$

Thus, we have calculated the integral (52). The same method is applied to calculate the auxiliary integral (53).

□

The proof of Theorem 2: Each coefficient of the Laurent series (26) can be represented by the contour integral formula

$$H_n = \frac{1}{2\pi i}\int_{\Gamma} z^{-n-1}A^{\#}(z)dz, \quad \Gamma \in D, \tag{54}$$

where Γ is a closed positively-oriented contour in the complex plane, which encloses zero but no other eigenvalues of A_0. Let us substitute (54) into the following expression

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n+i-1}A^{\#}(z_1)dz_1 A_k \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-i+k-2}A^{\#}(z_2)dz_2.$$

As in Lemma 4, we assume without loss of generality that the contour Γ_2 lies inside the contour Γ_1. Then, we can rewrite the above expressions as double integrals

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} z_1^{-n+i-1}z_2^{-m-i+k-2}A^{\#}(z_1)A_k A^{\#}(z_2)dz_2 dz_1$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} z_1^i z_2^{k-i-1} A^{\#}(z_1) A_k A^{\#}(z_2) dz_2 dz_1$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} \sum_{k=1}^{\infty} (z_2^k - z_1^k) A^{\#}(z_1) A_k A^{\#}(z_2) dz_2 dz_1.$$

Using the resolvent-like identity (32), we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i} A_k H_{m+i-k+1} =$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \left[\frac{A^{\#}(z_1) - A^{\#}(z_2)}{z_2 - z_1} - \frac{A^{\#}(z_1) P_0(z_2)}{z_2 - z_1} + \frac{P_0(z_1) A^{\#}(z_2)}{z_2 - z_1}\right] dz_2 dz_1.$$

Thus, we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i} A_k H_{m+i-k+1} = I_1 - I_2 + I_3,$$

with:

$$\begin{aligned}
I_1 &:= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{A^{\#}(z_1) - A^{\#}(z_2)}{z_2 - z_1} dz_2 dz_1, \\
I_2 &:= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{A^{\#}(z_1) P_0(z_2)}{z_2 - z_1} dz_2 dz_1, \\
I_3 &:= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{P_0(z_1) A^{\#}(z_2)}{z_2 - z_1} dz_2 dz_1.
\end{aligned}$$

Let us separately calculate the integrals I_1, I_2 and I_3. The integral I_1 can be written as

$$\begin{aligned}
I_1 &= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} A^{\#}(z_1) dz_2 dz_1 - \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} A^{\#}(z_2) dz_2 dz_1 \\
&= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_1} \left[\int\limits_{\Gamma_2} \frac{z_2^{-m-1}}{z_2 - z_1} dz_2\right] z_1^{-n-1} A^{\#}(z_1) dz_1 \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int\limits_{\Gamma_2} \left[\int\limits_{\Gamma_1} \frac{z_1^{-n-1}}{z_2 - z_1} dz_1\right] z_2^{-m-1} A^{\#}(z_2) dz_2
\end{aligned}$$

In the last equality we used Fubini Theorem to change the order of integration. Using the auxiliary integrals (50) and (51), we obtain

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i} \int\limits_{\Gamma_1} (-\eta_m z_1^{-m-1}) z_1^{-n-1} A^{\#}(z_1) dz_1 - \frac{1}{2\pi i} \int\limits_{\Gamma_2} (-(1-\eta_n) z_2^{-n-1}) z_2^{-m-1} A^{\#}(z_2) dz_2 \\
&= -\frac{\eta_n + \eta_m - 1}{2\pi i} \int\limits_{\Gamma_1} z_1^{-n-m-2} A^{\#}(z_1) dz_1 = -(\eta_n + \eta_m - 1) H_{n+m+1},
\end{aligned}$$

where the second integral can be taken over Γ_1 by the principle of deformation of contours. We calculate the second integral I_2 as follows:

$$
\begin{aligned}
I_2 &= \frac{1}{2\pi i}\int_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)\frac{1}{2\pi i}\int_{\Gamma_2}\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1}dz_2dz_1 \\
&= \begin{cases} \frac{1}{2\pi i}\int\limits_{\Gamma_1} 0z_1^{-n-1}dz_1, & m<0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)z_1^{-m-1}[P_{00}+z_1P_{01}+...+z_1^mP_{0m}]dz_1, & m\ge 0 \end{cases} \\
&= \begin{cases} 0, & m<0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-m-2}A^{\#}(z_1)[P_{00}+z_1P_{01}+...+z_1^mP_{0m}]dz_1, & m\ge 0 \end{cases} \\
&= \begin{cases} 0, & m<0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-m-2}A^{\#}(z_1)[P_0(z_1)-z_1^{m+1}P_{0m+1}-z_1^{m+2}P_{0m+2}-...]dz_1, & m\ge 0 \end{cases} \\
&= \begin{cases} 0, & m<0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-m-2}A^{\#}(z_1)[z_1^{m+1}P_{0m+1}+z_1^{m+2}P_{0m+2}+...]dz_1, & m\ge 0 \end{cases} \\
&= \begin{cases} 0, & m<0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1}+z_1P_{0m+2}+...]dz_1, & m\ge 0 \end{cases}
\end{aligned}
$$

where, in the above expressions, the auxiliary integral (52) and the property $A^{\#}(z)P_0(z)=0$ has been used. Now, we calculate the last integral I_3 with the help of the auxiliary integral (53).

$$
\begin{aligned}
I_3 &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{z_1^{-n-1}P_0(z_1)}{z_2-z_1}dz_1z_2^{-m-1}A^{\#}(z_2)dz_2 \\
&= \begin{cases} -\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-n-m-2}P_0(z_2)A^{\#}(z_2)dz_2, & n<0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+z_2^2P_{0n+3}+...]A^{\#}(z_2)dz_2, & n\ge 0, \end{cases} \\
&= \begin{cases} 0, & n<0, \\ -\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+z_2^2P_{0n+3}+...]A^{\#}(z_2)dz_2, & n\ge 0. \end{cases}
\end{aligned}
$$

Finally, summing up the three integrals I_1, I_2 and I_3, we obtain the relation (33).

□

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